Bull. Math. Soc. Sc. Math. Roumanie
Tome 48(96) No. 2, 2005, 103–118

Thermoelastic Plate in Frictional Contact

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Abstract

A model for dynamic frictional contact between a thermoelastic plate and a moving obstacle, which includes frictional heat generation, is presented. The obstacle may be reactive or rigid, and so contact is modeled by the normal compliance or the Signorini conditions. The existence of the unique weak solution for the problem with normal compliance is established by using approximations involving set-valued pseudo-monotone operators, a priori estimates, and Gronwall's inequality.

Key Words: Thermoelastic plate, contact with friction, frictional heat generation, weak solution, existence and uniqueness.
2000 Mathematics Subject Classification: Primary: 74M10 , Secondary: 74K20, 74H20, 74H25.

1 Introduction

We study a model for the dynamic frictional contact between a thermoelastic plate and a moving deformable obstacle which is situated under it. Under the action of applied forces the plate comes into contact with the moving surface and the resulting friction force generates heat which is conducted throughout the plate. Contact is described by the normal compliance condition. We establish the existence of the unique weak solution for the model. The proof is based on approximate problems, results from the theory of set-valued pseudo-monotone operators, a priori estimates on the approximate solutions, and Gronwall's inequality.

The Mathematical Theory of Contact Mechanics has made considerable progress recently (see the monographs [10, 11, 16, 19] and a host of references therein). However, the inclusion of thermal effects is rather rare ([1, 2, 8, 9] and references therein), in spite of the simple observation that frictional contact is usually accompanied by frictional heat generation, which can be substantial. Indeed, the application of the car brakes can generate over 100 HP of heat. This, in turn, may raise the system temperature to unacceptable levels and, therefore, the analysis and control of these processes are of considerable applied interest.

The novelty in the model is the inclusion of frictional contact and the accompanying frictional heat generation and heat conduction. Since the problem is two-dimensional both processes enter the equations as sources, and not as boundary conditions, which is the case in three dimensions.

This is a first step and the model here deals only with transversal motion. In later stages of this research we will consider the problem when the in-plane displacements and shear stresses are taken into account, too. Furthermore, there exists a considerable interest, both applied and theoretical, in the observability and controllability for such models. In subsequent work, we will consider local and global thermal null controllability properties of the thermoelastic variables. Some ideas we shall use can be found in [14, 20], where the Fréchet differentiability, in addition to the Lipschitz continuity of the normal compliance function are required. The observability of the state of the system from the boundary will allow to obtain important detailed information about the contact processes which is very difficult to measure experimentally.

2 The model

We consider a thermoelastic plate that in its reference configuration occupies the set $\Omega \subset \mathbb{R}^2$ and is of thickness h. Ω is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, and we denote by $\mathbf{n} = (n_1, n_2)$ the outer unit vector on Γ. The vertical displacement of the center plane of the plate is denoted by $w = w(x, y, t)$, and $\theta = \theta(x, y, t)$ is the temperature moment. A vertical force of surface density $f = f(x, y, t)$ acts on the plate and as a result it may come into contact with a moving obstacle which is situated below it, and is represented by $z = \phi(x, y)$, where ϕ is a given function. We assume that the obstacle, say a conveyor belt, is moving horizontally with a fixed velocity $\mathbf{v}$, and when the plate is in contact with the obstacle the frictional traction causes heat generation. However, we assume that the in-plate stresses are small and in this work they are neglected. This issue will be revisited in later stages of this study. Our main interest lies in the evolution of the thermomechanical state of the plate.

The setting of the problem is depicted in Fig. 1 where, for the sake of convenience, we set $z = \phi = -H_*$.

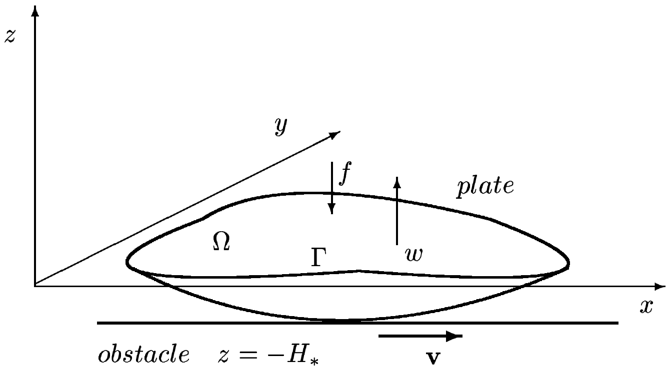

Fig. 1. The plate and the moving obstacle.

To take contact into account we note that if the obstacle is assumed rigid we have to impose the nonpenetration condition

$$w \geq \phi.$$

If, on the other-hand, we allow some interpenetration of surface asperities we may use the normal compliance condition

$$p = p_{nc}(w-\phi),$$

where the function $p = p(x,y,t)$ denotes the contact pressure, which is positive when the argument is negative, and is globally nonnegative. Additional properties of p_{nc} will be described shortly. We recall that in the literature, see e.g., [11, 16] and references therein, a common choice of p_{nc} is the power law

$$p_{nc}(w-\phi) = c_{nc}|(w-\phi)_-|^{m_{nc}},$$

where $(r)_- = \max\{0,-r\}$ is the negative part function, c_{nc} is the normal compliance constant, and m_{nc} is the normal compliance exponent. For technical reasons it is always assumed that $1 \leq m_{nc}$.

Actually, in this work we assume that p_{nc} is globally Lipschitz, nonnegative, and vanishes when its argument is positive, i.e.,

$$\begin{array}{rcl} |p_{nc}(r_1) - p_{nc}(r_2)| & \leq & K_{nc}|r_1 - r_2|, \\ p_{nc}(r) & \geq & 0, \\ p_{nc}(r) & = & 0 \quad \text{if } r \geq 0, \end{array} \tag{1}$$

for some positive constant K_{nc}. Next, we define

$$\Phi(w,x,y) \equiv \int_{\phi(x,y)}^{w} -p_{nc}(r-\phi(x,y))\,dr,$$

and note that there exist two positive constants A and B such that

$$\left(p_{nc}\left(r-\phi\left(x,y\right)\right)\right)^{2}\leq A\Phi\left(r,x,y\right)+B. \tag{2}$$

From now on we do not show explicitly the dependence of the various functions on (x, y).

The frictional heat generated during contact is given by

$$J=\mu p\|\mathbf{v}\|=\widetilde{\gamma}p,$$

and is proportional to the coefficient of friction μ, the contact pressure p, and the obstacle speed $\|\mathbf{v}\|$. Here, we set $\widetilde{\gamma}=\mu\|\mathbf{v}\|$ which is assumed to be a positive constant. When we use the normal compliance condition we have

$$J=\widetilde{\gamma}p_{nc}(w-\phi).$$

Next, for the sake of simplicity, we assume that the plate is clamped on Γ and has the ambient temperature, which is set as zero, thus,

$$w=\frac{\partial w}{\partial n}=0,\qquad \theta=0\qquad \text{on } \Gamma,$$

where $\partial w/\partial n=\mathbf{n}\cdot\nabla w$. Other boundary conditions are easy to incorporate and study.

We denote by $w_0=w_0(x,y)$ and $w_1=w_1(x,y)$ the initial conditions for w and for the velocity w', respectively, where here and below a prime denotes a partial time derivative.

The problem we study is the following *frictional contact of a thermoelastic plate with a moving reactive obstacle.*

Problem 2.1. *Find a pair of functions* $\{w,\theta\}$ *such that*

$$\rho h w''+D\triangle^{2}w+\alpha\triangle\theta=f+p_{nc}(w-\phi),\qquad in\ \Omega\times(0,T), \tag{3}$$

$$\rho c\theta'-\kappa\triangle\theta-\alpha\triangle w'=\widetilde{\gamma}p_{nc}(w-\phi),\qquad in\ \Omega\times(0,T), \tag{4}$$

$$w=\frac{\partial w}{\partial n}=\theta=0,\qquad on\ \Gamma\times(0,T), \tag{5}$$

$$w(x,y,0)=w_0,\quad w'(x,y,0)=w_1,\quad \theta(x,y,0)=0,\qquad in\ \Omega. \tag{6}$$

Here, ρ is the material density (per unit area), h is the thickness, α is a scaled, positive, and bounded thermal expansion coefficient, c is the heat capacity per unit area, and D is the flexural rigidity

$$D=\frac{Eh^{3}}{12(1-\nu^{2})},$$

where E is the Young modulus and ν the Poisson ratio. The foundation is assumed reactive, and its reaction is given by p_{nc}.

We note that in the equation of motion (3) the reaction force of the foundation is active only when $w < \phi$, and then the reaction is in the positive direction, upward. Similarly, the frictional heat generation in (4) is active only when $w < \phi$.

In the present paper, we consider the wellposedness of the model (3)-(6), under the general assumption in (1) on the nonlinearity p_{nc}. In a subsequent work, we will study the (global) *thermal null controllability* properties of the thermoelastic variables, for a specific choice of the form of the nonlinearity p_{nc} in (3). (This work is in the embryonic stage, but we anticipate that such "controllable" nonlinearities will include the term $(w-\phi)_-$). By the thermal null controllability, we refer to the following problem: "Given the mechanical initial data (w_0, w_1), in (6), with finite energy, is it possible, by means of a forcing control term u (say) on the right-hand side of (4), to have that the corresponding solution of (3)-(6) to satisfy $(w(T), w_t(T), \theta(T)) = (0,0,0)$?"

Such a controllability result for globally Lipschitz nonlinearities p_{nc} – with no size restrictions placed upon the initial data – will be in line with the strongest possible results available for global exact controllability of general PDE systems, see e.g., [14, 20]. We note that in these works, the authors impose in addition the Fréchet differentiability upon the given Lipschitz nonlinearity. For results concerning *local* null controllability properties of non-globally Lipschitz nonlinearities, see [3] and [5]. The global null controllability work for the present model (3)-(6) is in progress.

For the sake of completeness, we also state the problem in which the foundation is rigid. Then interpenetration is not allowed and we must enforce the condition $\phi \leq w$. In addition, when there is no contact the reaction force is zero, while if there is contact it is upward. Thus, we use the so-called Signorini unilateral contact condition

$$\phi \leq w, \qquad 0 \leq p, \qquad (w-\phi)p = 0.$$

The last part guarantees that one of the equalities must hold, since the point is either in contact and $w = \phi$, or is not in contact and then $p = 0$.

The *frictional contact of a thermoelastic plate with a moving obstacle with the Signorini contact condition* is as follows.

Problem 2.2. *Find a triple of functions* $\{w, p, \theta\}$ *such that*

$$\rho h w'' + D\Delta^2 w + \alpha\Delta\theta - f \equiv p \geq 0, \qquad \textit{in } \Omega \times (0,T), \tag{7}$$

$$\phi \leq w, \quad p(\phi - w) = 0, \qquad \textit{in } \Omega \times (0,T), \tag{8}$$

$$\rho c\theta' - \kappa\Delta\theta - \alpha\Delta w' = \widetilde{\gamma} p, \qquad \textit{in } \Omega \times (0,T), \tag{9}$$

$$w = \frac{\partial w}{\partial n} = \theta = 0, \qquad \textit{on } \Gamma \times (0,T), \tag{10}$$

$$w(x,y,0) = w_0, \quad w'(x,y,0) = w_1, \quad \theta(x,y,0) = 0, \qquad \textit{in } \Omega. \tag{11}$$

We note that normal compliance may be considered as a regularization of the Signorini condition. Indeed, at least formally, if we write

$$p_\varepsilon(w-\phi)=\frac{1}{\varepsilon}p_{nc}(w-\phi),$$

which physically means that the rigidity or stiffness of the foundation is proportional to $1/\varepsilon$, then we obtain the Signorini condition in the limit $\varepsilon \to 0$. Alternatively, we may consider the Signorini condition as an idealization of the normal compliance condition, as there are no real perfectly rigid objects.

It turns out that we are unable to deal with the problem with the Signorini condition at this stage. Therefore, we pose it as an open problem, left for future study.

3 Variational formulation and results

We turn to the variational or abstract formulation of the problem, following [7], where the details can be found. Let

$$(u,v)=\int_\Omega uv\,dx, \qquad \|v\|^2=(v,v),$$

denote the inner product and the norm on $H \equiv L^2(\Omega)$, respectively. We shall seek the displacements w in $\mathcal{W}$, where

$$W=\{v\in H^2(\Omega): v=\frac{\partial v}{\partial n}=0 \text{ on } \Gamma\} \quad \text{and} \quad \mathcal{W}=L^2(0,T;W). \tag{12}$$

We shall seek the temperature θ in $\mathcal{V}$, where

$$V=\{\theta\in H_0^1(\Omega)\} \quad \text{and} \quad \mathcal{V}=L^2(0,T;V). \tag{13}$$

We equip W and V with the equivalent norms

$$\|v\|_W=(\triangle v,\triangle v), \qquad \|v\|_V=(\nabla v,\nabla v),$$

respectively. We note that

$$W\subseteq V\subseteq H=H^*\subseteq V^*\subseteq W^*,$$

where the stars denote the dual spaces, H is identified with its dual, and the inclusions are compact. We also denote by $\langle w^*,w\rangle_{W^*,W}$ and $\langle v^*,v\rangle_{V^*,V}$ the duality pairings, and we shall omit the subscripts if the meaning is clear from the context.

We proceed formally to obtain a variational formulation of Problem 2.1 . We multiply (3) with $v\in\mathcal{W}$, use Green's formulas and the boundary conditions (5), and obtain

$$\int_0^T \langle \rho h w'', v\rangle \, dt + D\int_0^T (\Delta w, \Delta v)\, dt - \alpha \int_0^T (\nabla\theta, \nabla v)\, dt = \int_0^T (f, v)\, dt + \int_0^T (p_{nc}(w-\phi), v)\, dt. \tag{14}$$

Similarly, we multiply (4) with $\zeta \in \mathcal{V}$, use the Gauss theorem and the boundary conditions (5), and find

$$\int_0^T \langle \rho c\theta', \zeta\rangle \, dt + \kappa \int_0^T (\nabla\theta, \nabla\zeta)\, dt + \alpha \int_0^T (\nabla w', \nabla\zeta)\, dt = \widetilde{\gamma} \int_0^T (p_{nc}(w-\phi), \zeta)\, dt. \tag{15}$$

We now define the operators $A : W \to W^*$, $L : V \to V^*$, $L_1 : W \to V^*$, and $L_2 : V \to W^*$ by

$$\begin{aligned} \langle Aw, v\rangle &\equiv \frac{D}{\pi h}\int_\Omega \Delta w \Delta v dx, \qquad \langle L\theta, \zeta\rangle \equiv \int_\Omega \nabla\theta\cdot\nabla\zeta dx, \\ \langle L_1 w, \zeta\rangle &\equiv \int_\Omega \nabla w\cdot\nabla\zeta dx, \qquad \langle L_2\theta, \zeta\rangle \equiv \int_\Omega \nabla\theta\cdot\nabla\zeta dx. \end{aligned}$$

The abstract or variational formulation of Problem 2.1 is as follows.

Problem 3.1. *Find a pair of functions* $\{w, \theta\} \in \mathcal{W}\times\mathcal{V}$ *such that*

$$w' \in \mathcal{W}, \qquad w'' \in \mathcal{W}^*, \qquad \theta' \in \mathcal{V}^*, \tag{16}$$

and

$$w(0) = w_0 \in W, \quad w'(0) = w_1 \in H, \quad \theta(0) = 0, \tag{17}$$

which satisfy the abstract evolution equations for a.e.t $\in (0, T)$,

$$w'' + Aw - a_1 L_2\theta - b_1 p_{nc}(w-\phi) = b_1 f, \tag{18}$$

$$\theta' + a_0 L\theta + a_2 L_1 w' = b_2 p_{nc}(w-\phi). \tag{19}$$

Here we introduced the parameters

$$\frac{\kappa}{\rho c} \equiv a_0, \quad \frac{\alpha}{\rho h} \equiv a_1, \quad \frac{1}{\rho h} \equiv b_1, \quad \frac{\alpha}{\rho c} \equiv a_2, \quad \frac{\widetilde{\gamma}}{\rho c} \equiv b_2.$$

We note that there is a technical difficulty with this formulation of the problem due to the presence of the term $L_1 w'$ in equation (19) while there is no comparable reference to w' which would provide sufficient regularity in (18). To remedy this difficulty we let $\Theta(t) \equiv \int_0^t \theta(s)\, ds$. Then in terms of Θ equation (19) becomes

$$\Theta' + a_0 L\Theta + a_2 Lw = -a_2\Delta w_0 + b_2 \int_0^{(\cdot)} p_{nc}(w-\phi)\, ds. \tag{20}$$

This equation holds in H with the initial condition $\Theta(0) = 0$. The difficulty is removed by formulating (19) in this way.

The main result in this work is the following theorem.

Theorem 3.2. *There exists a unique solution to Problem* 3.1 *in which* (19) *is replaced with* (20). *The solution satisfies*

$$w'' \in \mathcal{W}^*, \quad w' \in L^\infty(0,T;H), \quad w \in \mathcal{W} \cap L^\infty\left(0,T;H^2(\Omega)\right), \tag{21}$$

$$w \in C\left([0,T]\times\overline{\Omega}\right), \quad \Theta \in L^\infty(0,T;V), \quad \Theta' \in \mathcal{V}. \tag{22}$$

The formulation that employs (20) is just a slightly weaker form of the one with (19). It is seen from the regularity of the solution above and from (20) that in the new formulation $(\theta + a_2 Lw)' \in \mathcal{V}'$.

We establish Theorem 3.2 in the next section.

To simplify slightly the presentation, from now on we will not distinguish between the operators L, L_1 and L_2, and will denote them by L, since the context easily determines which one is meant.

The variational formulation of Problem 2.2 is obtained in a similar manner, only now we use the test functions from the set

$$\mathcal{K} = \{v \in \mathcal{V} : \ \phi \le v \ \text{ a.e. in } \Omega \times (0,T)\}. \tag{23}$$

Then, we multiply (7) with $v - w$, for $v \in \mathcal{K}$, and after some manipulations find that

$$\int_0^T \langle \rho h w'', v - w\rangle \, dt + D\int_0^T (\triangle w, \triangle(v-w))\, dt - \alpha \int_0^T (\nabla\theta, \nabla(v-w))\, dt$$
$$- \int_0^T (f, v-w)\, dt = \int_0^T (p, v-w)\, dt \ge 0. \tag{24}$$

Here, we used the fact that

$$(p, v-w) = (p, v-\phi+\phi-w) = (p, v-\phi) + (p, \phi - w) = (p, v-\phi) \ge 0,$$

since $(p, \phi - w) = 0$ by (8) and $v \in \mathcal{K}$.

The variational formulation of Problem 2.2 is the following.

Problem 3.3. *Find a triple of functions* $\{w, p, \theta\} \in \mathcal{W} \times L^2(0,T;L^2(\Omega)) \times \mathcal{V}$ *such that*

$$w' \in \mathcal{W}, \qquad w'' \in \mathcal{W}^*, \quad p \ge 0 \ \text{ a.e.}, \qquad \theta' \in \mathcal{V}^*,$$

and

$$w = w_0, \quad w' = w_1, \quad \theta = 0, \qquad \text{in } \Omega \times \{0\},$$

which satisfy (24) *and* (15), *where in the latter* p_{nc} *has been replaced with* p, *for each* $v \in \mathcal{K}$ *and* $\zeta \in \mathcal{V}$.

As was noted above, this problem remains unsolved.

4 The solution of Problem 3.1

In this section we prove Theorem 3.2, thus establishing the existence of the unique weak solution to Problem 2.1.

We begin with the following regularized problem in which a viscosity term has been added and the leading term regularized.

Problem 4.1. *Find a pair of functions* $\{w, \theta\} \in \mathcal{W} \times \mathcal{V}$ *such that*

$$w' \in \mathcal{W}, \qquad w'' \in \mathcal{W}, \qquad \theta' \in \mathcal{V}^*, \tag{25}$$

and

$$w(0) = w_0 \in W, \quad w'(0) = w_1 \in W, \quad \theta(0) = 0, \tag{26}$$

which, for $\varepsilon > 0$*, satisfy the abstract evolution equations*

$$((I + \varepsilon A) w')' + \varepsilon A w' + A w - a_1 L\theta - b_1 p_{nc}(w - \phi) = b_1 f, \tag{27}$$

$$\theta' + a_0 L\theta + a_2 L w' = b_2 p_{nc}(w - \phi). \tag{28}$$

The problem can be reformulated as an implicit first order system of evolution equations as follows. Letting $z = w'$, the system can be written as

$$\left(\begin{pmatrix} I + \varepsilon A & 0 & 0 \\ 0 & I + A & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} z \\ w \\ \theta \end{pmatrix} \right)' + \begin{pmatrix} \varepsilon A z + A w - a_1 L\theta - b_1 p_{nc}(w - \phi) \\ -(I + A) z \\ a_0 L\theta - b_1 p_{nc}(w - \phi) + a_2 L z \end{pmatrix}$$
$$= \begin{pmatrix} b_1 f \\ 0 \\ 0 \end{pmatrix},$$

along with the initial conditions

$$\begin{pmatrix} I + \varepsilon A & 0 & 0 \\ 0 & I + A & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} z \\ w \\ \theta \end{pmatrix} (0) = \begin{pmatrix} I + \varepsilon A & 0 & 0 \\ 0 & I + A & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} w_1 \\ w_0 \\ 0 \end{pmatrix}.$$

This system can be written in the form

$$(\mathcal{B}\mathbf{y})' + \mathcal{A}(\mathbf{y}) = \mathbf{f}, \qquad \mathcal{B}\mathbf{y}(0) = \mathcal{B}\mathbf{y}_0,$$

where $\mathbf{y} = (z, w, \theta)^T$, $\mathcal{B}$ is the linear operator in the first term on the left-hand side, and $\mathcal{A}$ is the nonlinear operator in the second term. Also, $\mathbf{f} = (b_1 f, 0, 0)^T$, and $\mathbf{y}_0 = (w_1, w_0, 0)^T$. As was noted above, we may replace $\mathbf{y}_0$ with $\mathbf{y}_0 = (w_1, w_0, \theta_0)^T$, with $\theta_0 \in V$, and all the results below hold, as well.

It is straightforward, by using an exponential shift in time, to show that when λ is sufficiently large the operator $\lambda\mathcal{B} + e^{-\lambda(\cdot)}\mathcal{A}\left(e^{\lambda(\cdot)}\right)$ is pseudomonotone, bounded, and coercive as a map from $\mathcal{W} \times \mathcal{W} \times \mathcal{V}$ to its dual. Therefore, it follows from the main existence theorem in [13] that for each $\varepsilon > 0$ there exists a solution $(z, w, \theta) \in \mathcal{W} \times \mathcal{W} \times \mathcal{V}$ to the problem, such that $(z', w', \theta') \in \mathcal{W} \times \mathcal{W} \times \mathcal{V}^*$. Standard monotonicity arguments, resulting from the assumption that p_{nc} is Lipschitz continuous, show that the solution is unique. We state these observations as the following theorem.

Theorem 4.2. *There exists, for each $\varepsilon > 0$, a unique solution to Problem* 4.1.

To prove Theorem 3.2 we proceed to obtain the necessary a priori estimates on the solutions of Problem 4.1, which are independent of ε. This will be done in the following steps.

First, we recall that

$$\Phi(w, \mathbf{x}) \equiv \int_{\phi(\mathbf{x})}^{w} -p_{nc}(r - \phi(\mathbf{x}))\, dr,$$

where from now on we write $\mathbf{x} = (x, y)$. It follows from assumptions (1) on p_{nc} that $\Phi(w, \mathbf{x}) > 0$ when $w < \phi(\mathbf{x})$, and $\Phi(w, \mathbf{x}) = 0$ when $w \geq \phi(\mathbf{x})$. In particular, $\Phi(w, \mathbf{x}) \geq 0$.

We multiply (27) by w' and integrate both sides over $(0, t)$. Letting C denote a generic constant independent of ε we find

$$\frac{1}{2}|w'(t)|_H^2 + \frac{1}{2}\varepsilon \langle Aw', w'\rangle(t) + \varepsilon \int_0^t \langle Aw'(s), w'(s)\rangle\, ds + \frac{1}{2}C|\Delta w(t)|_H^2 - \frac{1}{2}C|\Delta w_0|^2$$
$$-a_1 \int_0^t (\nabla\theta, \nabla w)\, ds + b_1 \int_\Omega \Phi(w, \phi(\mathbf{x}))\, dx \leq C \int_0^t |f|_H^2\, ds.$$

Dividing both sides by a_1 and adjusting the constants yields

$$\frac{1}{2a_1}|w'(t)|_H^2 + \frac{1}{2a_1}\varepsilon \langle Aw', w'\rangle(t) + \frac{\varepsilon}{a_1}\int_0^t \langle Aw'(s), w'(s)\rangle\, ds + C|\Delta w(t)|_H^2$$
$$-C|\Delta w_0|^2 - \int_0^t (\nabla\theta, \nabla w')\, ds + \frac{b_1}{a_1}\int_\Omega \Phi(w, \phi(\mathbf{x}))\, dx \leq C\int_0^t |f|_H^2\, ds. \quad (29)$$

Next, we multiply (28) by θ, integrate both sides over $(0, t)$, divide by a_2 and obtain

$$\frac{1}{2a_2}|\theta(t)|_H^2 + \frac{a_0}{a_2}\int_0^t ||\theta(s)||_V^2\, ds + \int_0^t (\nabla\theta, \nabla w')\, ds$$
$$-\frac{b_2}{a_2}\int_0^t \int_\Omega p_{nc}(w - \phi)\,\theta dx ds = 0. \quad (30)$$

Adding (30) and (29) and the adjusting constants yields

$$|w'(t)|_H^2 + |\theta(t)|_H^2 + \varepsilon\langle Aw', w'\rangle(t) + \int_0^t ||\theta(s)||_V^2\, ds + \varepsilon\int_0^t \langle Aw'(s), w'(s)\rangle\, ds$$
$$+|\Delta w(t)|_H^2 + \int_\Omega \Phi(w(t), \phi(\mathbf{x}))\, dx \leq C + C\int_0^t \int_\Omega p_{nc}(w - \phi)\,|\theta|\, dx ds.$$

Then (2) implies

$$\begin{aligned} &|w'(t)|_H^2+|\theta(t)|_H^2+\varepsilon\langle Aw',w'\rangle(t)+\int_0^t\|\theta(s)\|_V^2\,ds \\ &+\varepsilon\int_0^t\langle Aw'(s),w'(s)\rangle\,ds+|\Delta w(t)|_H^2+\int_\Omega\Phi(w(t),\phi(\mathbf{x}))\,dx \\ &\leq C+C\int_0^t\int_\Omega p_{nc}(w-\phi)^2\,dxds+C\int_0^t\int_\Omega|\theta|_H^2\,dxds \\ &\leq C+C\int_0^t\int_\Omega\Phi(w,\mathbf{x})\,dxds+C\int_0^t\int_\Omega|\theta|_H^2\,dxds. \end{aligned}$$

An application of Gronwall's inequality yields

$$\begin{aligned} &|w'(t)|_H^2+|\theta(t)|_H^2+\varepsilon\langle Aw',w'\rangle(t)+\int_0^t\|\theta(s)\|_V^2\,ds+|\Delta w(t)|_H^2 \\ &+\varepsilon\int_0^t\langle Aw'(s),w'(s)\rangle\,ds+\int_\Omega\Phi(w(t),\phi(\mathbf{x}))\,dx\leq C_1, \end{aligned} \tag{31}$$

where C_1 is independent of ε. It follows that there is a set of positive measure $S(t)$ such that for $\mathbf{x}\in S(t)$,

$$\Phi(w(t)(\mathbf{x}),\phi(\mathbf{x}))<2C_1/m(\Omega).$$

Consequently, there exists a constant C_2, which is independent of t, such that for $\mathbf{x}\in S(t)$,

$$|w(t)(\mathbf{x})|<C_2.$$

It follows from (31) that $w(t)\in H^2(\Omega)$ and then the Sobolev embedding theorem implies that there exists a positive constant K_Ω, which depends only on Ω, such that for each $t\in[0,T]$

$$\|w(t)\|_{C^{0,1/2}(\overline{\Omega})}\leq K_\Omega C_1.$$

Indeed, since Ω is a two-dimensional region, $H^2(\Omega)$ embeds continuously into $W^{1,p}(\Omega)$ for every $p>1$. In particular, this is true for $p=4$ and $W^{1,4}(\Omega)$ embeds continuously into $C^{0,1/2}(\overline{\Omega})$. Therefore, if $\mathbf{x}\in S(t)$ and $\mathbf{x}_*$ is an arbitrary point of Ω then

$$|w(t)(\mathbf{x})-w(t)(\mathbf{x}_*)|<K_\Omega C_1|\mathbf{x}-\mathbf{x}_*|^{1/2}\leq K_\Omega C_1\sqrt{diam(\Omega)}.$$

Thus,

$$|w(t)(\mathbf{x}_*)|\leq K_\Omega C_1\sqrt{diam(\Omega)}+C_2\equiv C_3. \tag{32}$$

It follows now from (27) that there exists a constant C, independent of ε, such that

$$\begin{aligned} &|w'(t)|_H^2 + |\theta(t)|_H^2 + \varepsilon \langle Aw', w' \rangle (t) + \int_0^t ||\theta(s)||_V^2 \, ds + \varepsilon \left|\left| (Aw')' \right|\right|_{\mathcal{W}^*} \\ &+ |\theta'|_{\mathcal{V}^*} + ||w''||_{\mathcal{W}^*} + \varepsilon \int_0^t \langle Aw'(s), w'(s) \rangle \, ds + |\Delta w(t)|_H^2 \\ &+ \int_\Omega \Phi(w(t), \phi(\mathbf{x})) \, dx + ||w(t)||_{L^\infty(\Omega)} \le C. \end{aligned} \tag{33}$$

Next, a difficulty arises in passing to the limit in (28) because of the term Lw'. To obtain the necessary estimate on this term, we consider an integrated version of this equation. First, we note that $Lw_0 = -\Delta w_0 \in H$ since $w_0 \in W$, and also estimate (33) implies that $Lw \in \mathcal{H}$. Let $\Theta(t) \equiv \int_0^t \theta(s) \, ds$ then, in terms of Θ, (28) becomes

$$\Theta' + a_0 L\Theta + a_2 Lw = -a_2 \Delta w_0 + b_2 \int_0^{(\cdot)} p_{nc}(w - \phi) \, ds. \tag{34}$$

The equation holds in H and the initial condition is $\Theta(0) = 0$. We now consider the version of Problem 4.1 in which the dependent variable θ is replaced with Θ. Using the definition of Θ and estimate (33) along with the equation, (34), the following additional estimate holds for Θ,

$$||\Theta(t)||_V + |\Theta'(t)|_H + ||L\Theta||_{\mathcal{V}^*} \le C. \tag{35}$$

The next task is to pass to the limit as $\varepsilon \to 0$. To this end, we denote by X_1, X_2, and X_3 three Hilbert spaces such that $X_1 \subseteq X_2 \subseteq X_3$, with compact inclusion map $X_1 \to X_2$ and continuous inclusion map $X_2 \to X_3$. We shall use of the following two theorems found in Lions [15] and Simon [18], respectively (see also [12] for the proofs).

Theorem 4.3. *Let* $p \ge 1$, $q > 1$, *and let*

$$S_R = \{\mathbf{u} \in L^p(0,T;X_1) : \mathbf{u}' \in L^q(0,T;X_3), \ ||\mathbf{u}||_{L^p(0,T;X_1)} + ||\mathbf{u}'||_{L^q(0,T;X_3)} < R\}.$$

Then S_R *is precompact in* $L^p(0,T;X_2)$.

Theorem 4.4. *Let*

$$S_{RT} = \{\mathbf{u} : ||\mathbf{u}(t)||_{X_1} + ||\mathbf{u}'||_{L^q(0,T;X_3)} \le R, \quad t \in [0,T]\},$$

for some $q > 1$. *Then* S_{RT} *is precompact in* $C(0,T;X_2)$.

We denote by $(w_\varepsilon, \Theta_\varepsilon)$ the solution to Problem 4.1 for $\varepsilon > 0$. Then, estimate (33), (35) and the two theorems imply that there exist $w \in \mathcal{W}$ and $\Theta \in \mathcal{V}$, and a subsequence, still denoted by ε, such that as $\varepsilon \to 0$ the following hold true:

$$\begin{array}{cr}
\varepsilon A w_\varepsilon'(t) \to 0 \text{ strongly in } W^*, & (a)\\
\varepsilon A w_\varepsilon' \to 0 \text{ strongly in } \mathcal{W}^*, & (b)\\
w_\varepsilon' \to w' \text{ weak} * \text{ in } L^\infty(0,T;H), & (c)\\
Aw_\varepsilon \to Aw \text{ weakly in } \mathcal{W}^*, & (d)\\
w_\varepsilon \to w \text{ strongly in } C([0,T];H^r(\Omega)),\ r \in (1,2), & (e)\\
w_\varepsilon \to w \text{ weak} * \text{ in } L^\infty\left(0,T;H^2(\Omega)\right), & (f)\\
w_\varepsilon'' \to w'' \text{ weakly in } \mathcal{W}^*, & (g)\\
Lw_\varepsilon \to Lw \text{ strongly in } \mathcal{V}^*, & (h)\\
\theta_\varepsilon \to \theta \text{ weakly in } \mathcal{V}, & (i)\\
L\theta_\varepsilon \to L\theta \text{ weakly in } \mathcal{W}^*, & (j)\\
\Theta_\varepsilon \to \Theta \text{ weak} * \text{ in } L^\infty(0,T;V), & (k)\\
L\Theta_\varepsilon \to L\Theta \text{ weakly in } \mathcal{V}^*, & (l)\\
\Theta_\varepsilon' \to \Theta' \text{ weakly in } L^\infty(0,T;H). & (m)
\end{array} \tag{36}$$

Now, $(36(b))$ implies that

$$(\varepsilon A w_\varepsilon')' \to 0 \text{ weakly in } \mathcal{W}^*. \tag{37}$$

Indeed, it follows from (33) that there exists a subsequence such that $(\varepsilon A w_\varepsilon')'$ converges weakly in $\mathcal{W}^*$ to an element ξ. Thus, if $v \in C_0^\infty(0,T;W)$ then using $(36(b))$ yields

$$\int_0^T \langle \xi, v\rangle\, ds = \lim_{\varepsilon\to 0}\int_0^T \left\langle (\varepsilon A w_\varepsilon')', v\right\rangle ds = \lim_{\varepsilon\to 0} -\int_0^T \langle \varepsilon A w_\varepsilon', v'\rangle\, dr = 0.$$

Since $C_0^\infty(0,T;W)$ is dense in $\mathcal{W}$, assertion (37) follows. Next, Ω is two dimensional, therefore, $H^r(\Omega)$ embeds continuously into $C\left(\overline{\Omega}\right)$, for $r \in (1,2)$. Therefore, (36) and (37) are sufficient to pass to the limit in (27) and (34) and obtain

$$w'' + Aw - a_1 L\theta - b_1 p_{nc}(w - \phi) = b_1 f, \tag{38}$$

and, upon recalling that $\theta = \Theta'$, we have (34) with the initial conditions

$$w(0) = w_0 \in W, \quad w'(0) = w_1 \in W, \quad \Theta(0) = 0. \tag{39}$$

Moreover, the limit functions also satisfy

$$\begin{gathered}
w'' \in \mathcal{W}^*, \quad w' \in L^\infty(0,T;H), \quad w \in \mathcal{W} \cap L^\infty\left(0,T;H^2(\Omega)\right),\\
w \in C\left([0,T]\times\overline{\Omega}\right), \quad \Theta \in L^\infty(0,T;V), \quad \Theta' \in \mathcal{V}.
\end{gathered} \tag{40}$$

This establishes the existence part of Theorem 3.2. It remains to verify the uniqueness of the solution.

Proof of uniqueness: Suppose that (w_i, Θ_i) are two solutions, for $i = 1,2$. From (34) we conclude

$$\frac{1}{2a_2}\left|\Theta_1(t) - \Theta_2(t)\right|_H^2 + \frac{a_0}{a_2}\int_0^t \|\Theta_1 - \Theta_2\|_V^2\, ds + \int_0^t \langle L(w_1 - w_2), \Theta_1 - \Theta_2\rangle\, ds$$

$$\leq C\int_0^t\left(\int_0^s |w_1(r)-w_2(r)|_H\,dr\right)|\Theta_1(s)-\Theta_2(s)|_H\,ds. \tag{41}$$

Next, it follows from (38) that each one of the w_i satisfies

$$w'+A\left(\int_0^{(\cdot)} w(s)\,ds\right)-a_1L\Theta-b_1\int_0^{(\cdot)} p_{nc}(w-\phi)\,ds=b_1\int_0^{(\cdot)} f(s)\,ds+w_1.$$

Therefore,

$$\frac{1}{2a_1}|w_1(t)-w_2(t)|_H^2+$$
$$\frac{1}{a_1}\int_0^t\left\langle A\left(\int_0^s w_1(r)\,dr-\int_0^s w_2(r)\,dr\right),w_1(s)-w_2(s)\right\rangle ds$$
$$-\int_0^t\langle L(\Theta_1-\Theta_2),w_1-w_2\rangle\,ds\leq$$
$$C\int_0^t\left(\int_0^s |w_1(r)-w_2(r)|_H\,dr\right)|\Theta_1(s)-\Theta_2(s)|_H\,ds. \tag{42}$$

Adding (42) and (41) yields

$$\frac{1}{2a_2}|\Theta_1(t)-\Theta_2(t)|_H^2+\frac{a_0}{a_2}\int_0^t ||\Theta_1-\Theta_2||_V^2\,ds+\frac{1}{2a_1}|w_1(t)-w_2(t)|_H^2$$

$$\begin{aligned}&+\frac{1}{a_1}\int_0^t\left\langle A\left(\int_0^s (w_1(r)-w_2(r))\,dr\right),w_1(s)-w_2(s)\right\rangle ds\\ \leq\ & C\left(\int_0^t |w_1(r)-w_2(r)|_H^2\,dr+\int_0^t|\Theta_1(s)-\Theta_2(s)|_H^2\,ds\right).\end{aligned}$$

Now,

$$\left\langle A\left(\int_0^s (w_1(r)-w_2(r))\,dr\right),w_1(s)-w_2(s)\right\rangle ds\geq 0,$$

and so, after adjusting the constants, we find

$$\begin{aligned}&|\Theta_1(t)-\Theta_2(t)|_H^2+|w_1(t)-w_2(t)|_H^2\\ &\quad\leq C\left(\int_0^t |w_1(r)-w_2(r)|_H^2\,dr+\int_0^t|\Theta_1(s)-\Theta_2(s)|_H^2\,ds\right).\end{aligned} \tag{43}$$

Hence, by using the Gronwall inequality we obtain $\Theta_1=\Theta_2$ and $w_1=w_2$. This completes the proof of Theorem 3.2.

As a closing remark we note that we were unable to use this method to establish the existence of a solution for the Signorini problem. A 'natural' approach, often used in the literature, would be to write the obstacle reaction force as $\frac{1}{\varepsilon}p_{nc}$

and pass to the limit $\varepsilon \to 0$, thus obtaining formally the Signorini condition. However, the estimates above are insufficient for passing to the limit. Indeed, the Signorini condition implies a body force which exactly prevents any interpenetration, and there is no bound on this force. Thus, it is not surprising that we can't handle a dynamic problem which involves the acceleration, since the velocity may be discontinuous upon contact. We conclude that the problem of proving the existence of a solution to the model remains unresolved.

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Received: 1 December, 2004

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