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A Piezoelectric Body in Frictional Contact

by
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Abstract

We consider a mathematical model which describes the static frictional contact between a piezoelectric body and a foundation. We use a nonlinear electro-elastic constitutive law to model the piezoelectric material and the static version of Tresca's law to model the friction. We derive a variational formulation for the model which is in a form of a coupled system involving the displacement and the electric potential fields. Then we provide the existence of a unique weak solution to the model. The proof is based on arguments of variational inequalities and fixed point.

Key Words: Static process, frictional contact, piezoelectric material, Tresca's law, variational inequality, fixed point, weak solution.
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1 Introduction

Piezoelectricity is a coupling between a material's mechanical and electrical behaviors. In the simplest of terms, when a piezoelectric material is squeezed, an electric charge collects on its surface. Conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene flouride (a polymer film).

On a nanoscopic scale, piezoelectricity results from a nonuniform charge distribution within a crystal's unit cells. When such a crystal is mechanically deformed, the positive and negative charge centers displace by differing amounts. So while the overall crystal remains electrically neutral, the difference in charge center displacements results in an electric polarization within the crystal. Electric polarization due to mechanical input is perceived as piezoelectricity.

General models for elastic materials with piezoelectric effects can be found in [6], [7], [11], [12] and, more recently, in [1], [10]. Currently, there is a considerable interest in frictional contact problems involving piezoelectric materials, see for instance [2], [5] and the references therein. Indeed, situations which involve contact phenomena abound in industry and everyday life. The contact of the braking pads with the wheel, the tire with the road and the piston with skirt are just three simple examples. Because of the importance of contact processes a considerable effort has been made in their modelling and the engineering literature concerning this topic is extensive. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies to models for contact with deformable bodies which include coupling between mechanical and electrical properties.

The aim of this paper is to provide such an extension. Indeed, we consider here a model for the process of frictional contact between an electrolastic body, which is acted upon by forces and electric charges, and a foundation. The process is static, the contact is frictional and it is modeled with a version of Tresca's law of dry friction in which the friction bound is given. Taking into account the piezoelectric behavior of the body consists the main trait of novelty of the model. We derive a variational formulation of the model then we prove its unique weak solvability. The proofs are based on results of the variational inequalities theory.

The paper is structured as follows. In Section 2 we state the model of the equilibrium process of the elastic piezoelectric body in frictional contact with a foundation. In Section 3 we introduce some preliminary material, list assumptions on the problem data and state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is presented in Section 4.

2 Problem statement

We consider the following physical setting. An elastic piezoelectric body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $meas\,\Gamma_1 > 0$, $meas\,\Gamma_a > 0$ and $\Gamma_3 \subseteq \Gamma_b$. We assume that the body is clamped on Γ_1 and surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body is in bilateral frictional contact with an obstacle, the so-called foundation. We model the contact with a version of Tresca's law of dry friction, used for instance in [3] and [8]. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Also, below $\boldsymbol{\nu}$ represents the unit outward normal on Γ while " $\cdot$ " and $\|\cdot\|$ denote the inner product and the

Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively.

With the assumption above, the problem of equilibrium of the electroelastic body in frictional contact with a foundation is the following.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, *an electric potential* $\varphi : \Omega \to \mathbb{R}$ *and an electric displacement field* $\boldsymbol{D} : \Omega \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^T \boldsymbol{E}(\varphi) \quad \text{in} \quad \Omega, \tag{1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3}$$

$$\mathrm{div}\, \boldsymbol{D} = q_0 \quad \text{in} \quad \Omega, \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{6}$$

$$\left\{ \begin{array}{l} u_\nu = 0, \quad \|\boldsymbol{\sigma}_\tau\| \le g, \\ \boldsymbol{\sigma}_\tau = -g \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if} \quad \boldsymbol{u}_\tau \neq \boldsymbol{0} \end{array} \right. \quad \text{on} \quad \Gamma_3, \tag{7}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{8}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_2 \quad \text{on} \quad \Gamma_b. \tag{9}$$

In (1)–(9) and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equations (1) and (2) represent the electroelastic constitutive law of the material in which $\mathcal{F}$ is a given nonlinear function, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the small strain tensor, $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^T$ is its transposite and $\boldsymbol{\beta}$ denotes the electric permitivitty tensor. Details of the linear version of the constitutive relations (1) and (2) can be find in [1] and [2]. Equations (3) and (4) represent the equilibrium equations for the stress and electric displacement fields, respectively, (5) and (6) are the displacement and traction boundary conditions, respectively, and (8), (9) represent the electric boundary conditions. Condition $u_\nu = 0$ in (7) shows that the contact is bilateral and the reminder conditions in (7) represent the Tresca friction law; here u_ν represents the normal displacement, $\boldsymbol{\sigma}_\tau$ is the tangential stress, $\boldsymbol{u}_\tau$ denotes the tangential displacement and g is the friction bound function, i.e., the magnitude of the limiting friction traction at which slip begins.

3 Variational formulations and main result

In this section we list the assumptions on the data, derive a variational formulation for the contact problem (1)–(9) and state our main existence and uniqueness result, Theorem 3.1. To this end we need to introduce notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i\ ,\qquad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}\qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}\ ,\qquad \|\tau\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}}\qquad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d.$$

Here and everywhere in this paper i,j,k,l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j}=\frac{\partial u_i}{\partial x_j}$.

Everywhere below we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces:

$$L^2(\Omega)^d=\{\ \boldsymbol{v}=(v_i)\mid v_i\in L^2(\Omega)\ \},\quad H^1(\Omega)^d=\{\ \boldsymbol{v}=(v_i)\mid v_i\in H^1(\Omega)\ \},$$
$$\mathcal{H}=\{\ \boldsymbol{\tau}=(\tau_{ij})\mid \tau_{ij}=\tau_{ji}\in L^2(\Omega)\ \},\quad \mathcal{H}_1=\{\ \boldsymbol{\tau}\in\mathcal{H}\mid \tau_{ij,j}\in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$, are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d}=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx,\quad (\boldsymbol{u},\boldsymbol{v})_{H^1(\Omega)^d}=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx+\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx,\quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}+(\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_{L^2(\Omega)^d}$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here $\boldsymbol{\varepsilon}:H_1\to\mathcal{H}$ and $\mathrm{Div}:\mathcal{H}_1\to L^2(\Omega)^d$ are the deformation and divergence operators, respectively, that is

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\quad \varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}(v_{i,j}+v_{j,i})\quad \forall\boldsymbol{v}\in H^1(\Omega)^d,$$
$$\mathrm{Div}\,\boldsymbol{\tau}=(\tau_{ij,j})\quad \forall\boldsymbol{\tau}\in\mathcal{H}_1.$$

For every element $\boldsymbol{v}\in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V=\{\ \boldsymbol{v}\in H^1(\Omega)^d\mid \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1,\ \ v_\nu=0\text{ on }\Gamma_3\ \}.$$

Since $meas\,(\Gamma_1)>0$, the following Korn's inequality holds:

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}\geq c_K\,\|\boldsymbol{v}\|_{H^1(\Omega)^d}\quad \forall\,\boldsymbol{v}\in V, \tag{10}$$

where $c_K>0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(\boldsymbol{u},\boldsymbol{v})_V=(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{11}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (10) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (10) and (11), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}\|_V \qquad \forall \boldsymbol{v} \in V. \tag{12}$$

We also introduce the spaces

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \},$$
$$\mathcal{W} = \{\ \boldsymbol{D} = (D_i) \mid D_i \in L^2(\Omega),\ \operatorname{div} \boldsymbol{D} \in L^2(\Omega)\ \}$$

where $\operatorname{div} \boldsymbol{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)}, \quad (\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = (\boldsymbol{D}, \boldsymbol{E})_{L^2(\Omega)^d} + (\operatorname{div} \boldsymbol{D}, \operatorname{div} \boldsymbol{E})_{L^2(\Omega)}.$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Notice also that, since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \ge c_F\, \|\psi\|_W \quad \forall\, \psi \in W, \tag{13}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a.

In the study of the mechanical problem (1)–(9) we assume that

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } M_{\mathcal{F}} > 0 \text{ such that} \\
\quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \le M_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\
\quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\
\quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1)) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\
\quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \xi) \text{ is Lebesgue measurable on } \Omega \\
\quad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\
\text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega).
\end{array}\right. \tag{15}$$

$$\left\{\begin{array}{l}
\text{(a) } \boldsymbol{\beta} = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\
\text{(b) } \beta_{ij} \in L^\infty(\Omega). \\
\text{(c) } \beta_{ij} = \beta_{ji}. \\
\text{(d) There exists } m_\beta > 0 \text{ such that } \beta_{ij}(\boldsymbol{x}) E_i E_j \ge m_\beta \|\boldsymbol{E}\|^2 \\
\quad \forall\, \boldsymbol{E} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{16}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_3)^d \tag{17}$$

$$q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b), \quad q_2 = 0 \text{ on } \Gamma_3, \tag{18}$$

$$g \in L^\infty(\Gamma_3), \quad g \geq 0\cdot \tag{19}$$

We make in what follows some comments on the assumptions (14)–(19).

First, we note that the condition (14) is satisfied in the case of the linear elastic constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ in which

$$\mathcal{F}\boldsymbol{\xi} = (f_{ijkl}\xi_{kl}),$$

provided that $f_{ijkl} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$f_{ijkl}(\boldsymbol{x})\xi_k\xi_l \geq \alpha\|\boldsymbol{\xi}\|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.$$

Examples of nonlinear constitutive laws which satisfy (14) can be find in [8] and [9].

Next, as it is shown in (15) and (16), we see that the piezoelectric operator $\mathcal{E}$ as well as the electric permitivitty operator $\boldsymbol{\beta}$ are assumed to be linear and, moreover, $\boldsymbol{\beta}$ is symmetric and positive definite. Recall also that the transposite tensor $\mathcal{E}^T$ is given by $\mathcal{E}^T = (e_{ijk}^T)$ where $e_{ijk}^T = e_{kij}$, and the following equality holds:

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^T\boldsymbol{v} \qquad \forall \boldsymbol{\sigma} \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d. \tag{20}$$

We also remark that (17) represent regularity assumptions on the densities of volume forces and surface tractions while (18) represent regularity assumptions on the densities of volume and surface electric charges. We need condition $q_2 = 0$ on Γ_3 since the contact is bilateral and the foundation is insulator.

Finally, by (19) it follows that the function g is positive and bounded.

We now turn to the variational formulation of Problem P and, to this end, we introduce further notation. Let $h : V \longrightarrow \mathbb{R}$ be the functional

$$h(\boldsymbol{u}) = \int_{\Gamma_3} g\, \|\boldsymbol{v}_\tau\| da, \quad \forall\, \boldsymbol{v} \in V \tag{21}$$

and, using Riesz's representation theorem, consider the elements $\boldsymbol{f} \in V$ and $q \in W$ given by

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v}\, \in\, V, \tag{22}$$

$$(q, \psi)_W = -\int_\Omega q_0\psi\, dx + \int_{\Gamma_b} q_2\psi\, da \qquad \forall\, \psi \in W. \tag{23}$$

Using integration by parts, it is straightforward to see that if $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ are sufficiently regular functions which satisfy (3)–(9) then

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + h(\boldsymbol{v}) - h(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall\, \boldsymbol{v} \in V, \tag{24}$$

$$(\boldsymbol{D}, \nabla\psi)_{L^2(\Omega)^d} = (q, \psi)_W \quad \forall \psi \in W. \tag{25}$$

We plugg (1) in (24), (2) in (25) and use the notation $\boldsymbol{E} = -\nabla\varphi$ to obtain the following variational formulation of Problem P, in the terms of displacement field and electric potential.

Problem P_V. *Find a displacement field $\boldsymbol{u} \in V$ and an electric potential $\varphi \in W$ such that*

$$(\mathcal{F}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}))_{\mathcal{H}} + (\mathcal{E}^T\nabla\varphi, \varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}))_{L^2(\Omega)^d} + h(\boldsymbol{v}) - h(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V \tag{26}$$

$$(\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} = (q, \psi)_W \quad \forall \psi \in W. \tag{27}$$

Our main existence and uniqueness result which we establish in Section 4 is the following.

Theorem 3.1 *Assume that* (14)–(19) *hold. Then Problem P_V has a unique solution $(\boldsymbol{u}, \varphi)$ which depends Lipschitz continuously on $\boldsymbol{f} \in V$ and $q \in W$.*

A "quadriple" of functions ($\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, $\boldsymbol{D}$) which satisfy (1), (2), (26) and (27) is called a *weak solution* of the piezoelectric contact problem P. We conclude by Theorem 3.1 that, under the assumptions (14)–(19), the piezoelectric contact problem (2)–(9) has at least a weak solution ($\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, $\boldsymbol{D}$) such that $\boldsymbol{u} \in V$, $\varphi \in W$. Moreover, it is easy to see that in this case $\boldsymbol{\sigma} \in \mathcal{H}_1$ and $\boldsymbol{D} \in \mathcal{W}$. The solution is unique and depends Lipschitz continuously on the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0 and q_2.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps. To present it we consider the product space $X = V \times W$ together

with the inner product

$$(x, y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{28}$$

and the associated norm $\|\cdot\|_X$. Everywhere below we assume that (14)–(19) hold.

We introduce the operator $A : X \to X$ defined by

$$\begin{aligned}(Ax, y)_X = (\mathcal{F}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} \\ +(\mathcal{E}^T\nabla\varphi, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} - (\mathcal{E}\varepsilon(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} \\ \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X\end{aligned} \tag{29}$$

and we extend the functional h defined by (21) to a functional j defined on X, that is

$$j(x) = h(\boldsymbol{u}) \qquad \forall x = (\boldsymbol{u}, \varphi) \in X. \tag{30}$$

Finally, we consider the element $f \in X$ given by

$$f = (\boldsymbol{f}, q) \in X. \tag{31}$$

We start with the following equivalence result.

Lemma 4.1 *The couple $x = (\boldsymbol{u}, \varphi)$ is a solution to Problem P_V if and only if*

$$(Ax, y - x)_X + j(y) - j(x) \geq (f, y - x)_X \quad \forall\, y \in X. \tag{32}$$

Proof: Let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to Problem P_V and let $y = (\boldsymbol{v}, \psi) \in Y$. We use the test function $\psi - \varphi$ in (26), add the corresponding inequality to (27) and use (28)–(31) to obtain (32). Conversely, let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to the elliptic variational inequality (32). We take $y = (\boldsymbol{v}, \varphi)$ in (32) where $\boldsymbol{v}$ is an arbitrary element of V and obtain (26); then we take successively $y = (\boldsymbol{v}, \varphi + \psi)$ and $y = (\boldsymbol{v}, \varphi - \psi)$ in (32), where ψ is an arbitrary element of W; as a result we obtain (27), which concludes the proof. □

Next, we start with the study of the the properties of the operator A given by (29).

Lemma 4.2 *The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.*

Proof: Consider two elements $x_1 = (\boldsymbol{u}_1, \varphi_1)$, $x_2 = (\boldsymbol{u}_2, \varphi_2) \in X$. Using (29) we have

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X = \\ (\mathcal{F}\varepsilon(\boldsymbol{u}_1) - \mathcal{F}\varepsilon(\boldsymbol{u}_2), \varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d} + \\ (\mathcal{E}^T\nabla\varphi_1 - \mathcal{E}^T\nabla\varphi_2, \varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2))_{\mathcal{H}} - (\mathcal{E}\varepsilon(\boldsymbol{u}_1) - \mathcal{E}\varepsilon(\boldsymbol{u}_1), \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}\end{aligned}$$

and, since it follows by (20) that $(\mathcal{E}^T\nabla\varphi, \varepsilon(\boldsymbol{u}))_{\mathcal{H}} = (\mathcal{E}\varepsilon(\boldsymbol{u}), \nabla\varphi)_{L^2(\Omega)^d}$ for all $x = (\boldsymbol{u}, \varphi) \in X$, we find

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X = \\ (\mathcal{F}\varepsilon(\boldsymbol{u}_1) - \mathcal{F}\varepsilon(\boldsymbol{u}_2), \varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}.\end{aligned}$$

We use now (14), (16) and Friedrichs-Poincaré inequality (13) to see that there exists $c_1 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$, Ω and Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2)$$

and, keeping in mind (28), we obtain

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1 \, \|x_1 - x_2\|_X^2. \tag{33}$$

In the same way, using (14)–(16), after some algebra it follows that there exists $c_2 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$ and $\mathcal{E}$ such that

$$(Ax_1 - Ax_2, y)_X \leq c_2(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V + \|\varphi_1 - \varphi_2\|_W \|\boldsymbol{v}\|_V + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\psi\|_W + \|\varphi_1 - \varphi_2\|_W \|\psi\|_W)$$

for all $y = (\boldsymbol{v}, \psi) \in V$. We use (28) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \leq 4c_2 \|x_1 - x_2\|_V \|y\|_V \qquad \forall y \in X$$

and, taking $y = Ax_1 - Ax_2 \in X$, we find

$$\|Ax_1 - Ax_2\|_X \leq 4c_2 \|x_1 - x_2\|_V. \tag{34}$$

Lemma 4.2 is now a consequence of inequalities (33) and (34). □

Next we investigate the properties of the functional j given by (30), (21).

Lemma 4.3 $j : X \to \mathbb{R}$ *is a convex and continuous functional.*

Proof: We first remark that j is a convex functional on X. Let $\{x_n\} = \{(\boldsymbol{u}_n, \varphi_n)\} \subset X$ such that $x_n \to x = (\boldsymbol{u}, \varphi) \in X$. It follows that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^2(\Gamma_3)^d$, which imply that

$$\|\boldsymbol{u}_{n\tau}\| \to \|\boldsymbol{u}_\tau\| \quad \text{in} \quad L^2(\Gamma_3). \tag{35}$$

Therefore, we use the definition of j and (35) to deduce that $j(x_n) \to j(x)$ as $n \to \infty$, which concludes the proof. □

We have now all the ingredients to prove the Theorem.

Proof of Theorem 3.1: Assume that (14)–(19) hold. Then, Lemmas 4.2 and 4.3 allow us to use the standard results for the theory of variational inequalities (see for example [4]); we obtain that the variational inequality (32) has a unique solution $x = (\boldsymbol{u}, \varphi) \in X$ and, using Lemma 4.1, we deduce that $(\boldsymbol{u}, \varphi) \in V \times W$ is a unique solution to Problem P_V. Moreover, the solution depends Lipschitz continuously on $\boldsymbol{f} \in V$ and $q \in W$, which concludes the proof.

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