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A quasistatic viscoplastic contact problem with adhesion and damage

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Abstract

In this work our main goal is to provide numerical simulations of a quasistatic frictionless contact problem arising in viscoplasticity taking into account the damage of the material and the adhesion to an obstacle. The mechanical damage, caused by excessive stress or strain, is modelled by an inclusion of parabolic type, and the adhesion by an ordinary differential equation. The contact is assumed with a deformable obstacle and then, a normal compliance contact condition is used. The variational formulation is provided for this mechanical problem and the existence of a unique solution is stated. Then, a fully discrete scheme is introduced using the finite element method to approximate the spatial domain and the Euler scheme to discretize the time derivatives. Error estimates are derived and, under suitable regularity assumptions, the linear convergence of the algorithm is deduced. Finally, some numerical examples are presented to show the performance of the method.

Key Words: Viscoplasticity, normal compliance, adhesion, damage, error estimates, numerical simulations.
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1 Introduction

Situations of frictionless or frictional contact abound in industry. Contacts of the braking pads with the wheel or the tire with the road are just a few simple examples. Because of its importance, the engineering literature concerning this topic is rather extensive (see, e.g., the monographs [8, 17, 20] and references therein).

In many industrial problems involving contact, there is a need to take into account the damage of the material, due to mechanical stress or strain. Early models, derived from the thermodynamical principles, were introduced in [11], where some numerical simulations were performed. Recently, damage models have been improved and studied in many engineering papers.

The effect of the adhesion on the contact surface has been also included into the model. According to [9, 10], a surface variable, the bonding field, is introduced, taking values between zero and one and describing the fractional density of active bonds on the contact surface. Contact problems with adhesion have been studied in some recent papers involving viscoelastic materials ([3, 4, 5, 7, 15, 18]).

In this paper a coupled contact problem involving a viscoplastic body in contact with a deformable obstacle taking into account the damage material and the adhesion to the obstacle is studied. The damage and adhesion phenomenons will be described more precisely in Section 2. The contact process is described with a version of the normal compliance condition which allows for the interpenetration of the surface asperities ([16]). We notice that this condition has been used extensively in numerical algorithms and computer codes and also as an approximation of the Signorini condition (which describes the contact with a rigid obstacle).

2 Mechanical and variational formulations

Let S_d denote the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. Let "$\cdot$" be the inner product on $\mathbb{R}^d$ or S_d, and $|\cdot|$ the Euclidean norms on these spaces.

Let us consider a viscoplastic body that occupies the domain $\Omega \subset \mathbb{R}^d$, and let the time interval of interest be $[0, T]$, $T > 0$. The outer surface $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_D, Γ_F and Γ_C. For a.e. $\boldsymbol{x} \in \Gamma$, we denote by $\boldsymbol{\nu}(\boldsymbol{x})$ the unit normal vector outward to Γ. A density of volume forces $\boldsymbol{f}_B$ acts in Ω and surface tractions of density $\boldsymbol{f}_F$ are given on Γ_F. The body is assumed to be clamped on Γ_D, and so the displacement field vanishes there. Finally, the body is assumed to be in frictionless contact with a foundation on the contact surface Γ_C (see Figure 1).

We denote by $\boldsymbol{u}$ the displacement field, $\boldsymbol{\sigma}$ the stress tensor and $\varepsilon(\boldsymbol{u})$ the linearized strain tensor. Moreover, let ζ be the damage field, which is defined in Ω and measures the density of the microcracks in the material. The material is assumed viscoplastic with the following constitutive law (see, e.g., [13, 19] and references therein),

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta),$$

where $\mathcal{E}$ and $\mathcal{G}$ are the elastic tensor and the viscoplastic constitutive function, respectively, whose properties will be described below. Here, a dot above a variable represents its partial time derivative.

The damage of the material, as a result of the tensile or compressive stresses in the body, has been considered. According to [11], it is modelled using the internal state variable ζ, taking values in $[0, 1]$, in such a way that when $\zeta = 1$ (damage-free) the material is elastic ($\mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) = \boldsymbol{0}$), when $\zeta = 0$ (fully damaged) the material has always plastic strains ($\mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) \neq \boldsymbol{0}$), and when $0 < \zeta < 1$ there is partial damage.

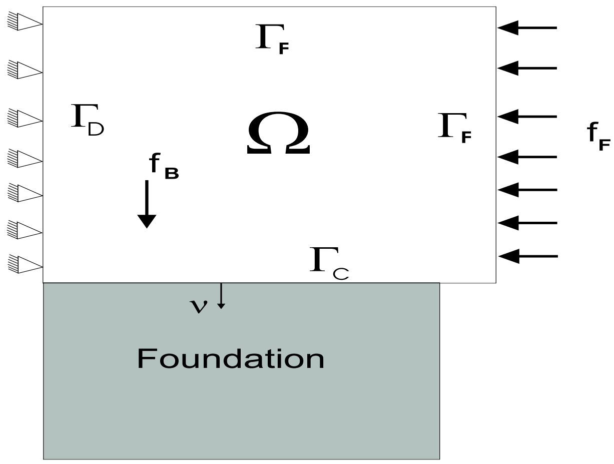

Figure 1: A viscoplastic body in frictionless contact with a deformable foundation.

Following [11], the evolution of the microscopic cracks responsible for the damage is described by the differential inclusion

$$\dot{\zeta} - \kappa \triangle \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \zeta).$$

Here, $\triangle$ is the Laplacian, $\kappa > 0$ is the damage diffusion constant, ϕ is the damage source function and $\partial I_{[0,1]}$ denotes the subdifferential of the indicator function $I_{[0,1]}$ of the interval $[0,1]$. It guarantees that $\zeta \in [0,1]$. Moreover, we assume that there is no damage influx throughout the boundary Γ, and therefore $\partial\zeta/\partial\boldsymbol{\nu} = 0$ on Γ.

Next, we describe the contact conditions with adhesion on Γ_C. Following [9, 10], we introduce the surface state variable β defined on Γ_C, which represents the intensity of the adhesion over the contact surface, with values $0 \leq \beta \leq 1$. When $\beta = 1$ at a point $\boldsymbol{x} \in \Gamma_C$, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive and there is no adhesion; and $0 < \beta < 1$ is the case of partial adhesion when only a fraction of the bonds is active.

We assume that the normal stress satisfies the following normal compliance contact condition with adhesion

$$\sigma_\nu = -p_\nu(u_\nu) + \gamma_\nu \beta^2(-R(u_\nu))_+ \quad \text{on} \quad \Gamma_C \times (0,T),$$

where $r_+ = \max\{0, r\}$, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ is the normal displacement which, when positive, represents the penetration of the surface asperities into the foundation, p_ν is a prescribed function such that $p_\nu(r) = 0 \quad \text{for } r \leq 0$, and $R : \mathbb{R} \to \mathbb{R}$ is the truncation operator

$$R(s) = \begin{cases} L & \text{if } s \geq L, \\ s & \text{if } |s| \leq L, \\ -L & \text{if } s \leq -L, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it stretches without offering any additional resistance ([4, 18]).

The contribution of the adhesive to the normal traction is represented by the second term on the right-hand side, namely $\gamma_\nu \beta^2(-R(u_\nu))_+$. Thus, the adhesive traction is nonnegative and proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion, and to the normal displacement, but as long as it does not exceed the bond length L. Once it exceeds it the normal traction remains constant. More general expressions for this condition can be found in [15]. General forms of the normal compliance contact condition with adhesion can be found in [5].

We assume that the tangential stiffness of the glue depends on the intensity of adhesion and on the tangential displacement, but only up to the bond length L, thus,

$$-\boldsymbol{\sigma}_\tau = p_\tau(\beta) R^*(\mathbf{u}_\tau) \quad \text{on } \Gamma_C \times (0, T),$$

where the truncation operator R^* is defined by

$$R^*(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \leq L, \\ \frac{\boldsymbol{v}}{|\boldsymbol{v}|} L & \text{if } |\boldsymbol{v}| \geq L. \end{cases}$$

Then, $p_\tau(\beta)$ acts as the stiffness or spring constant, and the traction is in direction opposite to the displacement. We note that the model can be extended to a more general adhesion condition (see [15]).

Following [9, 10] the evolution of the adhesion field is governed by the following differential equation

$$\dot{\beta} = -(\gamma_\nu \beta (R(u_\nu))^2 - \epsilon_a)_+ \quad \text{on } \Gamma_C \times (0, T),$$

where ϵ_a represents a limit bound energy, below which there is no change in the bonding. We note that this condition does not allow for rebonding, once debonding takes place, since $\dot{\beta} \leq 0$.

We assume that the forces acting on the system vary slowly in time so that the process is quasistatic, and we denote by $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, ζ_0 and β_0 the initial displacements, stresses, damage and adhesion, respectively. Therefore, the mechanical formulation corresponding to this quasistatic viscoplastic contact problem with adhesion and damage is written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$, *a damage field* $\zeta : \Omega \times [0, T] \to [0, 1]$ *and an adhesion field*

$\beta : \Omega \times [0,T] \to [0,1]$ *such that,*

$$
\begin{aligned}
&\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_B = \mathbf{0} \quad \text{in} \quad \Omega \times (0,T),\\
&\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T),\\
&\dot{\zeta} - \kappa\,\triangle\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) \quad \text{in} \quad \Omega \times (0,T),\\
&\frac{\partial\zeta}{\partial\boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma \times (0,T),\\
&\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D \times (0,T),\\
&\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0,T),\\
&-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu\beta^2(-R(u_\nu))_+ \quad \text{on } \Gamma_C \times (0,T),\\
&-\boldsymbol{\sigma}_\tau = p_\tau(\beta)R^*(\mathbf{u}_\tau) \quad \text{on } \Gamma_C \times (0,T),\\
&\dot{\beta} = -(\gamma_\nu\beta(R(u_\nu))^2 - \epsilon_a)_+ \quad \text{on } \Gamma_C \times (0,T),\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \zeta(0) = \zeta_0 \ \text{ in } \Omega, \quad \beta(0) = \beta_0 \ \text{ on } \Gamma_C.
\end{aligned}
$$

In order to obtain the variational formulation of the above problem, we need to introduce additional notation and assumptions on the problem data.

Let Q be the following variational space:

$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d}\,;\, \tau_{ij} = \tau_{ji},\, i,j = 1,\ldots,d\},$$

define the space of admissible displacements V by

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\, \boldsymbol{v} = \mathbf{0} \quad \text{on} \quad \Gamma_D\},$$

and denote by $\mathcal{K}$ the following closed convex subset of $H^1(\Omega)$,

$$\mathcal{K} = \{\xi \in H^1(\Omega)\,;\, 0 \le \xi \le 1 \quad \text{in} \quad \Omega\}.$$

For a Banach space X, let us denote by $(\cdot,\cdot)_X$ the inner product defined in X and by $\|\cdot\|_X$ its associated norm.

The elastic tensor $\mathcal{E} : \Omega \times S_d \to S_d$ is a fourth-order symmetric definite positive tensor; that is,

$$
\left.
\begin{aligned}
&\text{(a) } \mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d.\\
&\text{(b) } \mathcal{E}(\boldsymbol{x})\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau},\boldsymbol{\sigma} \in S_d.\\
&\text{(c) There exists } C_\mathcal{E} > 0 \text{ such that}\\
&\qquad \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge C_\mathcal{E}|\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_d.
\end{aligned}
\right\} \tag{1}
$$

The viscoplastic function $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R} \to S_d$ satisfies

$$
\left.
\begin{aligned}
&\text{(a) There exists } L_\mathcal{G} > 0 \text{ such that}\\
&\quad |\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varepsilon_1, \beta_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \beta_2)|\\
&\qquad\quad \le L_\mathcal{G}\,(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\varepsilon_1 - \varepsilon_2| + |\beta_1 - \beta_2|)\\
&\quad\ \forall\,\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d, \quad \beta_1, \beta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(b) } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \varepsilon, \beta) \ \text{ is a Lebesgue measurable function in } \ \Omega,\\
&\quad\ \forall\,\boldsymbol{\sigma}, \varepsilon \in S_d, \quad \beta \in \mathbb{R}.\\
&\text{(c) } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0) \in Q.
\end{aligned}
\right\} \tag{2}
$$

The damage source function $\phi : \Omega \times S_d \times S_d \times \mathbb{R} \to \mathbb{R}$ verifies

$$\left.\begin{array}{l}
\text{(a) There exists } L_\phi > 0 \text{ such that} \\
\quad |\phi(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varepsilon_1, \beta_1) - \phi(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \beta_2)| \\
\qquad\qquad \le L_\phi \left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\varepsilon_1 - \varepsilon_2| + |\beta_1 - \beta_2|\right) \\
\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d, \quad \beta_1, \beta_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(b) } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{\sigma}, \varepsilon, \beta) \text{ is a Lebesgue measurable function in } \Omega, \\
\qquad \forall\, \boldsymbol{\sigma}, \varepsilon \in S_d, \quad \beta \in \mathbb{R}. \\
\text{(c) } \boldsymbol{x} \mapsto \phi(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \in L^2(\Omega).
\end{array}\right\} \tag{3}$$

The normal compliance function $p_\nu : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ and the tangential function $p_\tau : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfy:

$$\left.\begin{array}{l}
\text{(a) There exists an } L_\nu > 0 \text{ such that} \\
\qquad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu\, |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_C, \ \forall r \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) = 0 \quad \text{for all } r \le 0. \\
\text{(d) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)) \cdot (r_1 - r_2) \ge 0 \quad \forall\, r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C..
\end{array}\right\} \tag{4}$$

$$\left.\begin{array}{l}
\text{(a) There exists an } L_\tau > 0 \text{ such that} \\
\qquad |p_\tau(\boldsymbol{x}, \beta_1) - p_\tau(\boldsymbol{x}, \beta_2)| \le L_\tau\, |\beta_1 - \beta_2| \quad \forall\, \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C. \\
\text{(b) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, \beta) \text{ is Lebesgue measurable on } \Gamma_C \quad \forall \beta \in \mathbb{R}. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0) \in L^\infty(\Gamma_C).
\end{array}\right\} \tag{5}$$

The body forces and surface tractions have the regularity

$$\boldsymbol{f}_B \in W^{1,2}(0, T; [L^2(\Omega)]^d), \quad \boldsymbol{f}_F \in W^{1,2}(0, T; [L^2(\Gamma_F)]^d). \tag{6}$$

Using Riesz's representation theorem, let $\boldsymbol{f}(t) \in V$ be given by the relation

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_B(t), \boldsymbol{v})_{[L^2(\Omega)]^d} + (\boldsymbol{f}_F(t), \boldsymbol{v})_{[L^2(\Gamma_F)]^d}, \quad \forall \boldsymbol{v} \in V.$$

The adhesion coefficient and the limit bound satisfy

$$\gamma_\nu \in L^\infty(\Gamma_C), \quad \gamma_\nu \ge 0, \qquad \epsilon_a \in L^\infty(\Gamma_C), \quad \epsilon_a \ge 0. \tag{7}$$

Let us define the following bilinear form $a : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ by

$$a(\xi, \psi) = \kappa \int_\Omega \nabla\xi \nabla\psi \, d\boldsymbol{x}, \quad \forall \xi, \psi \in \mathcal{K},$$

where $\kappa > 0$. We denote by $j : L^\infty(\Gamma_C) \times V \times V \to \mathbb{R}$ the functional

$$\begin{aligned}
j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \ & \int_{\Gamma_C} p_\nu(u_\nu)\, v_\nu \, da - \int_{\Gamma_C} \gamma_\nu \beta^2 (-R(u_\nu))_+ \, v_\nu \, da \\
& + \int_{\Gamma_C} p_\tau(\beta)\, R^*(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \, da \qquad \forall \beta \in L^\infty(\Gamma_C), \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,
\end{aligned}$$

where, for a function $\boldsymbol{v} \in [H^1(\Omega)]^d$, let $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ be its normal and tangential components, respectively.

Finally, let the initial data $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, β_0 and ζ_0 be chosen in such a way that

$$\left.\begin{array}{l} \boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \quad \beta_0 \in L^\infty(\Gamma_C), \quad 0 < \beta_0 \le 1 \ \text{ a.e. on } \Gamma_C, \\ \zeta_0 \in H^1(\Omega), \quad 0 < \zeta_0 \le 1 \text{ a.e. in } \Omega. \end{array}\right\} \tag{8}$$

Moreover, we also assume the following compatibility condition

$$(\boldsymbol{\sigma}_0, \varepsilon(\boldsymbol{v}))_Q + j(\beta_0, \boldsymbol{u}_0, \boldsymbol{v}) = (\boldsymbol{f}(0), \boldsymbol{v})_V, \quad \forall \boldsymbol{v} \in V. \tag{9}$$

Applying a Green's formula, we obtain the following variational formulation of Problem **P**.

Problem VP. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$*, a stress field* $\boldsymbol{\sigma} : [0,T] \to Q$*, a damage field* $\zeta : [0,T] \to \mathcal{K}$ *and an adhesion field* $\beta : [0,T] \to L^\infty(\Gamma_C)$ *such that,*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \zeta(t)) \quad \text{a.e.} \quad t \in (0,T), \tag{10}$$

$$\dot{\beta}(t) = -(\gamma_\nu \beta (R(u_\nu))^2 - \epsilon_a)_+ \quad \text{a.e. } t \in (0,T), \tag{11}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}))_Q + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V, \quad \forall \boldsymbol{v} \in V, \quad \text{a.e.} \quad t \in (0,T), \tag{12}$$

$$\begin{aligned} &(\dot{\zeta}(t), \xi - \zeta(t))_{L^2(\Omega)} + a(\zeta(t), \xi - \zeta(t)) \ge (\phi(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \zeta(t)), \xi - \zeta(t))_{L^2(\Omega)}, \\ &\qquad \forall \xi \in \mathcal{K}, \quad \text{a.e.} \quad t \in (0,T), \end{aligned} \tag{13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \zeta(0) = \zeta_0 \ \text{ in } \ \Omega, \quad \beta(0) = \beta_0 \ \text{ on } \ \Gamma_C. \tag{14}$$

The existence of a unique solution to Problem **VP** is stated in the following.

Theorem 1. *Assume that (1)-(9) hold. Then there exists a unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta, \zeta)$ *to Problem* **VP**. *Moreover, the solution satisfies,*

$$\begin{aligned} &\boldsymbol{u} \in W^{1,2}(0,T;V), \quad \boldsymbol{\sigma} \in W^{1,2}(0,T;Q), \quad \beta \in W^{1,2}(0,T;L^\infty(\Gamma_C)), \\ &\zeta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \end{aligned}$$

The proof of Theorem 1 is carried out in several steps using similar arguments as those used in [19]. First, the differential equation in (11) for the adhesion field is solved for a given displacement field $\boldsymbol{u}$, and the continuous dependence of the adhesion solution with respect to $\boldsymbol{u}$ is obtained. Secondly, using fixed point arguments, the existence of a unique solution to (10), (12)-(13), for a given adhesion field β, is proved. Finally, using the Banach fixed point theorem, the existence of a unique solution to Problem **VP** is derived.

3 Numerical analysis of a fully discrete scheme

In this section we consider a fully discrete approximation of Problem **VP** by discretizing the spatial domain using the finite element method, and the time derivatives by the Euler scheme. Assume that Ω is a polygonal domain and let

$\mathcal{T}^h$ be a regular finite element triangulation of the domain Ω compatible with the boundary partition $\Gamma = \Gamma_D \cup \Gamma_F \cup \Gamma_C$. We denote by θ^h the triangulation induced by $\mathcal{T}^h$ on Γ_C. Let us define the following finite element spaces:

$$V^h = \{\boldsymbol{v}^h \in [C(\overline{\Omega})]^d \,;\, \boldsymbol{v}^h_{|_T} \in [P_1(T)]^d \quad \forall T \in \mathcal{T}^h, \quad \boldsymbol{v}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$

$$Y^h = \{\xi^h \in C(\overline{\Omega}) \,;\, \xi^h_{|_T} \in P_1(T) \quad \forall T \in \mathcal{T}^h\},$$

$$Q^h = \{\boldsymbol{\tau}^h \in Q \,;\, \boldsymbol{\tau}^h_{|_T} \in [P_0(T)]^{d\times d} \quad \forall T \in \mathcal{T}^h\},$$

$$B^h = \{r^h \in C(\overline{\Omega}) \,;\, r^h_{|_C} \in P_1(C) \quad \forall C \in \theta^h\},$$

to approximate the spaces V, $L^2(\Omega)$, Q and $L^2(\Gamma_C)$, respectively. Moreover, let $\mathcal{K}^h = Y^h \cap \mathcal{K}$. For the sake of simplicity, we assume the following assumption on the viscoplasticity operator

$$\mathcal{G}(Q^h, Q^h, Y^h) \subset Q^h,$$

which is obtained in the more usual examples (see Section 4).

Let $\mathcal{P}_{B^h} : L^\infty(\Gamma_C) \subset L^2(\Gamma_C) \to B^h$ be the orthogonal projection operator defined by

$$(\mathcal{P}_{B^h}\gamma, \gamma^h)_{L^2(\Gamma_C)} = (\gamma, \gamma^h)_{L^2(\Gamma_C)} \quad \forall \gamma \in L^2(\Gamma_C),\ \gamma^h \in B^h.$$

In order to discretize the time derivatives, let $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition of the time interval $[0,T]$ and denote by k the time step, $k = T/N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we denote by $\delta w_n = (w_n - w_{n-1})/k$. In this section, no summation is considered over the repeated index n and, everywhere in the sequel, c will denote positive constants which are independent of the discretization parameters h and k.

We denote by $\boldsymbol{u}_0^h$, $\boldsymbol{\sigma}_0^h$, ζ_0^h and β_0^h appropiate approximations of the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, ζ_0 and β_0, respectively. The fully discrete approximation is based on the forward Euler scheme and it has the following form.

Problem VP^{hk}. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$, a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=0}^N \subset Q^h$, a discrete damage field $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset \mathcal{K}^h$ and a discrete adhesion field $\beta^{hk} = \{\beta_n^{hk}\}_{n=0}^N \subset B^h$ such that,*

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \quad \boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h, \quad \zeta_0^{hk} = \zeta_0^h, \quad \beta_0^{hk} = \beta_0^h,$$

and for $n = 1, \dots, N$,

$$\delta\beta_n^{hk} = -\mathcal{P}_{B^h}(\gamma_\nu \beta_{n-1}^{hk}[R((\boldsymbol{u}_{n-1}^{hk})_\nu)]^2 - \epsilon_a)_+, \tag{15}$$

$$\delta\boldsymbol{\sigma}_n^{hk} = \mathcal{E}\varepsilon(\delta\boldsymbol{u}_n^{hk}) + \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \varepsilon(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \tag{16}$$

$$(\boldsymbol{\sigma}_n^{hk}, \varepsilon(\boldsymbol{v}^h))_Q + j(\beta_{n-1}^{hk}, \boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h) = (\boldsymbol{f}_n, \boldsymbol{v}^h)_V, \quad \forall \boldsymbol{v}^h \in V^h, \tag{17}$$

$$(\delta\zeta_n^{hk}, \xi^h - \zeta_n^{hk})_{L^2(\Omega)} + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \\ \geq (\phi(\boldsymbol{\sigma}_{n-1}^{hk}, \varepsilon(\boldsymbol{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_{L^2(\Omega)}, \quad \forall \xi^h \in \mathcal{K}^h. \tag{18}$$

Using classical results on variational inequalities (see, e.g., [12]), the existence of a unique solution to Problem $\mathbf{VP}^{hk}$ is obtained.

We notice that Problem $\mathbf{VP}^{hk}$ is solved, at each time step, as follows. First, β_n^{hk} is obtained from (15). Then, plugging (16) into (17), we have that $\boldsymbol{u}_n^{hk}$ is the solution of a discrete linear variational equation which can be seen as a linear system and then, Cholesky's method is applied. Finally, the discrete damage field, ζ_n^{hk}, is obtained solving the discrete variational inequality (18), applying a penalty-duality algorithm introduced in [1] and already used in other contact problems ([2, 4]).

Using similar arguments to those employed in [2, 3, 4], we obtain the following error estimates result.

Theorem 2. *Let the assumptions of Theorem 1 still hold. If the continuous solution has the additional regularity*

$$\boldsymbol{u} \in C^1([0,T];V), \quad \boldsymbol{\sigma} \in C^1([0,T];Q), \\ \zeta \in C([0,T];H^2(\Omega)) \cap H^2(0,T;L^2(\Omega)), \quad \beta \in C^1([0,T];L^2(\Gamma_C)),$$

the following error estimate is obtained for all $\{\boldsymbol{v}_j^h\}_{j=0}^N \subset V^h$ *and* $\{\zeta_j^h\}_{j=0}^N \subset \mathcal{K}^h$,

$$\max_{0\leq n\leq N} \Big\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q^2 + \|\zeta_n - \zeta_n^{hk}\|_{L^2(\Omega)}^2 + \|\beta_n - \beta_n^{hk}\|_{L^2(\Gamma_C)}^2 \Big\} \\ + k\sum_{j=1}^N \|\nabla(\zeta_j - \zeta_j^{hk})\|_{[L^2(\Omega)]^d}^2 \leq c\Big(\|\zeta_1 - \xi_1^h\|_{L^2(\Omega)}^2 + k\sum_{j=1}^N I_j^2 + e_0 \\ + k^2[\|\boldsymbol{u}\|_{C^1([0,T];V)}^2 + \|\boldsymbol{\sigma}\|_{C^1([0,T];Q)}^2 + \|\zeta\|_{H^2(0,T;L^2(\Omega))}^2] \\ + \frac{1}{k}\sum_{j=1}^{N-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_{L^2(\Omega)}^2 + \max_{0\leq n\leq N} \|\zeta_n - \xi_n^h\|_{L^2(\Omega)}^2$$

$$+k\sum_{j=1}^{N}\|\phi(\boldsymbol{\sigma}_j,\boldsymbol{\varepsilon}(\boldsymbol{u}_j),\zeta_j)-\delta\zeta_j+\kappa\Delta\zeta_j\|_{L^2(\Omega)}\cdot\|\zeta_j-\xi_j^h\|_{L^2(\Omega)}$$
$$+k\sum_{j=1}^{N}\|(I-\mathcal{P}_{B^h})(\gamma_\nu\beta_j[R((\dot{\boldsymbol{u}}_j)_\nu)]^2-\epsilon_a)_+\|_{L^2(\Gamma_C)}$$
$$+k\sum_{j=1}^{N}\|\zeta_j-\xi_j^h\|_{H^1(\Omega)}^2+k\sum_{j=1}^{N}\|\boldsymbol{u}_j-\boldsymbol{v}_j^h\|_V^2\Big),$$

where

$$I_j=\Big\|\int_0^{t_j}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\zeta(s))\,ds-k\sum_{i=1}^{j-1}\mathcal{G}(\boldsymbol{\sigma}_i,\boldsymbol{\varepsilon}(\boldsymbol{u}_i),\zeta_i)\Big\|_Q$$

is an integration error, and

$$e_0=\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q^2+\|\beta_0-\beta_0^h\|_{L^2(\Gamma_C)}^2+\|\zeta_0-\zeta_0^h\|_{L^2(\Omega)}^2$$

is the approximation error on the initial conditions.

We notice that the above estimate is the main tool to obtain different rates of convergence depending on the solution regularity. Therefore, if we assume that

$$\boldsymbol{u}\in C([0,T];[H^2(\Omega)]^d),\quad \dot{\zeta}\in L^2(0,T;H^1(\Omega)),\tag{19}$$

we obtain the following theorem which states the linear convergence of the algorithm.

Theorem 3. *Let the assumptions of Theorem 2 still hold. Under the additional regularity condition (19), the numerical scheme provided by Problem* $\mathbf{VP}^{hk}$ *is linearly convergent, i.e., there exists* $c>0$*, independent of* h *and* k*, such that*

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q+\|\zeta_n-\zeta_n^{hk}\|_{L^2(\Omega)}$$
$$+\|\beta_n-\beta_n^{hk}\|_{L^2(\Gamma_C)}\Big\}\le c(h+k).$$

4 Numerical examples

In order to see the performance of the numerical scheme described in the previous section, some two-dimensional numerical simulations have been done. In all the examples presented below, the elastic tensor was assumed homogeneous and satisfying the plane stress hypothesis. Then, it has the following form

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta}=\frac{E\,r}{1-r^2}(\tau_{11}+\tau_{22})\delta_{\alpha\beta}+\frac{E}{1+r}\tau_{\alpha\beta},\qquad 1\le\alpha,\beta\le 2,$$

where $\delta_{\alpha\beta}$ denotes the Kronecker's symbol, and $E, r > 0$ are the Young's modulus and the Poisson's ratio of the material occupying Ω, respectively.

The function $\mathcal{G}$ was a version of the Maxwell function defined as

$$\mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) = -\left(\frac{1-\zeta}{100}\right) \Phi(\boldsymbol{\sigma}),$$

where $\Phi : S_2 \to S_2$ is a truncation operator given by

$$\forall \boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in S_2, \quad (\Phi(\boldsymbol{\tau}))_{ij} = \begin{cases} L_* & \text{if} \quad \tau_{ij} > L_*, \\ \tau_{ij} & \text{if} \quad \tau_{ij} \in [-L_*, L_*], \\ -L_* & \text{if} \quad \tau_{ij} < -L_*. \end{cases}$$

Truncation values $L = 1000$ and $L_* = 1000$ were employed in the simulations.

The normal compliance function p_ν was chosen in the following form,

$$p_\nu(r) = \frac{1}{\mu} r_+,$$

where μ is a positive constant which represents a deformability coefficient, and the tangential function p_τ was given by

$$p_\tau(\gamma) = \gamma.$$

Finally, the following damage source function ϕ was used in the examples,

$$\phi(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \zeta) = -\frac{1}{500} \|\boldsymbol{\sigma}\|_{VM},$$

where we recall that the von Mises norm, for a two-dimensional stress field $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2 \in S_2$, is defined by

$$\|\boldsymbol{\tau}\|_{VM}^2 = \tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2.$$

Moreover, the damage diffusion constant κ was taken as 1 and the initial adhesion as $\beta_0 = 1$.

4.1 First example: compression forces

As a first example, we consider the physical setting depicted in Figure 2. A viscoplastic body occupying the domain $\Omega = [0,3] \times [0,1]$ is clamped on its left vertical boundary ($\Gamma_D = \{0\} \times [0,1]$). No volume forces are acting in Ω, a compression force $\boldsymbol{f}_F = (0, -100t)\, N/m^2$ acts on the part $(0,3) \times \{1\}$ and the part $\{3\} \times [0,1]$ is traction free. Finally, we assume that the body is in contact with a deformable obstacle, the so-called foundation, on $\Gamma_C = [0,3] \times \{0\}$. The deformability coefficient μ was taken as 10^{-4} in such a way that the body is almost rigid and no penetration can be produced.

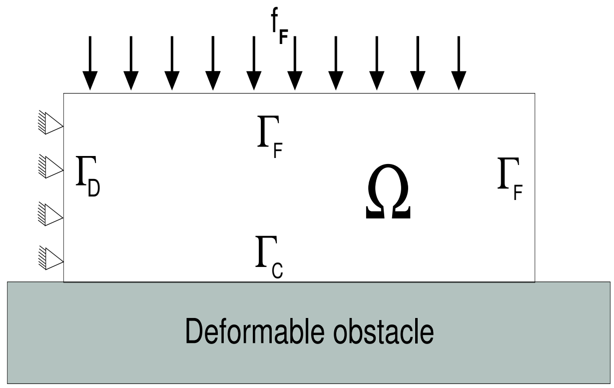

Figure 2: Physical setting of the first example.

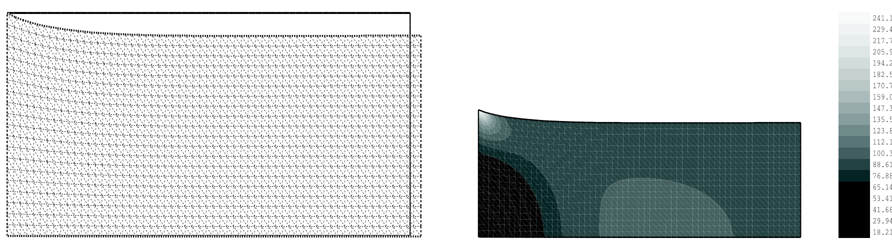

Figure 3: Final deformed mesh with initial configuration and von Mises stress norm.

The following data were employed in the simulation:

$$T = 1\,s, \quad \boldsymbol{f}_B = 0\,N/m^3, \quad E = 10^4\,N/m^2, \quad r = 0.3$$
$$\gamma_\nu = 10^3, \quad \epsilon_a = 0, \quad \beta_0 = 1$$

Using $k = 0.01$ as the time discretization parameter and the finite element spaces V^h, Q^h, Y^h and B^h for $h = 0.01$, the corresponding discrete problem $\mathbf{VP}^{hk}$ was solved. Then, the deformed mesh at final time with the initial configuration are plotted in Figure 3 (left), and the von Mises stress norm is shown in the deformed configuration (right). The corresponding damage field is depicted in Figure 4. We notice that, because of the compression forces, there is no separation of the body from the obstacle and so the adhesion effects take place only in the tangential component.

4.2 Second example: influence of the adhesion

Secondly, a two-dimensional viscoplastic body which occupies the domain $\Omega = [0,3] \times [0,1]$ is considered. The body is assumed clamped on $\Gamma_D = \{3\} \times [0,1]$

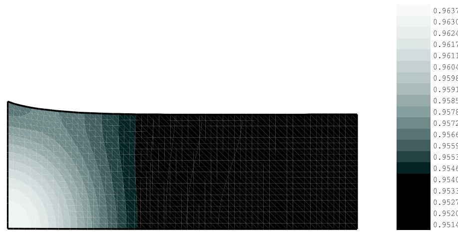

Figure 4: Damage field at final time.

and in frictionless contact on the part $\Gamma_C = (0,3) \times \{0\}$. No volume forces act in Ω, traction forces $\boldsymbol{f}_F = (30,30)sin(\frac{t\pi}{2})\, N/m^2$ act on the boundary $\{0\} \times [0,1]$ and the boundary $[0,3] \times \{1\}$ is assumed traction free (see Figure 5).

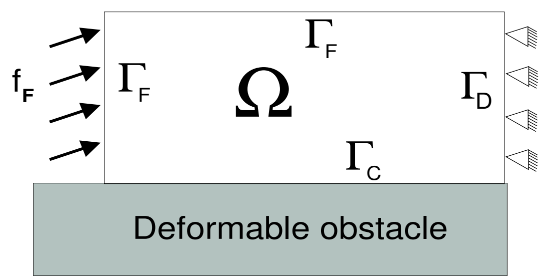

Figure 5: Physical setting of the second example.

The following data were employed in the simulation:

$$T = 5\,s, \quad \boldsymbol{f}_B = 0\,N/m^3, \quad E = 5 \times 10^5\,N/m^2, \quad r = 0.3, \quad \epsilon_a = 0.$$

Again, we used the time discretization parameter $k = 0.01$ and the finite element spaces V^h, Q^h, Y^h and B^h for $h = 0.01$. Our aim here was to show the influence of the adhesion in the deformations. Therefore, in Figure 6 the deformed mesh (multiplied by a factor 10) at final time and the initial configuration are shown on the left-hand side where adhesion takes place (value $\gamma_\nu = 10^3$ was employed) and without taking into account the adhesion on the right-hand side. As we can see, the adhesion prevents the body to separate from the foundation.

In Figures 7 and 8, the associated stress norms and the damage fields are plotted in the deformed configuration for the case with adhesion (left) or without it (right).

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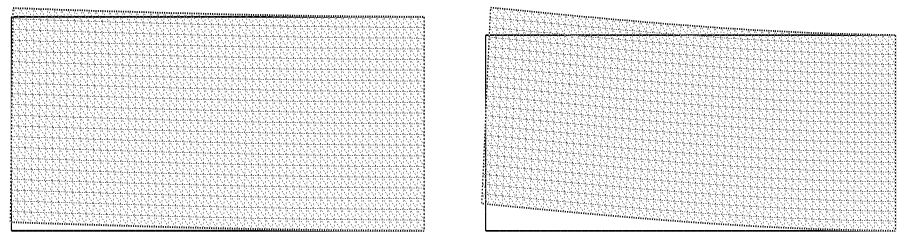

Figure 6: Deformed meshes at final time and initial configuration with adhesion (left) and without it (right).

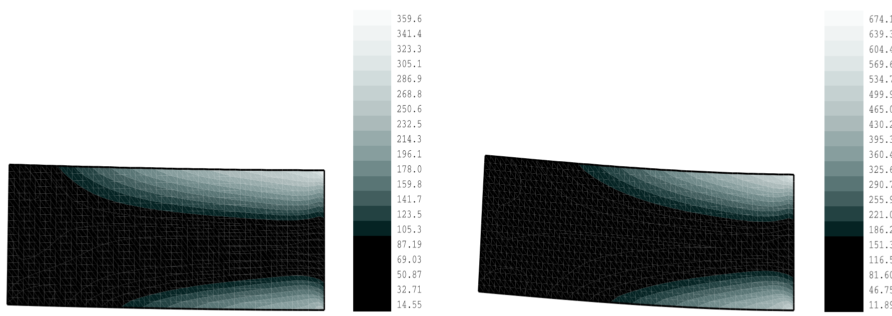

Figure 7: von Mises stress norm at final time with adhesion (left) and without it (right).

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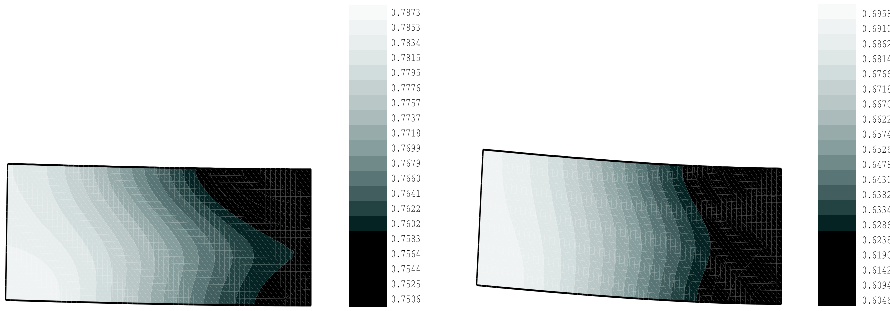

Figure 8: Damage field at final time with adhesion (left) and without it (right).

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