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Internal Approximation of Obstacle Problems[*]

by
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Abstract

In this paper, we consider internal finite element solutions of obstacle problems. An optimal order error estimate is derived for the internal solutions. Numerical results are presented showing the performance of the internal finite element methods and a comparison between the internal solutions and external solutions.

Key Words: Obstacle problem, finite element method, internal approximation, optimal order error estimate.
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1 Introduction

One family of variational inequalities are of the form

$$u \in K, \quad a(u, v-u) \geq \ell(v-u) \quad \forall\, v \in K, \tag{1.1}$$

where K is a non-empty, convex, closed set in a real Hilbert space V, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is bilinear, continuous and V-elliptic, $\ell \in V'$. It is well-known that the problem (1.1) has a unique solution (see, e.g., [5, 6]). When the set K is of the form

$$K = \{v \in V \mid v \geq \psi \text{ a.e. in } \Omega\},$$

the variational inequality (1.1) represents an obstacle problem. The obstacle function ψ is assumed such that $K \neq \emptyset$. Many references can be found in the literature on analysis of obstacle problem, see, e.g. [7].

For definiteness, in this paper, we consider the following concrete setting of the obstacle problem. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, and its boundary $\partial\Omega$ is

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split into a relatively closed part Γ_0 and the remaining part $\Gamma_1 = \partial\Omega\backslash\Gamma_0$. Then the space V is

$$V = H^1_{\Gamma_0}(\Omega) = \left\{v \in H^1(\Omega) \mid v = 0 \text{ a.e. on } \Gamma_0\right\}.$$

We allow the extreme cases where $\Gamma_0 = \emptyset$ or $\partial\Omega$, then the space V is $H^1(\Omega)$ or $H^1_0(\Omega)$. The bilinear form is given by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^2 b_i \frac{\partial u}{\partial x_i} v + b_0 u\, v \right) dx.$$

We assume $a_{ij} \in C(\overline{\Omega})$ and $b_i \in L^2(\Omega)$ are such that $a(\cdot,\cdot)$ is V-elliptic; sufficient conditions for the V-ellipticity can be found in [1, Section 7.4]. The linear form is given by

$$\ell(v) = \int_\Omega f\, v\, dx + \int_{\Gamma_1} g\, v\, ds$$

with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_1)$. Moreover, we assume $\psi \in W^{2,\infty}(\Omega)$, and $\psi(\boldsymbol{x}) < 0$ for $\boldsymbol{x}$ in a neighboehood of $\Gamma_0 \cap \overline{\Omega}$.

Let $K_h \subset V$ be a finite element set, intended as an approximation of K. Then the finite element method for solving (1.1) is

$$u_h \in K_h, \quad a(u_h, v_h - u_h) \ge \ell(v_h - u_h) \quad \forall\, v_h \in K_h. \tag{1.2}$$

This discrete problem has a unique solution.

For error estimation, the following inequality holds (e.g. [2]):

$$\begin{aligned} a(u-u_h, u-u_h) \le{}& a(u-u_h, u-v_h) + a(u, v_h - u) - \ell(v_h - u) \\ &+ a(u, v - u_h) - \ell(v - u_h) \quad \forall\, v_h \in K_h,\ v \in K. \end{aligned} \tag{1.3}$$

In particular, when the approximation is internal in the sense that $K_h \subset K$, it follows from (1.3) that

$$a(u-u_h, u-u_h) \le a(u-u_h, u-v_h) + a(u, v_h - u) - \ell(v_h - u) \quad \forall\, v_h \in K_h. \tag{1.4}$$

Using the continuity and V-ellipticity of the bilinear form $a(\cdot,\cdot)$, we obtain from (1.3) and (1.4) that

$$\begin{aligned} \|u-u_h\|_V \le{}& c \inf_{v_h \in K_h} \left[\|u - v_h\|_V + |a(u, u-v_h) - \ell(u - v_h)|^{1/2} \right] \\ &+ c \inf_{v \in K} |a(u, v - u_h) - \ell(v - u_h)|^{1/2} \quad \forall\, v_h \in K_h,\ v \in K, \end{aligned} \tag{1.5}$$

and

$$\|u-u_h\|_V \le c \inf_{v_h \in K_h} \left[\|u - v_h\|_V + |a(u, u-v_h) - \ell(u - v_h)|^{1/2} \right] \quad \forall\, v_h \in K_h \tag{1.6}$$

in the case of an internal approximation.

For simplicity, we assume $\overline{\Omega} \subset \mathbb{R}^2$ is a polygonal domain; the case of a general domain can be treated by using the approach given in [2]. Let $\{\mathcal{T}_h\}_h$ be a regular family of finite element triangulations of $\overline{\Omega}$. We require $\mathcal{T}_h$ to be compatible with Γ_0, i.e., if a side of an element $T \in \mathcal{T}_h$ intersects Γ_0 at more than one point, then the side is entirely on Γ_0. For an element $T \in \mathcal{T}_h$, we denote h_T the diameter of T, which is the length of the longest side of the triangle T. As usual,

$$h = \max_{T \in \mathcal{T}_h} h_T.$$

We further assume the triangulations are quasiuniform along Γ_0 in the sense that there is a constant $\sigma > 0$ such that if $T \in \mathcal{T}_h$ has a non-empty intersection with Γ_0, then

$$h_T \geq \sigma\, h.$$

For each partition $\mathcal{T}_h$, let X_h be the corresponding linear finite element subspace of $H^1(\Omega)$. Denote $\mathcal{N}_h$ the set of the nodes, and in the case $V = H^1_{\Gamma_0}(\Omega)$, denote $\mathcal{N}_{h,0}$ the subset of the nodes on Γ_0. For any function $v \in C(\overline{\Omega})$, we denote $\Pi_h v \in X_h$ the finite element interpolant of v. In particular, we denote $\psi_h = \Pi_h \psi$. Let $V_h = X_h \cap V$, which consists of linear element functions that vanish at the nodes in $\mathcal{N}_{h,0}$.

A natural finite element set approximating the constraint set K is

$$K_h^{(1)} = \{v_h \in V_h \mid v_h \geq \psi_h \text{ in } \Omega\}.$$

Note that the constraint

$$v_h \geq \psi_h \text{ in } \Omega$$

is equivalent to

$$v_h(\boldsymbol{a}) \geq \psi_h(\boldsymbol{a}) \quad \forall\, \boldsymbol{a} \in \mathcal{N}_h.$$

Let $u_h^{(1)} \in K_h^{(1)}$ be the corresponding finite element solution of (1.2). Under the solution regularity assumptions $u \in H^2(\Omega)$ and $u|_{\Gamma_1} \in \tilde{H}^2(\Gamma_1)$, it can be shown, following and extending the arguments in [2, 4] (for the case $\Gamma_0 = \partial\Omega$), that

$$\|u - u_h^{(1)}\|_V \leq c\, h. \tag{1.7}$$

Here, $\tilde{H}^2(\Gamma_1)$ is defined as follows. Write $\overline{\Gamma_1} = \cup_{i=1}^{i_0} \overline{\Gamma_{1,i}}$ with each $\overline{\Gamma_{1,i}}$ a line segment. Then $v|_{\Gamma_1} \in \tilde{H}^2(\Gamma_1)$ if and only if $v|_{\Gamma_{1,i}} \in H^2(\Gamma_{1,i})$, $1 \leq i \leq i_0$. Denote

$$\|v\|_{\tilde{H}^2(\Gamma_1)} = \left[\sum_{i=1}^{i_0} |v|^2_{H^2(\Gamma_{1,i})}\right]^{1/2}.$$

Notice that when ψ is a convex function, $K_h^{(1)} \subset K$. In general, $K_h^{(1)} \not\subset K$, $K_h^{(1)}$ is said to be an external approximation of K, and $u_h^{(1)}$ is an external finite element solution.

In this paper, we consider an internal approximation of the constraint set. In the case where the variational inequality arises from an optimization problem, this has the advantage that the energy functional is finite over the internal finite element constraint set.

2 Internal finite element method

A basic tool is the following pointwise error bound.

Proposition 2.1. *Let* $v \in W^{2,\infty}(\Omega)$ *and denote by* $\Pi_h v \in X_h$ *the linear finite element interpolant. Then we have the error bound*

$$\|v - \Pi_h v\|_{\infty,\Omega} \le \frac{4}{9}\, h^2 |v|_{2,\infty,\Omega}. \tag{2.1}$$

Proof: For a general triangular element T, denote by $\boldsymbol{a}_i = (a_{i,1}, a_{i,2})^T$, $1 \le i \le 3$, its vertices and by p_i, $1 \le i \le 3$, the corresponding linear basis functions. We have the property

$$p_i(\boldsymbol{x}) \ge 0,\ i = 1,2,3,\ \sum_{i=1}^{3} p_i(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in T. \tag{2.2}$$

The linear interpolant has the following expression on T:

$$\Pi_h v(\boldsymbol{x}) = \sum_{i=1}^{3} v(\boldsymbol{a}_i)\, p_i(\boldsymbol{x}), \quad \boldsymbol{x} \in T.$$

Let us bound the interpolation error $v - \Pi_h v$ over T. By the Taylor theorem, we have

$$v(\boldsymbol{a}_i) = v(\boldsymbol{x}) + \nabla v(\boldsymbol{x}) \cdot (\boldsymbol{a}_i - \boldsymbol{x}) + R_i \tag{2.3}$$

for $1 \le i \le 3$, where the remainder

$$\begin{aligned} R_i = \frac{1}{2} \big[&(a_{i,1} - x_1)^2 v_{x_1x_1}(\boldsymbol{\xi}_i) + 2\,(a_{i,1} - x_1)\,(a_{i,2} - x_2)\, v_{x_1x_2}(\boldsymbol{\xi}_i) \\ &+ (a_{i,2} - x_2)^2 v_{x_2x_2}(\boldsymbol{\xi}_i) \big] \end{aligned}$$

with $\boldsymbol{\xi}_i$ a point between $\boldsymbol{a}_i$ and $\boldsymbol{x}$. Easily,

$$|R_i| \le \frac{1}{2} |v|_{2,\infty,T} \left[(a_{i,1} - x_1)^2 + 2\,|a_{i,1} - x_1|\,|a_{i,2} - x_2| + (a_{i,2} - x_2)^2 \right],$$

and so

$$|R_i| \le |v|_{2,\infty,T} \|\boldsymbol{a}_i - \boldsymbol{x}\|^2. \tag{2.4}$$

Here and below, $\|\boldsymbol{x}\|$ is the ordinary Euclidean norm for a vector $\boldsymbol{x} \in \mathbb{R}^2$. Multiply (2.3) by $p_i(\boldsymbol{x})$ and sum over $i = 1,2,3$ to obtain

$$\Pi_h v(\boldsymbol{x}) = v(\boldsymbol{x}) + \sum_{i=1}^{3} R_i p_i(\boldsymbol{x}). \tag{2.5}$$

Using the bound (2.4), we have, for $\boldsymbol{x} \in T$,

$$\left|\sum_{i=1}^{3} R_i p_i(\boldsymbol{x})\right| \le |v|_{2,\infty,T} \sum_{i=1}^{3} p_i(\boldsymbol{x})\, \|\boldsymbol{a}_i - \boldsymbol{x}\|^2.$$

Write

$$\boldsymbol{x} = \sum_{j=1}^{3} \boldsymbol{a}_j p_j(\boldsymbol{x}), \quad \boldsymbol{a}_i = \sum_{j=1}^{3} \boldsymbol{a}_i p_j(\boldsymbol{x}).$$

Then

$$\begin{aligned} \|\boldsymbol{a}_i - \boldsymbol{x}\|^2 &= \left\|\sum_{j\neq i} (\boldsymbol{a}_i - \boldsymbol{a}_j)\, p_j(\boldsymbol{x})\right\|^2 \\ &\le \left[\sum_{j\neq i} h\, p_j(\boldsymbol{x})\right]^2 \\ &= h^2 \left[\sum_{j\neq i} p_j(\boldsymbol{x})\right]^2 \\ &= h^2 \left[1 - p_i(\boldsymbol{x})\right]^2. \end{aligned}$$

Hence,

$$\left|\sum_{i=1}^{3} R_i p_i(\boldsymbol{x})\right| \le h^2 |v|_{2,\infty,T} \sum_{i=1}^{3} p_i(\boldsymbol{x}) \left[1 - p_i(\boldsymbol{x})\right]^2. \tag{2.6}$$

A tedious but elementary calculation shows that the maximum value of

$$\sum_{i=1}^{3} p_i(\boldsymbol{x}) \left[1 - p_i(\boldsymbol{x})\right]^2$$

for $\boldsymbol{x} \in T$ is $4/9$. Therefore, from (2.5), we see that the maximum interpolation error on T, $\|v - \Pi_h v\|_{\infty,T}$, is bounded by $(4/9)\, h^2 |v|_{2,\infty,T}$. Then the error estimate (2.1) holds. □

From now on, we denote

$$c_0 = \frac{4}{9}.$$

A consequence of Proposition 2.1 is the pointwise inequality

$$v(\boldsymbol{x}) \le \Pi_h v(\boldsymbol{x}) + c_0 h^2 |v|_{2,\infty,\Omega}.$$

We introduce the following finite element set

$$K_h^{(2)} = \left\{ v_h \in V_h \mid v_h \geq \psi_h + c_0 h^2 |\psi|_{2,\infty,\Omega} \text{ in } \Omega \right\}.$$

Then for $v_h \in K_h^{(2)}$, we have

$$v_h(\boldsymbol{x}) \geq \psi_h(\boldsymbol{x}) + c_0 h^2 |\psi|_{2,\infty,\Omega} \geq \psi(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in \Omega.$$

Therefore, $K_h^{(2)} \subset K$. So $K_h^{(2)}$ is an internal approximation of K. The corresponding internal finite element method is

$$u_h^{(2)} \in K_h^{(2)}, \quad a(u_h^{(2)}, v_h - u_h^{(2)}) \geq \ell(v_h - u_h^{(2)}) \quad \forall\, v_h \in K_h^{(2)}. \tag{2.7}$$

We remark that for $V = H^1(\Omega)$, $K_h^{(2)}$ is always nonempty. When $V = H^1_{\Gamma_0}(\Omega)$, then $K_h^{(2)}$ is guaranteed to be non-empty if h is sufficiently small.

Regarding the error of the internal finite element solution $u_h^{(2)}$ defined in (2.7), we have an error estimate similar to (1.7).

Theorem 2.2. *Assume $u \in H^2(\Omega)$ and $u|_{\Gamma_1} \in \tilde{H}^2(\Gamma_1)$. Then we have the optimal order error estimate*

$$\|u - u_h^{(2)}\|_V \leq c\, h. \tag{2.8}$$

Proof: Since the approximation is internal, we have the bound (1.6). We have

$$a(u,v) - \ell(v) = (Au - f, v)_{0,\Omega} + (Bu - g, v)_{0,\Gamma_1} \quad \forall\, v \in V,$$

where

$$\begin{aligned} Au &= -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i} + b_0 u \in L^2(\Omega), \\ Bu &= \sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\nu_j \in L^2(\Gamma_1), \end{aligned}$$

with $\boldsymbol{\nu} = (\nu_1, \nu_2)^T$ the unit outward normal vector, defined a.e. on $\partial\Omega$. Then from (1.6), we obtain

$$\|u - u_h^{(2)}\|_V \leq c \inf_{v_h \in K_h^{(2)}} \left[\|u - v_h\|_V + \|u - v_h\|_{0,\Omega}^{1/2} + \|u - v_h\|_{0,\Gamma_1}^{1/2} \right] \quad \forall\, v_h \in K_h^{(2)}. \tag{2.9}$$

In the case $V = H^1(\Omega)$, $\Gamma_1 = \partial\Omega$ and we take $v_h \in V_h$ by specifying

$$v_h(\boldsymbol{a}) = u(\boldsymbol{a}) + c_0 h^2 |\psi|_{2,\infty,\Omega}, \quad \boldsymbol{a} \in \mathcal{N}_h.$$

Then, by the finite element interpolation error estimates ([3]),

$$\begin{aligned}
|u - v_h|_{1,\Omega} &\le c\,h\,|u|_{2,\Omega},\\
\|u - v_h\|_{0,\Omega} &\le c\,h^2(|u|_{2,\Omega} + |\psi|_{2,\infty,\Omega})\\
\|u - v_h\|_{0,\Gamma_1} &\le c\,h^2(|u|_{\tilde{H}^2(\Gamma_1)} + |\psi|_{2,\infty,\Omega}).
\end{aligned}$$

Then from (2.9), we get the error estimate (2.8).

In the general case $V = H^1_{\Gamma_0}(\Omega)$, we take $v_h \in X_h$ by specifying

$$v_h(\boldsymbol{a}) = \begin{cases} u(\boldsymbol{a}) + c_0 h^2 |\psi|_{2,\infty,\Omega}, & \boldsymbol{a} \in \mathcal{N}_h \backslash \mathcal{N}_{h,0},\\ 0, \quad \boldsymbol{a} \in \mathcal{N}_{h,0}. \end{cases}$$

Then when h is small enough, $v_h \in K_h^{(2)}$. Consider any $T \in \mathcal{T}_h$. If T is an interior element, i.e., none of its vertices lie on $\partial\Omega$, or if T does not have any point on Γ_0, then

$$\begin{aligned}
|u - v_h|_{1,T} &\le c\,h\,|u|_{2,T},\\
\|u - v_h\|_{0,T} &\le c\,h^2(|u|_{2,T} + h_T|\psi|_{2,\infty,\Omega}).
\end{aligned}$$

If T intersects Γ_0 at one point or one side, then

$$\begin{aligned}
|u - v_h|_{1,T} &\le c\,h\,(|u|_{2,T} + h_T|\psi|_{2,\infty,\Omega}),\\
\|u - v_h\|_{0,T} &\le c\,h^2(|u|_{2,T} + h_T|\psi|_{2,\infty,\Omega}).
\end{aligned}$$

Finally, if T has a side on $\overline{\Gamma_1}$, then

$$\|u - v_h\|_{0,\partial T\cap\Gamma_1} \le c\,h^2(|u|_{H^2(\partial T\cap\Gamma_1)} + h_T^{1/2}|\psi|_{2,\infty,\Omega}).$$

So once again from (2.9), we get the error estimate (2.8). □

3 Numerical examples

We present some numerical results to show the performance of the internal approximation of obstacle problems. We introduce a model problem with a parameter, for which we have a closed-form solution formula; moreover, we can compute all the integrals in forming the discrete systems and in evaluating numerical solution errors so that the only errors are due to the finite element discretization.

Consider a one-dimensional obstacle problem over $\Omega = (0,1)$. Let $\kappa > 4$ be a parameter, and let the obstacle be given as

$$\psi(x) = \kappa\, x(1-x) - 1, \quad 0 \le x \le 1.$$

The space $V = H^1_0(0,1)$, $\|v\|_V = \|v'\|_{L^2(0,1)}$, and

$$K = \{v \in V \mid v \ge \psi \text{ a.e. in } \Omega\}.$$

It can be verified that the unique solution of the variational inequality

$$u \in K, \quad \int_0^1 u'\,(v-u)'\,dx \ge 0 \quad \forall\, v \in K$$

is

$$u(x) = \begin{cases} (\kappa - 2\kappa^{1/2})\,x, & 0 \le x \le \kappa^{-1/2}, \\ \kappa\,x(1-x) - 1, & \kappa^{-1/2} \le x \le 1 - \kappa^{-1/2}, \\ (\kappa - 2\kappa^{1/2})\,(1-x), & 1 - \kappa^{-1/2} \le x \le 1. \end{cases}$$

We note that as $\kappa \to 4+$, the influence of the obstacle on the solution becomes weaker, and as κ increases, the influence of the obstacle on the solution becomes stronger. In the following, we report numerical results corresponding to $\kappa = 4.1$, 4.5, and 10. We compute the external and internal numerical solutions using linear finite elements on uniform meshes with $h = 2^{-l}$, $l = 2, \ldots, 10$.

In the context of one-dimension, the interpolation error bound (2.1) is replaced by

$$\|v - \Pi_h v\|_{\infty,\Omega} \le \frac{1}{8}\,h^2 |v|_{2,\infty,\Omega}.$$

Correspondingly, the internal approximation set is

$$K_h^{(2)} = \left\{ v_h \in V_h \mid v_h \ge \psi_h + (1/8)\,h^2 |\psi|_{2,\infty,\Omega} \text{ in } \Omega \right\}.$$

Figure 1 shows a comparison of the errors of the internal and external finite element solutions of the model problem with $\kappa = 4.1$. For the external solution, there is a range of l ($l \le 6$) where refining the mesh does not lead to better solution accuracy. When l passes this range, the external and internal finite element solutions converge at essentially the same speed. Indeed, for the error line segments between $h = 2^{-9}$ and 2^{-10}, the slope is 0.9921 for the internal approximation, and is 0.9920 for the external approximation.

Figure 2 reports the internal and external solution errors in solving the model problem with $\kappa = 4.5$. We have similar convergence behavior as for the problem with $\kappa = 4.1$, except that the range of l, for which refining the mesh does not lead to better solution accuracy, is smaller. For the error line segments between $h = 2^{-9}$ and 2^{-10}, the slopes for both the internal approximation and the external approximation are around 1.0026.

Then we increase the value κ to 10. The results are given in Figure 3. The convergence behaviors of the two finite element solutions are very similar. For the error line segments between $h = 2^{-9}$ and 2^{-10}, the slopes for both the internal approximation and the external approximation are around 0.9999.

We have also computed the L^2 norm errors for both the internal and external finite element solutions. The results are given in Figures 4, 5, and 6.

4 Concluding statement

In this paper, we study an internal finite element method for solving obstacle problems. One advantage of the method is that the energy functional associated

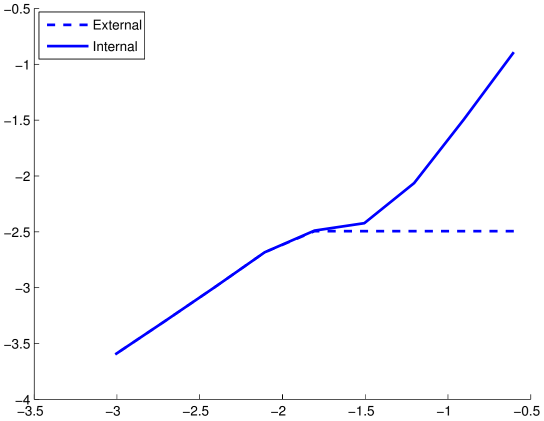

Figure 1: log-log plot of $\| \cdot \|_V$ errors for the internal and external solutions, $\kappa = 4.1$

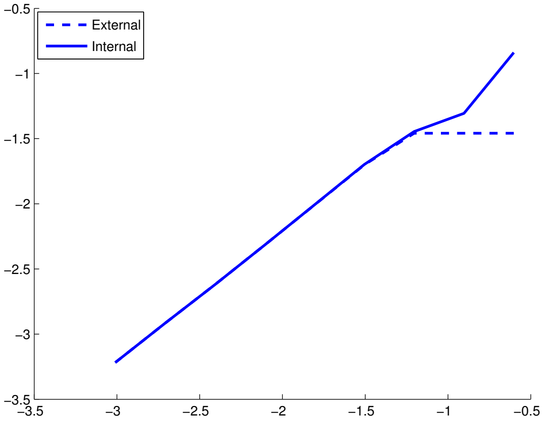

Figure 2: log-log plot of $\| \cdot \|_V$ errors for the internal and external solutions, $\kappa = 4.5$

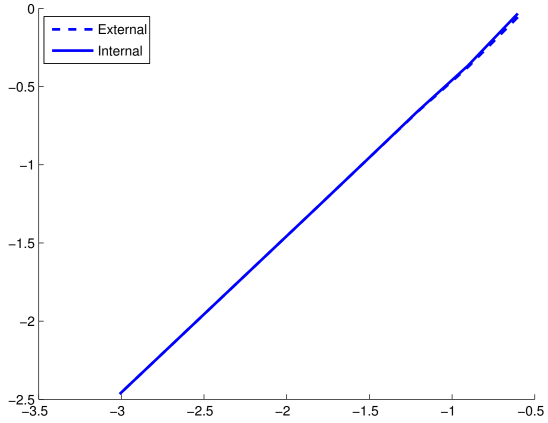

Figure 3: log-log plot of $\|\cdot\|_V$ errors for the internal and external solutions, $\kappa = 10$

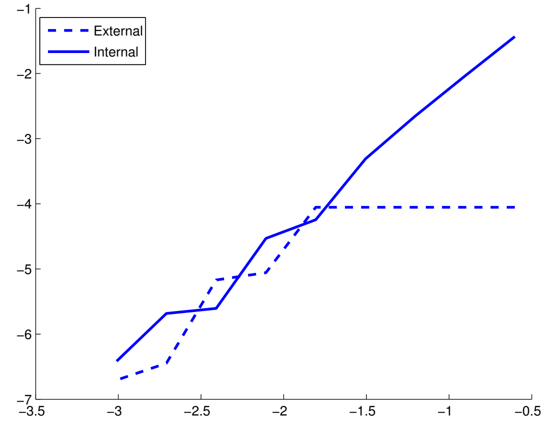

Figure 4: log-log plot of $\|\cdot\|_{L^2}$ errors for the internal and external solutions, $\kappa = 4.1$

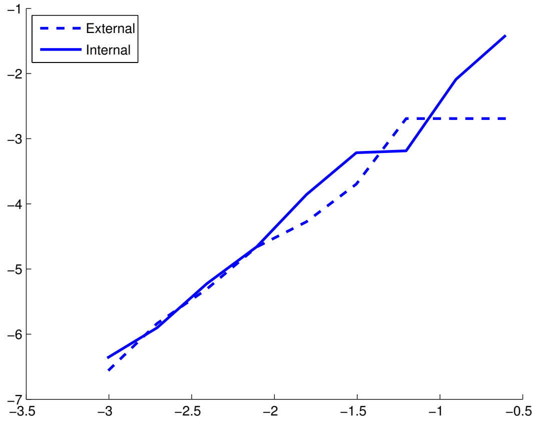

Figure 5: log-log plot of $\|\cdot\|_{L^2}$ errors for the internal and external solutions, $\kappa = 4.5$

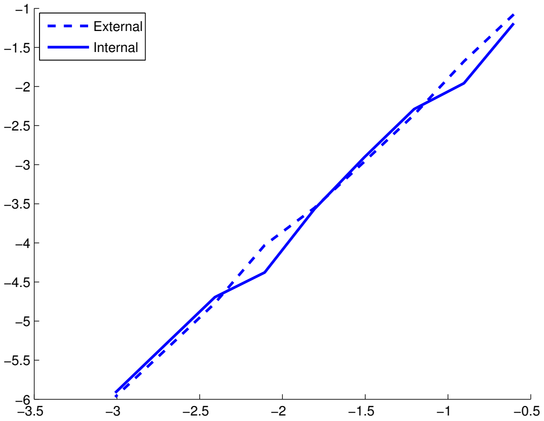

Figure 6: log-log plot of $\|\cdot\|_{L^2}$ errors for the internal and external solutions, $\kappa = 10$

with an obstacle problem is finite over the internal finite element set. An optimal order error estimate is derived for the internal finite element solution. Numerical results show good performance of the internal method. When the influence of the obstacle is strong (for large κ in the model problem solved in Section 3), the internal method and the external method provide solutions with nearly the same accuracy.

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