Bull. Math. Soc. Sc. Math. Roumanie
Tome 48(96) No. 2, 2005, 139–163

On the Navier-Stokes system in a thin film flow with Tresca free boundary condition and its asymptotic behavior

by
Mahdi Boukrouche, and Rachid El Mir

Abstract

We study first the existence and uniqueness of solutions to a lubrication problem governed by the stationary Navier-Stokes system and subject to Tresca free boundary conditions. Then we establish the asymptotic behavior of its solutions.

Key Words: Free boundary; Tresca law; Navier-Stokes problem; Lubrication; Inertial effects; Asymptotic behavior of weak solutions.
2000 Mathematics Subject Classification: Primary: 35R35, Secondary: 76D05, 76D03.

1 Introduction

In [1] the authors studied the influence of the Reynolds number with respect to the thin film ratio parameter ε in the justification of the well known equation of the lubrication theory, which was the basic Reynolds equation. They considered the widely assumed no-slip boundary conditions.

This widely assumed no-slip boundary conditions when the fluid has the same velocity as surrounding solid boundary is not respected any more since the shear rate becomes too high [9], [5] , [2].

We are interested here to generalize the results obtained in [1] to the case taking into account this phenomenon described by the Tresca free boundary conditions.

For ε small but fixed, we study first the influence of the Reynolds number with respect to the thin film thickness to obtain the existence and uniqueness of velocity and pressure solutions of our problem. Then we study the rigorous justification of the limit problem when $\varepsilon \to 0$.

We show in this study, as it is usually assumed, that the inertial effects in the asymptotic behavior disappear. In the subsequent work [7] we also obtain as in [1] the influence of the inertial effects at the second order in the asymptotic expansions of the pressure and velocity.

The plan of this paper is as follows, we present in section 2 the basic equations and assumptions, in section 3 we give the weak formulation of the problem, in section 4 we give the main results on existence result, the needed estimates on velocity and pressure then the uniqueness result, in section 5 we study the limit problem by the asymptotic analysis.

2 Basic equations and assumptions

Let ω be a rectangular domain in $\mathbb{R}^{n-1}$

$$\omega = \prod_{i=1}^{n-1}]0, a_i[\times \{0\}$$

$n = 2$ or 3 and $a_i > 0$ for $i = 1, .., n-1$. We denote $x' = (x_1, .., x_{n-1})$ and $x = (x', x_n)$. We suppose that the domain ω is the bottom of the gap whose the top Γ_1^ε is defined by $x_n = H(x') = \varepsilon h(x')$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h is a function such that $h \in \mathcal{C}^0(\mathbb{R})$ and $0 < h_{min} \leq h(x') \leq h_{max}$ $\forall (x', 0) \in \omega$. We denote by Ω^ε the domain of the flow,

$$\Omega^\varepsilon = \{(x', x_n) \in \mathbb{R}^n \ : (x', 0) \in \omega \ , \quad 0 < x_n < \varepsilon h(x)\}.$$

Let Γ^ε be the boundary of Ω^ε. We have $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_L^\varepsilon \cup \bar{\Gamma}_1^\varepsilon$ where Γ_L^ε is the lateral boundary. For a given body forces $f^\varepsilon = (f_1^\varepsilon, .., f_n^\varepsilon)$ the motion of the fluid is described by:

- The incompressibility equation

$$div(u^\varepsilon) = u_{i,i} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \ \ in \ \ \Omega^\varepsilon. \tag{2.1}$$

- The law of conservation of momentum, with the Reynolds number ε^γ

$$\varepsilon^\gamma u_j u_{i,j} = f_i + \sigma_{ij,j} \ \ in \ \ \Omega^\varepsilon, \ \ \gamma \in \mathbb{R} \tag{2.2}$$

where the stress tensor σ^ε is decomposed as follows

$$\sigma_{ij}^\varepsilon = -p^\varepsilon \delta_{ij} + 2\mu d_{ij}(u^\varepsilon), \tag{2.3}$$

where $u^\varepsilon = (u_1^\varepsilon, .., u_n^\varepsilon)$ is the velocity field, p^ε is the pressure, and δ_{ij} is the Krönecker symbol. The components of the symmetric deformation velocity tensor is given by

$$d_{ij}(u^\varepsilon) = \frac{1}{2}(u_{i,j}^\varepsilon \ + \ u_{j,i}^\varepsilon). \tag{2.4}$$

To describe the boundary conditions, we introduce first a vector function $g =$

$(g_1, .., g_n)$ such that

$$\int_{\Gamma^\varepsilon} g.\mathbf{n}\ ds = 0, \tag{2.5}$$

and

$$\int_{\Gamma_L^\varepsilon} (g)^2\ \ g.\mathbf{n}\, ds = 0. \tag{2.6}$$

The actual velocities on the boundary, except for the components of the tangential velocity on ω, are given in terms of g.

- On Γ_1^ε, no slip condition is given. The upper surface being assumed to be fixed so

$$u^\varepsilon = g = 0. \tag{2.7}$$

- On Γ_L^ε, the velocity is known and parallel to the ω-plane

$$u^\varepsilon = \varepsilon^\beta g \quad with\ \ g_n = 0\ \ and\ \beta \in \mathbb{R} \tag{2.8}$$

- On ω, there is no flux condition across ω so that

$$u_n^\varepsilon = g_n = 0. \tag{2.9}$$

The tangential velocity on ω is unknown and satisfies the Tresca friction law [6] at ω with k^ε upper limit for stress

$$\left.\begin{array}{l} |\ \sigma_T^\varepsilon\ | < \ k^\varepsilon \ \Rightarrow\ u_T^\varepsilon\ = \varepsilon^\beta s \\ |\ \sigma_T^\varepsilon\ | = \ k^\varepsilon \ \Rightarrow\ \exists\ \lambda \geq 0 \ \text{ such that }\ u_T^\varepsilon = \varepsilon^\beta s - \lambda \sigma_T^\varepsilon \end{array}\right\} \ \text{on}\ \ \omega, \tag{2.10}$$

where $s = g$ on ω, $|\ .\ |$ is the euclidean norm in $\mathbb{R}^{n-1}$, $\mathbf{n} = (\mathbf{n}_1, .., \mathbf{n}_n)$ is the unit outward normal to Γ^ε. Using Einstein's summations we have

$$u_{\mathbf{n}}^\varepsilon = u^\varepsilon.\mathbf{n} = u_i^\varepsilon \mathbf{n}_i \quad ; \quad u_{T_i}^\varepsilon = u_i^\varepsilon - u_{\mathbf{n}}^\varepsilon \mathbf{n}_i, \tag{2.11}$$

$$\sigma_{\mathbf{n}}^\varepsilon = (\sigma^\varepsilon.\mathbf{n}).\mathbf{n} = \sigma_{ij}^\varepsilon \mathbf{n}_i \mathbf{n}_j \quad ; \quad \sigma_{T_i}^\varepsilon = \sigma_{ij}^\varepsilon \mathbf{n}_j - \sigma_{\mathbf{n}}^\varepsilon \mathbf{n}_i, \tag{2.12}$$

which are respectively, the normal and the tangential velocity on ω, and the components of the normal and the tangential stress tensor on ω.

3 Weak formulation

We denote $H^1(\Omega^\varepsilon)$ the Sobolev space and $H_0^1(\Omega^\varepsilon)$ the closure of $\mathcal{D}(\Omega^\varepsilon)$ in $H^1(\Omega^\varepsilon)$. The dual space of $H_0^1(\Omega^\varepsilon)$ is denoted by $H^{-1}(\Omega^\varepsilon)$. We assume that the function g is in $(H^{\frac{1}{2}}(\Gamma^\varepsilon))^n$, the space of traces on Γ^ε of functions from $(H^1(\Omega^\varepsilon))^n$. Due to (2.5) it is well known [8] (Lemma 2.2) that there exists a function G such that

$$G \in (H^1(\Omega^\varepsilon))^n \ \text{ with } div(G) = 0 \ \text{ in } \ \ \Omega^\varepsilon,\ and \quad G = g \ \text{ on } \Gamma^\varepsilon. \tag{3.1}$$

We denote

$$G^\varepsilon = \varepsilon^\beta G. \tag{3.2}$$

We consider the functional framework on Ω^ε

$$H^1_{\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon}(\Omega^\varepsilon) = \{\psi \in H^1(\Omega^\varepsilon) : \ \psi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon\},$$

$$V^\varepsilon = \{v \ \in (H^1(\Omega^\varepsilon))^n \ : \ v = G^\varepsilon \ \text{ on } \Gamma_L^\varepsilon, \ v = 0 \ \text{ on } \Gamma_1^\varepsilon, \ v.n = 0 \ \text{ on } \omega\},$$

$$V^\varepsilon_{div} = \{ \ v \in V^\varepsilon : \ div(v) = 0\},$$

$$L^2_0(\Omega^\varepsilon) = \{ \ q \in L^2(\Omega^\varepsilon) \ : \int_{\Omega^\varepsilon} q \, dx = 0\},$$

Using the following notations:

$$a : \ V^\varepsilon \times V^\varepsilon \to \mathbb{R}$$

$$(u, v) \to a(u, v) = \int_{\Omega^\varepsilon} d_{ij}(u) d_{ij}(v) dx' dx_n$$

$$b : \ V^\varepsilon \times V^\varepsilon \times V^\varepsilon \to \mathbb{R}$$

$$(u, v, w) \to b(u, v, w) = \int_{\Omega^\varepsilon} u_i v_{j,i} w_j dx' dx_n$$

$$j^\varepsilon : \ V^\varepsilon \to \mathbb{R}^+$$

$$v \to j^\varepsilon(v) = \int_\omega k^\varepsilon \mid v - \varepsilon^\beta s \mid d\sigma.$$

A formal application of (2.1)-(2.10) leads to the following variational problem as in [6]:

Problem 1. *Find* $u^\varepsilon \in V^\varepsilon_{div}, \ \ p^\varepsilon \in L^2_0(\Omega^\varepsilon)$, such that

$$2\mu a(u^\varepsilon, \phi - u^\varepsilon) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, \phi - u^\varepsilon) - (p^\varepsilon, div(\phi)) + j^\varepsilon(\phi) - j^\varepsilon(u^\varepsilon) \geq (f^\varepsilon, \phi - u^\varepsilon) \ \ \forall \phi \in V^\varepsilon, \tag{3.3}$$

we remark that a, b and j^ε satisfy the following properties: i) a is a bilinear form

- continuous: $\mid a(u, v) \mid \leq \parallel u \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel v \parallel_{(H^1(\Omega^\varepsilon))^n}$
- coercive, by Korn's and Poincaré's inequalities $\exists \alpha > 0$ such that:

$$a(v, v) \geq \alpha \parallel v \parallel^2_{(H^1(\Omega^\varepsilon))^n} \quad \forall v \in (H^1(\Omega^\varepsilon))^n \ \ , \ \ v = 0 \ \text{ on } \ \Gamma_1^\varepsilon$$

ii) b is a trilinear form

- continuous: $\exists K_1 > 0$ such that :

$$\mid b(u, v, w) \mid \leq K_1 \parallel u \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel v \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel w \parallel_{(H^1(\Omega^\varepsilon))^n},$$

- from (2.6) b is antisymmetric so

$$b(u,v,w)+b(u,w,v)=0,\ \ \forall (u,v,w)\in (V^{\varepsilon})^3.$$

iii) j^{ε} is convex continuous but non differentiable in $(H^1(\Omega^{\varepsilon}))^n$.
The constants K_1 and α do not depend on ε. We introduce the change of scale $z=\frac{x_n}{\varepsilon}$. We get a fixed domain Ω:

$$\Omega=\{(x',x_n)\in\mathbb{R}^n\ :\ (x',0)\in\omega,\ 0<x_n<h(x')\},$$

and we denote by $\Gamma=\bar{\omega}\cup\bar{\Gamma}_1\cup\bar{\Gamma}_L$ its boundary, then we define the following functions in Ω:

$$u_i^{\varepsilon}(x',x_n)=\varepsilon^{\beta}\hat{u_i^{\varepsilon}}(x',z)\ \text{ for }\ i=1,..,n-1;\ \ u_n^{\varepsilon}(x',x_n)=\varepsilon^{\beta+1}\hat{u_n^{\varepsilon}}(x',z);$$

$$p^{\varepsilon}(x',x_n)=\varepsilon^{-2+\beta}\hat{p^{\varepsilon}}(x',z).$$

For the data we suppose that depend on ε in the following way:

$$f^{\varepsilon}(x',x_n)=\varepsilon^{-2+\beta}\hat{f}(x',z);\ \ k^{\varepsilon}=\varepsilon^{-1+\beta}\hat{k}$$

$$G_i^{\varepsilon}(x',x_n)=\varepsilon^{\beta}\hat{G}_i(x',z_n)\ \text{ pour }\ i=1,..,n-1\ ;\ G_n^{\varepsilon}(x',x_n)=\varepsilon^{\beta+1}\hat{G}_i(x',z_n).$$

We define the functional framework on Ω by:

$$H^1_{\Gamma_1\cup\Gamma_L}(\Omega)=\{\psi\in H^1(\Omega)\ :\ \psi=0 \text{ on } \Gamma_1\cup\Gamma_L\},$$

$$V=\{v\ \in (H^1(\Omega)^n\ :\ v=\hat{G}\ \text{ on }\ \Gamma_L,\ v=0\ \text{ on }\ \Gamma_1,\ v.n=0\ \text{ on }\ \omega\},$$

$$V_{div}=\{\ v\in V\ :\ div(v)=0\},$$

$$L_0^2(\Omega)=\{\ q\in L^2(\Omega)\ :\ \int_{\Omega} q\,dx=0\}.$$

4 Existence and uniqueness results

4.1 Existence

Theorem 1 There exists μ_0 such that for $\mu>\mu_0$, the problem (3.3) has at least one solution $(u^{\varepsilon},p^{\varepsilon})$, under the condition $\beta=\frac{1}{2}-\gamma$.
Proof. Here we will apply the Schauder fixed point theorem. At the beginning we look for a constant $C>0$ such that the following application will be well defined

$$\Lambda\ \ :B_C\rightarrow\ B_C$$

$$\xi\rightarrow u^{\varepsilon}$$

where B_C is the $(H^1(\Omega^\varepsilon))^n$ closed ball of radius C, u^ε is the unique solution of the following variational inequality:

$$2\mu a(u^\varepsilon,\phi-u^\varepsilon)+\varepsilon^\gamma b(\xi,u^\varepsilon,\phi-u^\varepsilon)+j^\varepsilon(\phi)-j^\varepsilon(u^\varepsilon) \geq (f^\varepsilon,\phi-u^\varepsilon) \;\; \forall\phi\in V^\varepsilon_{div}. \tag{4.1}$$

In particular for $\phi=G^\varepsilon$, as $G^\varepsilon=\varepsilon^\beta s$ on ω and from (2.6) $b(\xi,u^\varepsilon,u^\varepsilon)=0$, we get

$$2\mu a(u^\varepsilon,u^\varepsilon)\leq 2\mu a(u^\varepsilon,G^\varepsilon)+\varepsilon^\gamma b(\xi,u^\varepsilon,G^\varepsilon)+(f^\varepsilon,G^\varepsilon-u^\varepsilon). \tag{4.2}$$

In the other hand using Poincaré's and Young's inequalities, we obtain

$$\begin{aligned} 2\mu a(u^\varepsilon,G^\varepsilon) &\leq \frac{\mu}{2}a(u^\varepsilon\, u^\varepsilon)+2\mu a(G^\varepsilon,G^\varepsilon)\\ &\leq \frac{\mu}{2}a(u^\varepsilon\, u^\varepsilon)+2\mu \parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\parallel f^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\parallel u^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\leq\\ &\leq h_{max}\varepsilon \parallel \nabla u^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\parallel f^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\\ &\leq \frac{\alpha\mu}{4}\parallel u^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n}+\frac{\varepsilon^2 h^2_{max}}{\alpha\mu}\parallel f^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\parallel f^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\parallel G^\varepsilon \parallel_{(L^2(\Omega^\varepsilon))^n}\leq\\ &\leq \frac{\alpha\mu}{4}\parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n}+\frac{\varepsilon^2 h^2_{max}}{\alpha\mu}\parallel f^\varepsilon \parallel^2_{L^2(\Omega^\varepsilon)} \end{aligned} \tag{4.5}$$

Also using the continuity of b, Young's inequality then Poincaré's inequality, and taking into account that $0<\varepsilon<1$, we obtain

$$\begin{aligned} \mid \varepsilon^\gamma b(\xi,u^\varepsilon,G^\varepsilon)\mid &\leq K_1\varepsilon^\gamma \parallel \xi \parallel_{(H^1(\Omega^\varepsilon))^n}\parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\parallel G^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\\ &\leq K_1\varepsilon^\gamma C \parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\parallel G^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\\ &\leq 2K_1\varepsilon^\gamma C\Big[\frac{1}{2}\sqrt{\frac{\alpha\mu}{K_1C}}\parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\Big]\Big[\sqrt{\frac{K_1C}{\alpha\mu}}\varepsilon^\gamma \parallel G^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n}\Big]\\ &\leq K_1C\Big(\frac{\alpha\mu}{4K_1C}\parallel u^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n}+\frac{K_1C\varepsilon^{2\gamma}}{\alpha\mu}\parallel G^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n}\Big)\\ &\leq K_1C\Big\{\frac{\alpha\mu}{4K_1C}\parallel u^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n}+\\ &\qquad +\frac{K_1C\varepsilon^{2\gamma}(1+h^2_{max})\varepsilon^2}{\alpha\mu}\parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n}\Big\}\\ &\leq K_1C\Big\{\frac{\alpha\mu}{4K_1C}\parallel u^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n}+\\ &\qquad +\frac{K_1C\varepsilon^{2\gamma}(1+h^2_{max})}{\alpha\mu}\parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n}\Big\} \end{aligned}$$

Using (4.2), we deduce

$$\begin{aligned}\alpha\mu \parallel u^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n} &\leq \left(\frac{(K_1C)^2\varepsilon^{2\gamma}(1+h^2_{max})}{\alpha\mu} + \frac{\alpha\mu}{4}\right) \parallel \nabla G^\varepsilon \parallel^2_{L^2(\Omega^\varepsilon)} + \\ &+ \frac{2\varepsilon^2 h^2_{max}}{\alpha\mu} \parallel f^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n} . \end{aligned} \tag{4.6}$$

To have $\parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n} \leq C$, it is enough to set

$$\begin{aligned}&\left(\frac{(K_1C)^2\varepsilon^{2\gamma}(1+h^2_{max})}{(\alpha\mu)^2} + \frac{1}{4}\right) \parallel \nabla G^\varepsilon \parallel^2_{L^2(\Omega^\varepsilon)} + \\ &\qquad + \frac{2\varepsilon^2 h^2_{max}}{(\alpha\mu)^2} \parallel f^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n} \leq C^2\end{aligned} \tag{4.7}$$

i.e.

$$\begin{aligned}&\frac{1}{4} \parallel \nabla G^\varepsilon \parallel^2_{L^2(\Omega^\varepsilon)} + \frac{2\varepsilon^2 h^2_{max}}{(\alpha\mu)^2} \parallel f^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n} \leq \\ &\leq C^2\left\{1 - \frac{(K_1)^2\varepsilon^{2\gamma}(1+h^2_{max})}{(\alpha\mu)^2} \parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n} \right\}.\end{aligned} \tag{4.8}$$

But from (3.2) we have:

$$\parallel \nabla G^\varepsilon \parallel^2_{(L^2(\Omega^\varepsilon))^n} \leq \varepsilon^{2\beta-1} \parallel \nabla \hat{G} \parallel^2_{(L^2(\Omega))^n},$$

to give a meaning to (4.8), and then to obtain $\parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n} \leq C$, it's enough to have

$$\beta = \frac{1}{2} - \gamma \quad \text{and} \quad \mu > \mu_0 = \frac{K_1\sqrt{1+h^2_{max}}}{\alpha} \parallel \nabla \hat{G} \parallel^2_{(L^2(\Omega))^n} .$$

• Let us show that Λ is Lipschitzian on B_C. Indeed, let u^ε_1 and u^ε_2 two solutions of (4.1) such that $u^\varepsilon_1 = \Lambda(\xi_1)$ and $u^\varepsilon_2 = \Lambda(\xi_2)$. Then

$$2\mu a(u^\varepsilon_1, u^\varepsilon_2 - u^\varepsilon_1) + \varepsilon^\gamma b(\xi_1, u^\varepsilon_1, u^\varepsilon_2 - u^\varepsilon_1) + j^\varepsilon(u^\varepsilon_2) - j^\varepsilon(u^\varepsilon_1) - (f^\varepsilon, u^\varepsilon_2 - u^\varepsilon_1) \geq 0,$$

$$2\mu a(u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2) + \varepsilon^\gamma b(\xi_2, u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2) + j^\varepsilon(u^\varepsilon_1) - j^\varepsilon(u^\varepsilon_2) - (f^\varepsilon, u^\varepsilon_1 - u^\varepsilon_2) \geq 0.$$

By adding both inequalities, we get

$$-2\mu a(u^\varepsilon_1 - u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2) + \varepsilon^\gamma b(\xi_1, u^\varepsilon_1, u^\varepsilon_2 - u^\varepsilon_1) + \varepsilon^\gamma b(\xi_2, u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2) \geq 0,$$

using that a is coercive, we obtain

$$\begin{aligned}2\mu\alpha \parallel u^\varepsilon_1 - u^\varepsilon_2 \parallel^2_{(H^1(\Omega^\varepsilon))^n} &\leq 2\mu a(u^\varepsilon_1 - u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2) \\ &\leq \varepsilon^\gamma b(\xi_1 - \xi_2, u^\varepsilon_1, u^\varepsilon_1 - u^\varepsilon_2) + \varepsilon^\gamma b(\xi_2, u^\varepsilon_1, u^\varepsilon_1 - u^\varepsilon_2) - \\ &- \varepsilon^\gamma b(\xi_2, u^\varepsilon_2, u^\varepsilon_1 - u^\varepsilon_2),\end{aligned}$$

as

$$\varepsilon^\gamma b(\xi_2, u_1^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) - \varepsilon^\gamma b(\xi_2, u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) = \varepsilon^\gamma b(\xi_2, u_1^\varepsilon - u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) = 0,$$

by Sobolev imbeddings, we get

$$\begin{aligned}\mu\alpha \parallel u_1^\varepsilon - u_2^\varepsilon \parallel^2_{(H^1(\Omega^\varepsilon))^n} &\leq \\ \leq K_1\varepsilon^\gamma \parallel u^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel u_1^\varepsilon - u_2^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel \xi_1 - \xi_2 \parallel_{(H^1(\Omega^\varepsilon))^n} \\ \leq K_1 C\varepsilon^\gamma \parallel u_1^\varepsilon - u_2^\varepsilon \parallel_{(H^1(\Omega^\varepsilon))^n} \parallel \xi_1 - \xi_2 \parallel_{(H^1(\Omega^\varepsilon))^n},\end{aligned} \tag{4.9}$$

whence

$$\parallel \Lambda(\xi_1) - \Lambda(\xi_2) \parallel_{(H^1(\Omega^\varepsilon))^n} \leq \frac{K_1 C\varepsilon^\gamma}{2\mu\alpha} \parallel \xi_1 - \xi_2 \parallel_{(H^1(\Omega^\varepsilon))^n} . \tag{4.10}$$

Thus by Schauder fixed point theorem, there exists at least one solution u^ε for the following variational inequality:

$$\begin{aligned}2\mu a(u^\varepsilon, \phi - u^\varepsilon) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, \phi - u^\varepsilon) + j^\varepsilon(\phi) - j^\varepsilon(u^\varepsilon) \geq \\ \geq (f^\varepsilon, \phi - u^\varepsilon) \;\; \forall \phi \in V^\varepsilon_{div}.\end{aligned} \tag{4.11}$$

We can prove the existence of the pressure as in theorem 3.2 [4].

4.2 Some needed estimates

Before showing the uniqueness of the weak solutions to problem 1, we need to establish some estimates on the gradient of the velocities field. These estimates with others on the pressure, will be useful to study the limit problem in section 5. First, we introduce two technical lemmas which will be used to obtain the needed estimates.

Lemma 1 [1]: *For all $\phi \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ we have the following estimates:*
For $n = 3$, we have

$$\parallel \phi \parallel_{L^4(\Omega)} \leq h_{max}^{\frac{1}{4}} \parallel \frac{\partial \phi}{\partial x_1} \parallel^{\frac{1}{4}}_{L^2(\Omega)} \parallel \frac{\partial \phi}{\partial x_2} \parallel^{\frac{1}{4}}_{L^2(\Omega)} \parallel \frac{\partial \phi}{\partial z} \parallel^{\frac{1}{2}}_{L^2(\Omega)} \tag{4.12}$$

$$\parallel \phi \parallel_{L^6(\Omega)} \leq 4 \parallel \frac{\partial \phi}{\partial x_1} \parallel^{\frac{1}{3}}_{L^2(\Omega)} \parallel \frac{\partial \phi}{\partial x_2} \parallel^{\frac{1}{3}}_{L^2(\Omega)} \parallel \frac{\partial \phi}{\partial z} \parallel^{\frac{1}{3}}_{L^2(\Omega)} \tag{4.13}$$

For $n = 2$, we have

$$\parallel \phi \parallel_{L^4(\Omega)} \leq \sqrt{2h_{max}} \parallel \frac{\partial \phi}{\partial x_1} \parallel^{\frac{1}{4}}_{L^2(\Omega)} \parallel \frac{\partial \phi}{\partial z} \parallel^{\frac{3}{4}}_{L^2(\Omega)} . \tag{4.14}$$

Lemma 2 [1]: $\forall \phi \in H^1(\Omega)$ *such that:* $\phi = \hat{G}$ *on* Γ_L *and* $\phi = 0$ *on* Γ_1.

For $n=3$, we have

$$\| \phi \|_{L^4(\Omega)} \leq C(\Omega, \hat{G}) \Big[\| \frac{\partial \phi}{\partial z} \|_{L^2(\Omega)}^{\frac{1}{2}} \big(max\{ \| \frac{\partial \phi}{\partial x_1} \|_{L^2(\Omega)}, \| \frac{\partial \phi}{\partial x_2} \|_{L^2(\Omega)} \} \big)^{\frac{1}{2}} + \\ + \| \frac{\partial \phi}{\partial z} \|_{L^2(\Omega)}^{\frac{1}{3}} \Big]. \tag{4.15}$$

For $n=2$, we have

$$\| \phi \|_{L^4(\Omega)} \leq C(\Omega, \hat{G}) \Big(\| \frac{\partial \phi}{\partial x_1} \|_{L^2(\Omega)}^{\frac{1}{4}} \| \frac{\partial \phi}{\partial z} \|_{L^2(\Omega)}^{\frac{3}{4}} + \| \frac{\partial \phi}{\partial z} \|_{L^2(\Omega)}^{\frac{1}{2}} \Big). \tag{4.16}$$

Theorem 2 *For any solution $(u^\varepsilon, p^\varepsilon)$ to (3.3), there exists a constant $C>0$ (which does not depend on ε) and ε_1 such that for $\varepsilon \leq \varepsilon_1$, we have*

$$\sum_{i,j=1}^{n-1} \| \varepsilon \frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} \|_{L^2(\Omega)}^2 + \| \varepsilon \frac{\partial \hat{u_n^\varepsilon}}{\partial z} \|_{L^2(\Omega)}^2 + \\ + \sum_{i=1}^{n-1} \Big(\| \frac{\partial \hat{u_i^\varepsilon}}{\partial z} \|_{L^2(\Omega)}^2 + \| \varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i} \|_{L^2(\Omega)}^2 \Big) \leq C. \tag{4.17}$$

$$\| \frac{\partial \hat{p^\varepsilon}}{\partial x_i} \|_{H^{-1}(\Omega)} \leq C, \quad i = 1,..,n-1, \tag{4.18}$$

$$\| \frac{\partial \hat{p^\varepsilon}}{\partial z} \|_{H^{-1}(\Omega)} \leq C\varepsilon, \tag{4.19}$$

Proof. From (4.11) with the choice $\phi = G^\varepsilon$, and as $b(u^\varepsilon, u^\varepsilon, u^\varepsilon) = 0$ and $j^\varepsilon(G^\varepsilon) = 0$, we get

$$2\mu a(u^\varepsilon, u^\varepsilon) \leq 2\mu a(u^\varepsilon, G^\varepsilon) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, G^\varepsilon) + (f^\varepsilon, u^\varepsilon - G^\varepsilon), \tag{4.20}$$

by Korn's inequality, there exists a constant C_K which does not depend on ε such that

$$a(u^\varepsilon, u^\varepsilon) \geq C_K \| \nabla u^\varepsilon \|_{(L^2(\Omega^\varepsilon))^{n^2}}, \tag{4.21}$$

using Poincaré's inequality, we get

$$\| u^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n} \leq h_{max} \ \varepsilon \ \| \nabla u^\varepsilon \|_{(L^2(\Omega^\varepsilon))^{n^2}},$$

and using Young's inequality, we obtain

$$\| f^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n} \| u^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n} \leq \frac{\mu C_K}{2} \| \nabla u^\varepsilon \|_{(L^2(\Omega^\varepsilon))^{n^2}}^2 + \frac{\varepsilon^2 h_{max}^2}{2\mu C_K} \| f^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n}^2,$$

in the same way we have:

$$\| f^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n} \| G^\varepsilon \|_{(L^2(\Omega^\varepsilon))^n} \leq \frac{\mu C_K}{2} \| \nabla G^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^{n^2}} + \frac{\varepsilon^2 h^2_{max}}{2\mu C_K} \| f^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^n} .$$

Using Cauchy-Schwarz inequality then Young's inequality, we obtain

$$\begin{aligned} | 2\mu a(u^\varepsilon, G^\varepsilon) | &\leq 2\mu \int_{\Omega^\varepsilon} | D(u^\varepsilon) || D(G^\varepsilon) | \, dx \\ &\leq \int_{\Omega^\varepsilon} \big(\sqrt{\mu C_K} \, | D(u^\varepsilon) | \big)\big(2\sqrt{\frac{\mu}{C_K}} \, | D(G^\varepsilon) | \big) dx \\ &\leq \frac{\mu C_K}{2} \int_{\Omega^\varepsilon} | D(u^\varepsilon) |^2 \, dx + \frac{2\mu}{C_K} \int_{\Omega^\varepsilon} | D(G^\varepsilon) |^2 \, dx, \end{aligned}$$

as $\| D(v) \|_{(L^2(\Omega^\varepsilon))^{n^2}} \leq \| \nabla v \|_{(L^2(\Omega^\varepsilon))^{n^2}} \quad \forall v \in V^\varepsilon$, we deduce

$$2\mu a(u^\varepsilon, G^\varepsilon) \leq \frac{\mu C_K}{2} \| \nabla u^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^{n^2}} + \frac{2\mu}{C_K} \| \nabla G^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^{n^2}} .$$

From (4.20), we deduce

$$\begin{aligned} \mu C_K \| \nabla u^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^{n^2}} \leq \big(\frac{\mu C_K}{2} + \frac{2\mu}{C_K}\big) \| \nabla G^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^{n^2}} + \\ + \frac{\varepsilon^2 h^2_{max}}{\mu C_K} \| f^\varepsilon \|^2_{(L^2(\Omega^\varepsilon))^n} + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, G^\varepsilon). \end{aligned} \tag{4.22}$$

By dividing by ε and passing to the fixed domain Ω, we get

$$\begin{aligned} \mu C_K \varepsilon^{2\beta} \Big\{ \sum_{i,j=1}^{n-1} \| \frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} \|^2_{L^2(\Omega)} + \| \frac{\partial \hat{u_n^\varepsilon}}{\partial z} \|^2_{L^2(\Omega)} + \\ + \sum_{i=1}^{n-1} \Big(\| \frac{\partial \hat{u_i^\varepsilon}}{\varepsilon \partial z} \|^2_{L^2(\Omega)} + \| \varepsilon \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i} \|^2_{L^2(\Omega)} \Big) \Big\} \leq \frac{K}{\varepsilon^2} \varepsilon^{2\beta} + \varepsilon^\gamma \frac{b(u^\varepsilon, u^\varepsilon, G^\varepsilon)}{\varepsilon}, \end{aligned} \tag{4.23}$$

where

$$K = \frac{h^2_{max}}{\mu C_K} \| \hat{f} \|^2_{(L^2(\Omega))^n} + \Big(\frac{\mu C_K}{2} + \frac{\mu}{2C_K}\Big) \| \nabla_z \hat{G} \|^2 .$$

Now, we define I^ε, J^ε by:

$$I^\varepsilon = \max\Big(\max_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1}} \| \frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} \|_{L^2(\Omega)} \; ; \; \max_{1 \leq j \leq n-1} \| \frac{\varepsilon \partial \hat{u_n^\varepsilon}}{\partial x_j} \|_{L^2(\Omega)} \Big), \tag{4.24}$$

$$J^\varepsilon = \max\Big(\| \frac{\varepsilon \partial \hat{u_n^\varepsilon}}{\partial z} \|_{L^2(\Omega)} \; ; \; \max_{1 \leq i \leq n-1} \| \frac{\varepsilon \partial \hat{u_i^\varepsilon}}{\partial z} \|_{L^2(\Omega)} \Big). \tag{4.25}$$

Thus

$$\mu C_K \varepsilon^{2\beta}\Big(\big(\frac{J^\varepsilon}{\varepsilon}\big)^2 + (I^\varepsilon)^2\Big) \leq \frac{K}{\varepsilon^2}\varepsilon^{2\beta} + \varepsilon^\gamma \frac{b(u^\varepsilon, u^\varepsilon, G^\varepsilon)}{\varepsilon}. \tag{4.26}$$

• In the other hand, if n=2, we have

$$b(u^\varepsilon, u^\varepsilon, G^\varepsilon) = \varepsilon^{3\beta+1}\int_\Omega \hat{u_1^\varepsilon}\frac{\partial \hat{u_1^\varepsilon}}{\partial x_1}\hat{G}_1 dx'dz + \varepsilon^{3\beta+1}\int_\Omega \hat{u_2^\varepsilon}\frac{\partial \hat{u_1^\varepsilon}}{\partial z}\hat{G}_1 dx'dz + $$
$$+\varepsilon^{3\beta+3}\int_\Omega \hat{u_1^\varepsilon}\frac{\partial \hat{u_2^\varepsilon}}{\partial x_1}\hat{G}_2 dx'dz + \varepsilon^{3\beta+3}\int_\Omega \hat{u_2^\varepsilon}\frac{\partial \hat{u_2^\varepsilon}}{\partial z}\hat{G}_2 dx'dz,$$

by Hölder inequality we can estimate any term of b in the preceding formula, as follows:

$$\int_\Omega \hat{u_1^\varepsilon}\frac{\partial \hat{u_1^\varepsilon}}{\partial x_1}\hat{G}_1 dx'dz \leq \| \hat{u_1^\varepsilon} \|_{L^4(\Omega)} \| \frac{\partial \hat{u_1^\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \| \hat{G}_1 \|_{L^4(\Omega)} .$$

Using Lemma 2, we get

$$\begin{aligned} \| \hat{u_1^\varepsilon} \|_{L^4(\Omega)} &\leq C(\Omega, \hat{G})\Big(\| \frac{\partial \hat{u_1^\varepsilon}}{\partial x_1} \|^{\frac14}_{L^2(\Omega)} \| \frac{\partial \hat{u_1^\varepsilon}}{\partial z} \|^{\frac34}_{L^2(\Omega)} + \| \frac{\partial \hat{u_1^\varepsilon}}{\partial z} \|^{\frac12}_{L^2(\Omega)} \Big) \\ &\leq C(\Omega, \hat{G})\Big(I^{\varepsilon\,\frac14} J^{\varepsilon\,\frac34} + J^{\varepsilon\,\frac12}\Big), \end{aligned}$$

in the same way

$$\begin{aligned} \| \varepsilon\hat{u_2^\varepsilon} \|_{L^4(\Omega)} &\leq C(\Omega, \hat{G})\Big(\| \varepsilon\frac{\partial \hat{u_2^\varepsilon}}{\partial x_1} \|^{\frac14}_{L^2(\Omega)} \| \varepsilon\frac{\partial \hat{u_2^\varepsilon}}{\partial z} \|^{\frac34}_{L^2(\Omega)} + \| \varepsilon\frac{\partial \hat{u_2^\varepsilon}}{\partial z} \|^{\frac12}_{L^2(\Omega)} \Big) \\ &\leq C(\Omega, \hat{G})\Big(I^{\varepsilon\,\frac14} J^{\varepsilon\,\frac34} + J^{\varepsilon\,\frac12}\Big). \end{aligned}$$

Consequently

$$\begin{aligned} | b(u^\varepsilon, u^\varepsilon, G^\varepsilon) | &\leq C\varepsilon^{3\beta}\big(\varepsilon I^\varepsilon + J^\varepsilon\big)\big(I^{\varepsilon\,\frac14} J^{\varepsilon\,\frac34} + J^{\varepsilon\,\frac12}\big) \\ &\leq C\varepsilon^{3\beta}\Big\{\varepsilon^{\frac74}\Big(I^{\varepsilon\,\frac54}\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac34} + I^{\varepsilon\,\frac14}\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac74}\Big) + \\ &+ \varepsilon^{\frac32}\Big(I^\varepsilon\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac12} + \big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac32}\Big)\Big\}. \end{aligned} \tag{4.27}$$

Using Young's inequality in (4.27) successively for

$$\Big(I^{\varepsilon\,\frac54}, \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac34}\Big)\Big(\frac85, \frac83\Big) \;\; ; \;\; \Big(I^{\varepsilon\,\frac14}, \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac74}\Big)\Big(8, \frac87\Big)$$

$$\Big(I^\varepsilon, \Big(\frac{J^\varepsilon}{\sqrt{\varepsilon}}\Big)^{\frac12}\Big)\Big(2, 2\Big) \;\; ; \;\; \Big(\Big(\frac{J^\varepsilon}{\varepsilon}, \Big)^{\frac32}, 1\Big)\Big(\frac43, 4\Big),$$

as $\beta = \frac{1}{2} - \gamma$, then we obtain

$$\begin{aligned} \frac{\varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, G^\varepsilon)}{\varepsilon} &\leq \varepsilon^{2\beta}\Big\{C\varepsilon^{\frac{5}{4}}\Big[\frac{5}{8}I^{\varepsilon\,2} + \frac{3}{8}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \frac{1}{8}I^{\varepsilon\,2} + \frac{7}{8}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2\Big] + \\ &+ C\varepsilon^{\frac{3}{4}}\Big[\frac{1}{2}I^{\varepsilon\,2} + \frac{1}{2}\frac{J^\varepsilon}{\sqrt{\varepsilon}}\Big] + C\varepsilon\Big[\frac{3}{4}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \frac{1}{4}\Big]\Big\} \\ &\leq \varepsilon^{2\beta}\Big\{C\varepsilon^{\frac{3}{4}}\Big[\frac{5}{4}I^{\varepsilon\,2} + \frac{5}{4}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2\Big] + C\varepsilon^{\frac{3}{4}} + \\ &+ \frac{1}{2}C\varepsilon^{\frac{5}{4}}\frac{J^\varepsilon}{\varepsilon} + \frac{3}{4}C\varepsilon^{\frac{3}{4}}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2\Big\}, \quad \text{(because } \varepsilon < 1\text{)}. \end{aligned}$$

We return at (4.26) and dividing by $\varepsilon^{2\beta}$, we get

$$\begin{aligned} &\Big(\mu C_K - \frac{5}{4}C\varepsilon^{\frac{3}{4}}\Big)I^{\varepsilon\,2} + \Big(\frac{\mu C_K}{2} - 2C\varepsilon^{\frac{3}{4}}\Big)\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \\ &\qquad + \Big\{\frac{\mu C_K}{2}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 - \frac{1}{2}C\varepsilon^{\frac{5}{4}}\frac{J^\varepsilon}{\varepsilon}\Big\} \leq \frac{K'}{\varepsilon^2}, \end{aligned} \tag{4.28}$$

whence

$$\begin{aligned} &\Big(\mu C_K - \frac{5}{4}C\varepsilon^{\frac{3}{4}}\Big)I^{\varepsilon\,2} + \Big(\frac{\mu C_K}{2} - 2C\varepsilon^{\frac{3}{4}}\Big)\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \\ &\qquad + \frac{\mu C_K}{2}\Big(\frac{J^\varepsilon}{\varepsilon} - \frac{C}{2\mu C_K}\varepsilon^{\frac{5}{4}}\Big)^2 \leq \frac{K''}{\varepsilon^2}. \end{aligned} \tag{4.29}$$

To obtain estimate (4.17), the left side of (4.29) must be strictly positive, which induce the choice $\varepsilon < \varepsilon_{0,2}$, where $\varepsilon_{0,2}^{\frac{3}{4}} = \dfrac{1}{4}\dfrac{\mu C_K}{C}$.

• When $n = 3$, we obtain in the same way

$$\begin{aligned} \mid b(u^\varepsilon, u^\varepsilon, G^\varepsilon) \mid &\leq C\varepsilon^{3\beta}\big(\varepsilon I^\varepsilon + J^\varepsilon\big)\big(I^{\varepsilon\,\frac{1}{2}}J^{\varepsilon\,\frac{1}{2}} + J^{\varepsilon\,\frac{1}{3}}\big) \\ &\leq C\varepsilon^{3\beta}\Big\{\varepsilon^{\frac{3}{2}}\Big(I^{\varepsilon\,\frac{3}{2}}\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac{1}{2}} + I^\varepsilon\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac{3}{2}}\Big) + \\ &+ \varepsilon^{\frac{4}{3}}\Big(I^\varepsilon\big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac{1}{2}} + \big(\frac{J^\varepsilon}{\varepsilon}\big)^{\frac{3}{2}}\Big)\Big\}, \end{aligned} \tag{4.30}$$

thus

$$\begin{aligned} \frac{\varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, G^\varepsilon)}{\varepsilon} &\leq \varepsilon^{2\beta}\Big\{C\varepsilon\Big[I^{\varepsilon\,\frac{3}{2}}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} + I^{\varepsilon\,\frac{1}{2}}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{3}{2}}\Big] + \\ &+ C\varepsilon^{\frac{5}{6}}\Big[I^\varepsilon\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{1}{3}} + \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{4}{3}}\Big]\Big\}. \end{aligned} \tag{4.31}$$

Using Young's inequality successively for

$$\Big(I^{\varepsilon\,\frac{3}{2}}, \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big)\Big(\frac{4}{3}, 4\Big) \;\; ; \;\; \Big(I^{\varepsilon\,\frac{1}{2}}, \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{3}{2}}\Big)\Big(4, \frac{4}{3}\Big)$$

$$\Big(I^\varepsilon, \Big(\frac{J^\varepsilon}{\varepsilon}\Big)^{\frac{1}{3}}, 1\Big)\Big(2,3,6\Big) \;\; ; \;\; \Big(\Big(\frac{J^\varepsilon}{\varepsilon},\Big)^{\frac{4}{3}}, 1\Big)\Big(\frac{3}{2}, 3\Big),$$

thus

$$\begin{aligned}\frac{\varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, G^\varepsilon)}{\varepsilon} &\leq \varepsilon^{2\beta}\Big\{C\varepsilon\Big[\frac{3}{4}I^{\varepsilon 2} + \frac{1}{4}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \frac{1}{4}I^{\varepsilon 2} + \frac{3}{4}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2\Big] + \\ &+ C\varepsilon^{\frac{5}{6}}\Big[\frac{1}{2}I^{\varepsilon 2} + \frac{1}{3}\frac{J^\varepsilon}{\varepsilon} + \frac{1}{6} + \frac{2}{3}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \frac{1}{3}\Big]\Big\} \\ &\leq \varepsilon^{2\beta}\Big\{C\varepsilon^{\frac{5}{6}}\Big[\frac{3}{2}I^{\varepsilon 2} + \frac{5}{3}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \frac{1}{3}\frac{J^\varepsilon}{\varepsilon}\Big) + \frac{1}{2}\Big]\Big\}. \qquad (4.32)\end{aligned}$$

We return at (4.23) and by dividing by $\varepsilon^{2\beta}$, we get

$$\begin{aligned}&\Big(\mu C_K - \frac{3}{2}C\varepsilon^{\frac{5}{6}}\Big)I^{\varepsilon 2} + \Big(\frac{\mu C_K}{2} - \frac{5}{3}C\varepsilon^{\frac{5}{6}}\Big)\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 + \\ &\qquad + \Big(\frac{\mu C_K}{2}\Big(\frac{J^\varepsilon}{\varepsilon}\Big)^2 - \frac{C\varepsilon^{\frac{5}{6}}}{3\mu C_K}\Big)^2 \leq \frac{K_3}{\varepsilon^2}. \qquad (4.33)\end{aligned}$$

To obtain estimate (4.17), it is enough that any term of the left of (4.29) must be strictly positive, which induce the choice $\varepsilon < \varepsilon_{0,3}$, where $\varepsilon_{0,3}^{\frac{5}{6}} = \dfrac{3}{10}\dfrac{\mu C_K}{C}$.

We put $\varepsilon_1 = min(\varepsilon_{0,2}, \varepsilon_{0,3})$, the estimate (4.17) is valid in both cases $n = 2$ and $n = 3$, when $\varepsilon < \varepsilon_1$.

• To prove (4.18) and (4.19), we choose $\varphi = u^\varepsilon \pm \psi$ with $\psi \in H_0^1(\Omega^\varepsilon)$ in (3.3). We obtain

$$\int_{\Omega^\varepsilon} p^\varepsilon div\psi dx' dx_n = a(u^\varepsilon, \psi) + \varepsilon^\gamma b(u^\varepsilon, u^\varepsilon, \psi) - (f^\varepsilon, \psi).$$

Using (4.17), and the following Sobolev embedding

$$\| v \|_{L^4(\Omega^\varepsilon)} \leq C \| v \|_{H^1(\Omega^\varepsilon)},$$

so passing to the fixed domain Ω, we obtain the estimates (4.18) and (4.19).

4.3 Uniqueness

Theorem 3 With the same assumptions as in theorems 1 and 2, there exist ε_1' such that for $\varepsilon \leq \varepsilon_1'$, then $\hat{u^\varepsilon}$, such that $(u^\varepsilon, p^\varepsilon)$ is a solution of (3.3), is unique.

Proof. Let $(u^{1,\varepsilon}, p^{1,\varepsilon})$, $(u^{2,\varepsilon}, p^{2,\varepsilon})$ be two different solutions for (3.3). We choose $\phi = u^{2,\varepsilon}$ then $\phi = u^{1,\varepsilon}$ in (3.3) when the solutions are respectively $(u^{1,\varepsilon}, p^{1,\varepsilon})$ and $(u^{2,\varepsilon}, p^{2,\varepsilon})$. Then

$$2\mu a(u^{1,\varepsilon}, u^{2,\varepsilon} - u^{1,\varepsilon}) + \varepsilon^\gamma b(u^{1,\varepsilon}, u^{1,\varepsilon}, u^{2,\varepsilon} - u^{1,\varepsilon}) + j^\varepsilon(u^{2,\varepsilon}) - j^\varepsilon(u^{1,\varepsilon}) \geq (f^\varepsilon, u^{2,\varepsilon} - u^{1,\varepsilon}),$$

$$2\mu a(u^{2,\varepsilon}, u^{1,\varepsilon} - u^{2,\varepsilon}) + \varepsilon^\gamma b(u^{2,\varepsilon}, u^{2,\varepsilon}, u^{1,\varepsilon} - u^{2,\varepsilon}) + j^\varepsilon(u^{1,\varepsilon}) - j^\varepsilon(u^{2,\varepsilon}) \geq (f^\varepsilon, u^{1,\varepsilon} - u^{2,\varepsilon}).$$

Adding both inequalities, we get

$$-2\mu a(u^{1,\varepsilon}-u^{2,\varepsilon},u^{1,\varepsilon}-u^{2,\varepsilon})+$$
$$+\varepsilon^{\gamma}\Big(b(u^{2,\varepsilon},u^{2,\varepsilon},u^{1,\varepsilon}-u^{2,\varepsilon})-b(u^{1,\varepsilon},u^{1,\varepsilon},u^{1,\varepsilon}-u^{2,\varepsilon})\Big)\geq 0.$$

Let $v^{\varepsilon}=u^{1,\varepsilon}-u^{2,\varepsilon}$. We have

$$\begin{aligned} b(u^{1,\varepsilon},u^{1,\varepsilon},v^{\varepsilon})-b(v^{\varepsilon},u^{1,\varepsilon},v^{\varepsilon}) &= b(u^{2,\varepsilon},u^{1,\varepsilon},v^{\varepsilon})\\ &= b(u^{2,\varepsilon},v^{\varepsilon},v^{\varepsilon})+b(u^{2,\varepsilon},u^{2,\varepsilon},v^{\varepsilon})\\ &= b(u^{2,\varepsilon},u^{2,\varepsilon},v^{\varepsilon},)\end{aligned}$$

from (2.6) $b(u^{2,\varepsilon},v^{\varepsilon},v^{\varepsilon})=0$, thus

$$2\mu a(v^{\varepsilon},v^{\varepsilon})+\varepsilon^{\gamma}b(v^{\varepsilon},u^{1,\varepsilon},v^{\varepsilon})\leq 0. \tag{4.34}$$

Using Korn's inequality, we obtain

$$2\mu C_K \parallel \nabla v^{\varepsilon} \parallel_{(L^2(\Omega^{\varepsilon}))^n} +\varepsilon^{\gamma}b(v^{\varepsilon},u^{1,\varepsilon},v^{\varepsilon})\leq 0. \tag{4.35}$$

As

$$\begin{aligned} b(v^{\varepsilon},u^{1,\varepsilon},v^{\varepsilon}) &= \sum_{1\leq i,j\leq n}\int_{\Omega^{\varepsilon}} v_i^{\varepsilon}\frac{\partial u_j^{1,\varepsilon}}{\partial x_i}v_j^{\varepsilon}dx'dx_n =\\ =\varepsilon^{3\beta}\Big\{\varepsilon &\sum_{1\leq i,j\leq n-1}\int_{\Omega}\hat{v_i^{\varepsilon}}\frac{\partial \hat{u}_j^{1,\varepsilon}}{\partial x_i}\hat{v_j^{\varepsilon}}dx'dz+\varepsilon\sum_{1\leq j\leq n-1}\int_{\Omega}\hat{v_n^{\varepsilon}}\frac{\partial \hat{u}_j^{1,\varepsilon}}{\partial z}\hat{v_j^{\varepsilon}}dx'dz+\\ &+\varepsilon^3\sum_{1\leq i\leq n-1}\int_{\Omega}\hat{v_i^{\varepsilon}}\frac{\partial \hat{u}_n^{1,\varepsilon}}{\partial x_i}\hat{v_n^{\varepsilon}}dx'dz+\varepsilon^3\int_{\Omega}(\hat{v_n^{\varepsilon}})^2\frac{\partial \hat{u}_n^{1,\varepsilon}}{\partial z}dx'dz\Big\}, \end{aligned} \tag{4.36}$$

multiplying inequality (4.35) by $\varepsilon^{1-2\beta}$ and passing to the fixed domain, and taking into account that $\gamma+\beta=\frac{1}{2}$, we obtain

$$\begin{aligned} 2\mu C_K\Big\{&\sum_{i,j=1}^{n-1}\parallel \varepsilon\frac{\partial \hat{v_i^{\varepsilon}}}{\partial x_j}\parallel^2_{L^2(\Omega)}+\parallel \varepsilon\frac{\partial \hat{v_n^{\varepsilon}}}{\partial z}\parallel^2_{L^2(\Omega)}+\\ &+\sum_{i=1}^{n-1}\Big(\parallel \frac{\partial \hat{v_i^{\varepsilon}}}{\partial z}\parallel^2_{L^2(\Omega)}+\parallel \varepsilon^2\frac{\partial \hat{v_n^{\varepsilon}}}{\partial x_i}\parallel^2_{L^2(\Omega)}\Big)\Big\}+\\ +\varepsilon^{\frac{1}{2}}\Big\{\varepsilon^2&\sum_{1\leq i,j\leq n-1}\int_{\Omega}\hat{v_i^{\varepsilon}}\frac{\partial \hat{u}_j^{1,\varepsilon}}{\partial x_i}\hat{v_j^{\varepsilon}}dx'dz+\varepsilon^2\sum_{1\leq j\leq n-1}\int_{\Omega}\hat{v_n^{\varepsilon}}\frac{\partial \hat{u}_j^{1,\varepsilon}}{\partial z}\hat{v_j^{\varepsilon}}dx'dz+\\ &+\varepsilon^4\sum_{1\leq i\leq n-1}\int_{\Omega}\hat{v_i^{\varepsilon}}\frac{\partial \hat{u}_n^{1,\varepsilon}}{\partial x_i}\hat{v_n^{\varepsilon}}dx'dz+\varepsilon^4\int_{\Omega}(\hat{v_n^{\varepsilon}})^2\frac{\partial \hat{u}_n^{1,\varepsilon}}{\partial z}dx'dz\Big\}\leq 0. \end{aligned} \tag{4.37}$$

Now we want to estimate all the terms which come from the trilinear form b. We have $\hat{v^\varepsilon} \in H^1_{\Gamma_1}(\Omega)$, so we can apply Lemma 1. Computation is very similar for all terms, for instance, for $n = 2$

$$\begin{aligned}
&\varepsilon^{\frac{5}{2}} \int_\Omega (\hat{v_1^\varepsilon})^2 \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} dx' dz \leq \\
&\leq 2h_{max}\varepsilon^{\frac{5}{2}} \| \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \| \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^{\frac{1}{2}}_{L^2(\Omega)} \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|^{\frac{3}{2}}_{L^2(\Omega)} \\
&\leq \frac{3}{2} h_{max}\varepsilon^{\frac{1}{2}} \| \varepsilon \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \left(\| \varepsilon \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} \right) \\
&\leq \frac{3}{2} h_{max}\varepsilon^{\frac{1}{2}} \| \varepsilon \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \left\{ \| \varepsilon \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} + \right. \\
&\qquad \left. + \| \varepsilon^2 \frac{\partial \hat{v}_2^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \| \varepsilon \frac{\partial \hat{v}_2^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} \right\}.
\end{aligned}$$

We add all the terms of b and using estimate (4.17), then we return to (4.37), we deduce

$$\begin{aligned}
&\left(2\mu C_K - \frac{3}{2} h_{max} C \varepsilon^{\frac{1}{2}}\right) \left\{ \| \varepsilon \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} + \right. \\
&\qquad \left. + \| \varepsilon^2 \frac{\partial \hat{v}_2^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \| \varepsilon \frac{\partial \hat{v}_2^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} \right\} \leq 0.
\end{aligned} \tag{4.38}$$

When $\varepsilon \leq \varepsilon_{1,2}$ with $\varepsilon_{1,2}$ defined by

$$\varepsilon_{1,2}^{\frac{1}{2}} = \frac{4}{3} \frac{\mu C_K}{C h_{max}},$$

the first term in (4.38) is positive, thus $\hat{v^\varepsilon} = 0$ in Ω, what implies that $v^\varepsilon = 0$ in Ω^ε.
When $n = 3$, a similar computation leads to

$$\begin{aligned}
&\varepsilon^{\frac{5}{2}} \int_\Omega (\hat{v_n^\varepsilon})^2 \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} dx' dz \leq 16 h_{max}^{\frac{1}{2}} \varepsilon^{\frac{5}{2}} \left\{ \| \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \cdot \right. \\
&\qquad \left. \cdot \| \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^{\frac{1}{2}}_{L^2(\Omega)} \| \frac{\partial \hat{v}_1^\varepsilon}{\partial x_2} \|^{\frac{1}{2}}_{L^2(\Omega)} \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|_{L^2(\Omega)} \right\} \\
&\qquad \leq 8 h_{max}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \| \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \left\{ \| \varepsilon \frac{\partial \hat{v}_1^\varepsilon}{\partial x_1} \|^2_{L^2(\Omega)} + \right.
\end{aligned}$$

$$+ \| \varepsilon \frac{\partial \hat{v}_1^\varepsilon}{\partial x_2} \|^2_{L^2(\Omega)} + \| \frac{\partial \hat{v}_1^\varepsilon}{\partial z} \|^2_{L^2(\Omega)} \Big\}$$

$$\leq 8 h_{max}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \| \frac{\partial \hat{u}_1^{1,\varepsilon}}{\partial x_1} \|_{L^2(\Omega)} \Big\{ \sum_{i,j=1}^{2} \| \varepsilon \frac{\partial \hat{v_i^\varepsilon}}{\partial x_j} \|^2_{L^2(\Omega)} +$$

$$+ \| \varepsilon \frac{\partial \hat{v_3^\varepsilon}}{\partial z} \|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \Big(\| \frac{\partial \hat{v_i^\varepsilon}}{\partial z} \|^2_{L^2(\Omega)} + \| \varepsilon^2 \frac{\partial \hat{v_3^\varepsilon}}{\partial x_i} \|^2_{L^2(\Omega)} \Big) \Big\}.$$

Then we obtain in the same way

$$\big(2\mu C_K - 8 h_{max}) C \varepsilon^{\frac{1}{2}}\big) \Big\{ \sum_{i,j=1}^{2} \| \varepsilon \frac{\partial \hat{v_i^\varepsilon}}{\partial x_j} \|^2_{L^2(\Omega)} + \| \varepsilon \frac{\partial \hat{v_3^\varepsilon}}{\partial z} \|^2_{L^2(\Omega)} +$$

$$+ \sum_{i=1}^{2} \Big(\| \frac{\partial \hat{v_i^\varepsilon}}{\partial z} \|^2_{L^2(\Omega)} + \| \varepsilon^2 \frac{\partial \hat{v_3^\varepsilon}}{\partial x_i} \|^2_{L^2(\Omega)} \Big) \Big\} \leq 0. \qquad (4.39)$$

The first term in (4.39) is positive when $\varepsilon \leq \varepsilon_{1,3}$, where

$$\varepsilon_{1,3}^{\frac{1}{2}} = \frac{1}{4} \frac{\mu C_K}{C h_{max}}.$$

For $\varepsilon \leq \varepsilon_1' = \min\{\varepsilon_1 \ , \ \varepsilon_{1,3}\}$, the proof of uniqueness of $\hat{u^\varepsilon}$ is complete. □

5 Limit problem

We multiply variational inequality (3.3) by $\varepsilon^{1-2\beta}$, and passing to the fixed domain Ω, we obtain

$$\sum_{i,j=1}^{n-1} \int_\Omega \Big[\varepsilon^2 \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} + \frac{\partial \hat{u_j^\varepsilon}}{\partial x_i} \Big) - \hat{p^\varepsilon} \delta_{i,j} \Big] \frac{\partial}{\partial x_j} (\hat{\phi}_i - \hat{u_i^\varepsilon}) dx' dz +$$

$$+ \sum_{i=1}^{n-1} \int_\Omega \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i} \Big) \frac{\partial}{\partial z} (\hat{\phi}_i - \hat{u_i^\varepsilon}) dx' dz +$$

$$+ \int_\Omega (2\mu \varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial z} - \hat{p^\varepsilon}) \frac{\partial}{\partial z} (\hat{\phi_n} - \hat{u_n^\varepsilon}) dx' dz +$$

$$+ \sum_{j=1}^{n-1} \int_\Omega \varepsilon^2 \mu (\varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_j} + \frac{\partial \hat{u_j^\varepsilon}}{\partial z}) \frac{\partial}{\partial x_j} (\hat{\phi_n} - \hat{u_n^\varepsilon}) dx' dz +$$

$$+\varepsilon^{\frac{5}{2}}\sum_{i,j=1}^{n-1}\int_\Omega \hat{u_i^\varepsilon}\frac{\partial\hat{u_j^\varepsilon}}{\partial x_i}(\hat{\phi_j}-\hat{u_j^\varepsilon})dx'dz+\varepsilon^{\frac{9}{2}}\sum_{i=1}^{n-1}\int_\Omega \hat{u_i^\varepsilon}\frac{\partial\hat{u_n^\varepsilon}}{\partial x_i}(\hat{\phi_n}-\hat{u_n^\varepsilon})dx'dz+$$

$$+\varepsilon^{\frac{5}{2}}\sum_{j=1}^{n-1}\int_\Omega \hat{u_n^\varepsilon}\frac{\partial\hat{u_j^\varepsilon}}{\partial z}(\hat{\phi_j}-\hat{u_j^\varepsilon})dx'dz+\varepsilon^{\frac{9}{2}}\int_\Omega \hat{u_n^\varepsilon}\frac{\partial\hat{u_n^\varepsilon}}{\partial z}(\hat{\phi_n}-\hat{u_n})dx'dz\geq$$

$$\geq\sum_{i=1}^{n-1}\int_\Omega \hat{f}_i(\hat{\phi_i}-\hat{u_i^\varepsilon})dx'dz+\int_\Omega \varepsilon\hat{f}_n(\hat{\phi_n}-\hat{u_n^\varepsilon})dx'dz+$$

$$+\int_\omega \hat{k}\Big(\mid\hat{\phi}-s\mid-\mid\hat{u^\varepsilon}-s\mid\Big)d\sigma \tag{5.1}$$

Let now

$$V_z=\{v\in(L^2(\Omega))^{n-1}:\ \text{such that}\ \frac{\partial v_i}{\partial z}\in L^2(\Omega),\ \ i=1,..,n-1\ \text{ and }\ v=0\ \text{on}\ \Gamma_1\}$$

be the Banach space with the norm

$$\| v\|_{V_z}=\Big(\sum_{i=1}^{n-1}\| v_i\|^2_{L^2(\Omega)}+\|\frac{\partial v_i}{\partial z}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

Following [3], we say that $v=(v_1,..,v_{n-1})\in(L^2(\Omega))^{n-1}$ satisfies the condition (D') if,

$$\int_\Omega\sum_{i=1}^{n-1}v_i\frac{\partial\theta}{\partial x_i}dx'dz=0.\quad\forall\theta\in\mathcal{C}_0^\infty(\omega)$$

Let

$$\tilde{V_z}=\{v\in V_z:\ \text{ such that } v \text{ satisfies the condition } (D')\}$$

be a subspace of V_z.
And

$$\Pi(K)=\{\bar{\phi}\in(H^1(\Omega))^{n-1}:\ \ \bar{\phi}=(\hat{\phi}_1,..,\hat{\phi}_{n-1})\quad\hat{\phi}_i=\hat{G}_i\ \text{on}\ \Gamma_1\cup\Gamma_L\}.$$

Theorem 4 *Under the same assumptions as in theorem 3, for any solution* $(u^\varepsilon,p^\varepsilon)$*, after extraction of a subsequence, there exist* $u^\star=(u_1^\star,..,u_{n-1}^\star)\in\tilde{V_z}$ *and* $p^\star\in L_0^2(\Omega)$ *such that*

$$\hat{u_i^\varepsilon}\rightharpoonup u_i{}^\star\quad(1\leq i\leq n-1)\quad\textit{weakly}\quad\textit{in}\ V_z, \tag{5.2}$$

$$\varepsilon\frac{\partial\hat{u_i^\varepsilon}}{\partial x_j}\rightharpoonup 0\ \ (1\leq i,j\leq n-1)\ \ \textit{weakly}\ \ \textit{in}\ L^2(\Omega), \tag{5.3}$$

$$\varepsilon\frac{\partial\hat{u_n^\varepsilon}}{\partial z}\rightharpoonup 0\quad\textit{weakly}\quad\textit{in}\quad L^2(\Omega), \tag{5.4}$$

$$\varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i} \rightharpoonup 0 \ \ (1 \leq i \leq n-1) \ \ weakly \ \ in \ \ L^2(\Omega), \tag{5.5}$$

$$\varepsilon \hat{u_n^\varepsilon} \rightharpoonup 0 \quad weakly \ \ in \ \ L^2(\Omega), \tag{5.6}$$

$$\hat{p^\varepsilon} \rightharpoonup p^\star \ \ weakly \ \ in \ \ L^2_0(\Omega). \tag{5.7}$$

Proof. From (4.17) there exists a fixed constant $C > 0$ which does not depend on ε such that

$$\| \frac{\partial \hat{u_i^\varepsilon}}{\partial z} \|_{L^2(\Omega)} \leq C \ \ (1 \leq i \leq n-1),$$

using this estimate and Poincaré's inequality in the domain Ω, we deduce (5.2). We have $div(\hat{u^\varepsilon}) = 0$ in Ω, we obtain for $q \in \mathcal{C}_0^\infty(\omega)$:

$$\int_\Omega q(x')\Big(\sum_{i=1}^{n-1} \frac{\partial \hat{u_i^\varepsilon}}{\partial x_i} + \frac{\partial \hat{u_n^\varepsilon}}{\partial z}\Big)dx'dz = \int_\Omega q(x') \sum_{i=1}^{n-1} \frac{\partial \hat{u_i^\varepsilon}}{\partial x_i} dx'dz = 0.$$

As $\hat{u_i^\varepsilon} \rightharpoonup u_i^\star \ \ i = 1, .., n-1$, thus $u^\star$ satisfies condition (D') :

$$\int_\Omega q(x') \sum_{i=1}^{n-1} \frac{\partial u_i^\star}{\partial x_i} dx'dz = 0 \ \ \forall q \in \mathcal{C}_0^\infty(\omega).$$

Also (5.3)-(5.5) follows from (4.17). We prove (5.6)-(5.7) as in Theorem 4.2 [4]. □

Theorem 5 Under the same assumptions as in the theorem 3, $(u^\star, p^\star)$ satisfy

$$p^\star \in H^1(\omega), \tag{5.8}$$

$$-\mu \frac{\partial^2 u_i^\star}{\partial z^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \ \text{ for } \ \ (i = 1, .., n-1) \ \text{ in } \ L^2(\Omega). \tag{5.9}$$

Proof. Choosing in (5.1) $\hat{\phi}_i = \hat{u_i^\varepsilon}$ for $i = 1, .., n-1$, $\hat{\phi_n} = \hat{u_n^\varepsilon} \pm \psi$ with ψ in $H_0^1(\Omega)$, we deduce

$$\sum_{j=1}^{n-1} \int_\Omega \varepsilon^2 \mu (\varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_j} + \frac{\partial \hat{u_j^\varepsilon}}{\partial z}) \frac{\partial \psi}{\partial x_j} dx'dz + \int_\Omega \Big(2\mu\varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial z} - \hat{p^\varepsilon}\Big) \frac{\partial \psi}{\partial z} dx'dz +$$
$$+\varepsilon^{\frac{9}{2}} \sum_{i=1}^{n-1} \int_\Omega \hat{u_i^\varepsilon} \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i} \psi dx'dz + \varepsilon^{\frac{9}{2}} \int_\Omega \hat{u_n^\varepsilon} \frac{\partial \hat{u_n^\varepsilon}}{\partial z} \psi dx'dz = \int_\Omega \varepsilon \hat{f}_n \psi dx'dz,$$

Using the results of convergence (5.2)-(5.7), we obtain

$$\int_\Omega p^\star \frac{\partial \psi}{\partial z} dx'dz = 0, \ \ \forall \psi \in H_0^1(\Omega), \tag{5.10}$$

then

$$\frac{\partial p^\star}{\partial z} = 0 \ \text{ in } \ H^{-1}(\Omega). \tag{5.11}$$

Choosing now $\phi_i = \hat{u_i^\varepsilon} \pm \psi_i$ (for $i = 1,..,n-1$) with ψ_i in $H_1^0(\Omega)$ and $\phi_n = \hat{u_n^\varepsilon}$, in (5.1), we obtain

$$\sum_{i,j=1}^{n-1} \int_\Omega \Big[\varepsilon^2 \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} + \frac{\partial \hat{u_j^\varepsilon}}{\partial x_i}\Big) - \hat{p^\varepsilon} \delta_{i,j}\Big] \frac{\partial \psi_i}{\partial x_j} \, dx'dz +$$
$$+ \sum_{i=1}^{n-1} \int_\Omega \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i}\Big) \frac{\partial}{\partial z} \psi_i dx'dz + \varepsilon^{\frac{5}{2}} \sum_{i,j=1}^{n-1} \int_\Omega \hat{u_i^\varepsilon} \frac{\partial \hat{u_j^\varepsilon}}{\partial x_i} \psi_j dx'dz +$$
$$+\varepsilon^{\frac{5}{2}} \sum_{j=1}^{n-1} \int_\Omega \hat{u_n^\varepsilon} \frac{\partial \hat{u_j^\varepsilon}}{\partial z} \psi_j dx'dz = \sum_{i=1}^{n-1} \int_\Omega \hat{f}_i \psi_i dx'dz. \tag{5.12}$$

Using (5.2)-(5.7), and choosing $\psi_1 \in H_0^1(\Omega)$ when $n = 2$, and $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$ then $\psi_1 \in H_0^1(\Omega)$ and $\psi_2 = 0$ when $n = 3$, then we obtain the following equality:

$$-\sum_{i=1}^{n-1} \int_\Omega p^\star \frac{\partial \psi_i}{\partial x_i} dx'dz + \sum_{i=1}^{n-1} \int_\Omega \frac{\partial u_i^\star}{\partial z} \frac{\partial \psi_i}{\partial z} = \sum_{i=1}^{n-1} \int_\Omega \hat{f}_i \psi_i dx'dz, \tag{5.13}$$

using the Green formula, we obtain

$$-\mu \frac{\partial^2 u_i^\star}{\partial z^2} + \frac{\partial p^\star}{\partial x_i} = \hat{f}_i \ \text{ for } \ (i = 1,..,n-1) \ \text{ in } \ H^{-1}(\Omega). \tag{5.14}$$

To prove that $p^\star$ is in $H^1(\omega)$, let us recall that $p^\star$ does not depend on z from (5.11), then following [3] we choose ψ_i in (5.13) such that $\psi_i(x', z) = z(z - h(x'))\theta(x')$ with θ in $H_0^1(\omega)$, and using the Green formula we deduce

$$\frac{1}{6} \int_\omega p^\star \frac{\partial (h^3 \theta)}{\partial x_i} dx' - 2\mu \int_\omega h \tilde{u_i^\star} \theta dx' = \int_\omega \tilde{f}_i \theta dx',$$

where

$$\tilde{u_i^\star} = \frac{1}{h(x')} \int_0^{h(x')} u_i^\star(x', z) dz \ \text{ and } \tilde{f}_i = \int_0^{h(x')} z(z - h(x')) \hat{f}_i(x', z) dz.$$

Whence

$$2\mu h \tilde{u_i^\star} - \frac{1}{6} h^3 \frac{\partial p^\star}{\partial x_i} = \tilde{f}_i \ \ (i = 1,..,n-1) \ \text{ in } \ H^{-1}(\omega). \tag{5.15}$$

As $\hat{f}_i$ is in $L^2(\Omega)$, $u_i^\star$ in V_z, in particular in $L^2(\omega)$, therefore $\tilde{u_i^\star}$ and $\tilde{f}_i$ are in $L^2(\omega)$. Then from (5.15) we get $p^\star \in H^1(\omega)$, and (5.11) follows. As $\hat{f}_i$ in $L^2(\omega)$, from (5.15) we have $\dfrac{\partial^2 u_i^\star}{\partial z^2}$ in $L^2(\Omega)$, whence (5.9) holds.

Remark 1 All the terms of the trilinear form b disappear when ε tends to zero, because of the great power of ε in (5.1) ($\frac{5}{2}$ and $\frac{9}{2}$).

Now we define the traces $(s^\star, \tau^\star)$ by:

$$s^\star = u^\star(x', 0) \;\; ; \;\; \tau^\star = \frac{\partial u^\star}{\partial z}(x', 0).$$

Theorem 6 Under the same assumptions as in theorem 3, $(u^\star, p^\star)$ satisfy the following limit inequality

$$\sum_{i=1}^{n-1} \int_\Omega \mu \frac{\partial u_i^\star}{\partial z} \frac{\partial(\hat{\phi}_i - u_i^*)}{\partial z} dx'dz - \sum_{i=1}^{n-1} \int_\Omega p^\star \frac{\partial \hat{\phi}_i}{\partial x_i} dx'dz + \\ + \int_\omega \hat{k}(\mid \hat{\phi} - s \mid - \mid u^\star - s \mid) d\sigma \geq \sum_{i=1}^{n-1} \int_\Omega \hat{f}_i(\hat{\phi}_i - u^\star) dx'dz. \quad \forall \hat{\phi} \in \Pi(K) \quad (5.16)$$

Also $(s^\star, \tau^\star)$ satisfy the following inequality

$$\int_\omega \hat{k}(\mid \psi + s^\star - s \mid - \mid s^\star - s \mid) d\sigma - \int_\omega \mu \tau^\star \psi d\sigma \geq 0, \;\; \forall \psi \in (L^2(\omega))^{n-1}, \quad (5.17)$$

$$\left.\begin{array}{l} \mu \mid \tau^\star \mid \; < \; \hat{k} \;\; \Rightarrow \;\; s^\star = s \\ \mu \mid \tau^\star \mid \; = \; \hat{k} \;\; \Rightarrow \;\; \exists \; \lambda \geq 0 \;\; \text{such that} \quad s^\star = s + \lambda \tau^\star \end{array}\right\} \text{a.e} \quad \text{on} \;\; \omega. \quad (5.18)$$

Proof. Using theorem 5 we can pass to the limit in (5.1), then using (5.11) to obtain (5.16).

Choosing $\hat{\phi}$ such that $\hat{\phi}_i = \hat{u_i^\varepsilon} + \psi_i$ (for $i = 1, .., n-1$) and $\psi_i \in H^1_{\Gamma_1 \cup \Gamma_L}$ in (5.1) and $\hat{\phi_n} = \hat{u_n^\varepsilon}$, where $H^1_{\Gamma_1 \cup \Gamma_L} = \{v \in H^1(\Omega) : v = 0 \;\; \text{on} \; \Gamma_1 \cup \Gamma_L\}$. We obtain

$$\sum_{i,j=1}^{n-1} \int_\Omega \Big[\varepsilon^2 \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} + \frac{\partial \hat{u_j^\varepsilon}}{\partial x_i}\Big) - \hat{p^\varepsilon} \delta_{i,j}\Big] \frac{\partial \psi_i}{\partial x_j} \; dx'dz + \\ + \sum_{i=1}^{n-1} \int_\Omega \mu \Big(\frac{\partial \hat{u_i^\varepsilon}}{\partial z} + \varepsilon^2 \frac{\partial \hat{u_n^\varepsilon}}{\partial x_i}\Big) \frac{\partial}{\partial z} \psi_i dx'dz + \\ + \varepsilon^{\frac{5}{2}} \sum_{i,j=1}^{n-1} \int_\Omega \hat{u_i^\varepsilon} \frac{\partial \hat{u_i^\varepsilon}}{\partial x_j} \psi_j dx'dz + \varepsilon^{\frac{9}{2}} \sum_{j=1}^{n-1} \int_\Omega \hat{u_n^\varepsilon} \frac{\partial \hat{u_n^\varepsilon}}{\partial x_j} \psi_j dx'dz + \\ + \int_\omega \hat{k}(\mid \psi + \hat{u^\varepsilon} - s \mid - \mid \hat{u^\varepsilon} - s \mid) d\sigma \geq \sum_{i=1}^{n-1} \int_\Omega \hat{f}_i \psi_i dx'dz. \quad (5.19)$$

Using theorem 4, we can pass to the limit in (5.19), we obtain

$$\sum_{i=1}^{n-1} \int_\Omega -p^\star \frac{\partial \psi_i}{\partial x_i} dx' dz + \sum_{i=1}^{n-1} \int_\Omega \mu \frac{\partial u_i^\star}{\partial z} \frac{\partial \psi_i}{\partial z} dx' dz + \\ + \int_\omega \hat{k}(\mid \psi + s^\star - s \mid - \mid s^\star - s \mid) d\sigma \\ \geq \sum_{i=1}^{n-1} \int_\Omega \hat{f}_i \psi_i dx' dz. \qquad (5.20)$$

using now the Green formula, equation (5.9), and the fact that $\psi_i = 0$ on $\Gamma_1 \cup \Gamma_L$ and $cos(n, x_i') = 0$ on ω, we deduce

$$\int_\omega \hat{k}(\mid \psi + s^\star - s \mid - \mid s^\star - s \mid) d\sigma - \int_\omega \mu \tau^\star \psi d\sigma \geq 0, \quad \forall \psi \in (H^1_{\Gamma_1 \cup \Gamma_L})^{n-1}. (5.21)$$

This inequality remains valid for any $\psi \in (D(\omega))^{n-1}$, and by density of $D(\omega)$ in $L^2(\omega)$, it remains also valid for any $\psi \in (L^2(\omega))^{n-1}$. Whence (5.17) follows.

To prove (5.18), we take $\psi = \pm(s^\star - s)$, in (5.17), and we obtain

$$\int_\omega (\hat{k} \mid s^\star - s \mid - \mu \tau^\star (s^\star - s)) d\sigma = 0, \qquad (5.22)$$

we take $\psi = \varphi - (s^\star - s)$ with $\varphi \in (L^2(\omega))^{n-1}$, in (5.17), we get

$$\int_\omega (\hat{k} \mid \varphi \mid - \mu \tau^\star \varphi) d\sigma \geq \int_\omega (\hat{k} \mid s^\star - s \mid - \mu \tau^\star (s^\star - s)) d\sigma. \qquad (5.23)$$

from equality (5.22), we deduce

$$\int_\omega (\hat{k} \mid \varphi \mid - \mu \tau^\star \varphi) d\sigma \geq 0, \quad \forall \varphi \in (L^2(\omega))^{n-1}, \qquad (5.24)$$

taking first $\varphi = (\varphi_1, .., \varphi_{n-1})$ such that $\varphi_i \geq 0, \quad i = 1, .., n-1$, in (5.24), we obtain

$$\int_\omega (\hat{k} \mid \varphi \mid - \mu \mid \tau^\star \mid cos(\tau^\star, \varphi)) \mid \varphi \mid d\sigma = \\ = \int_\omega (\hat{k} - \mu \mid \tau^\star \mid cos(\tau^\star, \varphi)) \mid \varphi \mid \geq 0, \qquad (5.25)$$

then

$$\mu \mid \tau^\star \mid cos(\tau^\star, \varphi) \leq \hat{k} \ \text{ a.e on } \ \omega. \qquad (5.26)$$

taking now $-\varphi$, with $\varphi_i \geq 0$ for $i = 1, .., n-1$, and (5.24), we obtain

$$\int_\omega (\hat{k} \mid \varphi \mid +\mu \mid \tau^\star \mid cos(\tau^\star, \varphi) \mid \varphi \mid) d\sigma = \int_\omega (\hat{k} + \mu \mid \tau^\star \mid cos(\tau^\star, \varphi)) \mid \varphi \mid \geq 0,$$

whence

$$\mu \mid \tau^\star \mid cos(\tau^\star, \varphi) \leq -\hat{k} \;\text{ a.e on }\; \omega. \tag{5.27}$$

>From (5.26)-(5.27) we obtain

$$\mu \mid \tau^\star \mid \leq \hat{k} \;\text{ a.e on }\; \omega, \tag{5.28}$$

then

$$\hat{k} \mid s^\star - s \mid \geq \mu \mid \tau^\star \mid . \mid s^\star - s \mid \geq \mu \tau^\star .(s^\star - s) \;\text{ a.e on }\; \omega,$$
$$\hat{k} \mid s^\star - s \mid - \mu \tau^\star .(s^\star - s) \geq 0 \;\text{ a.e on }\; \omega,$$

from (5.22) we deduce

$$\hat{k} \mid s^\star - s \mid - \mu \tau^\star .(s^\star - s) = 0. \tag{5.29}$$

If $\mu \mid \tau^\star \mid = \hat{k}$, then from (5.29) we have

$$\mu \mid \tau^\star \mid . \mid s^\star - s \mid = \mu \tau^\star .(s^\star - s) = 0 \text{ a.e on } \omega;$$

then $cos(s^\star - s, \mu \tau^\star) = 1$, which implies existence of $\lambda \geq 0$ such that $s^\star - s = \lambda \mu \tau^\star$.

If $\mu \mid \tau^\star \mid < \hat{k}$, from (5.29) we have

$$\hat{k} \mid s^\star - s \mid - \mu \tau^\star .(s^\star - s) = 0 \geq (\hat{k} - \mid \mu \tau^\star \mid) \mid s^\star - s \mid \text{ a.e on } \omega,$$

whence $s^\star - s = 0$ a.e on ω, then (5.18) follows.

Theorem 7 The pair $(u^\star, p^\star)$ in theorem 4, following weak form of the Reynolds equation

$$\int_\omega (\frac{h^3}{12\mu} \nabla p^\star - \frac{h}{2} s^\star + \tilde{F}) \nabla \phi dx' = - \int_{\partial\omega} \phi h \tilde{\hat{G}}.n \;\; \forall \phi \in H^1(\omega), \tag{5.30}$$

where $\tilde{F}(x') = (\tilde{F}_1(x'), \tilde{F}_2(x'))$ with

$$\tilde{F}_i(x') = \frac{1}{\mu} \int_0^{h(x')} F_i(x', z) dz - \frac{h}{2\mu} F_i(x', h(x'))$$
$$F_i(x', z) = \int_0^z \int_0^\eta \hat{f}_i(x', t) dt d\eta \;\; ; \;\; \tilde{\hat{G}} = \frac{1}{h(x')} \int_0^{h(x')} \hat{G}(x', z) dz.$$

Proof. Integrate twice (5.9) between 0 and z, we obtain

$$\mu u^\star(x',z) = \frac{z^2}{2}\nabla p^\star(x') + \mu s^\star(x') + \mu z \tau^\star(x') - F(x',z), \tag{5.31}$$

As $u^\star(x',h(x')) = 0$, in particular

$$\frac{h^2}{2}\nabla p^\star(x') + \mu s^\star(x') + \mu h(x')\tau^\star(x') - F(x',h(x')) = 0. \tag{5.32}$$

Integrate equation (5.31) with respect to z, in the interval $(0,h(x'))$, we obtain

$$h\mu\tilde{u^\star}(x') = \frac{h^3}{6}\nabla p^\star(x') + \mu h s^\star(x') + \mu\frac{h^2}{2}\tau^\star(x') - h\tilde{F}(x'), \tag{5.33}$$

where for any function ψ, we set

$$\tilde{\psi}(x') = \frac{1}{h(x')}\int_0^{h(x')}\psi(x',z)dz, \quad \forall x' \in \omega.$$

In the other hand, for all $\phi \in H^1(\omega)$, we have

$$\begin{aligned}\int_\Omega \phi div(\hat{u^\varepsilon})dx'dz = 0 &= \int_\omega \phi(x')\int_0^{h(x')}\Big(\sum_{i=1}^{n-1}\frac{\partial\hat{u_i^\varepsilon}}{\partial x_i} + \frac{\partial\hat{u_n^\varepsilon}}{\partial z}\Big)dx'dz \\ &= \int_\omega \phi(x')\Big(\sum_{i=1}^{n-1}\frac{\partial(h\tilde{\hat{u_i^\varepsilon}})}{\partial x_i} + \hat{u_n^\varepsilon}(x',h(x')) - \hat{u_n^\varepsilon}(x',0)\Big)dx'\end{aligned}$$

as $\hat{u_n^\varepsilon}(x',0) = 0$ on $\partial\Omega = \bar{\omega}\cup\bar{\Gamma}_1\cup\bar{\Gamma}_L$, we obtain

$$\int_\omega \phi(x')\int_0^{h(x')}\sum_{i=1}^{n-1}\frac{\partial(h\tilde{\hat{u_i^\varepsilon}})}{\partial x_i}dx' = 0.$$

Using Green formula, we get

$$\sum_{i=1}^{n-1}\int_\omega h\tilde{\hat{u_i^\varepsilon}}\frac{\partial\phi}{\partial x_i}dx' = \sum_{i=1}^{n-1}\int_{\partial\omega} h\tilde{\hat{u_i^\varepsilon}}\phi.cos(n,x_i') = \sum_{i=1}^{n-1}\int_{\partial\omega} h\tilde{\hat{G}}_i\phi.cos(n,x_i').$$

As $\hat{u_i^\varepsilon} \rightharpoonup u_i^\star$ in V_z consequently in $L^2(\omega)$, therefore $\tilde{\hat{u_i^\varepsilon}} \rightharpoonup u_i^\star$ in $L^2(\omega)$, and as $\partial\omega \in \partial\Omega$, we deduce

$$\sum_{i=1}^{n-1}\int_\omega h\tilde{u_i^\star}\frac{\partial\phi}{\partial x_i}dx' = \sum_{i=1}^{n-1}\int_{\partial\omega}\phi h\tilde{\hat{G}}_i.cos(n,x_i') \;\; \forall\phi \in H^1(\omega). \tag{5.34}$$

Now, multiplying (5.33) by $\nabla\phi$, then integrate it in ω, using (5.34), we obtain

$$\int_\omega\Big(\frac{h^3}{6\mu}\nabla p^\star + hs^\star + \frac{h^2}{2}\tau^\star - \frac{h}{\mu}\tilde{F}\Big)\nabla\phi dx' = \int_{\partial\omega}\phi h\tilde{\hat{G}}.n. \tag{5.35}$$

Using (5.32) to eliminate the term containing $\tau^\star$ from (5.35), the weak form of the Reynolds (5.30) follows.

Theorem 8 *Under the same hypothesis of theorem 3, there exists a unique solution* $(u^\star, p^\star)$ *in* $\tilde{V}_z \times \left(L_0^2(\omega) \cap H^1(\omega)\right)$ *of inequality (5.16).*

Proof. Let (U^1, p^1), (U^2, p^2) two solutions of (5.16). Taking $\hat{\phi} = U^2$ and $\hat{\phi} = U^1$ respectively, as test function in (5.16). Using Lemma 5.1 [4], we get

$$\mu \int_\Omega \mid \frac{\partial(U^1 - U^2)}{\partial z} \mid^2 dx'dz \leq 0. \tag{5.36}$$

Using Poincaré's inequality, we deduce

$$\| U^1 - U^2 \|_{V_z} = 0. \tag{5.37}$$

The uniqueness of $p^\star$ in $L_0^2(\omega) \cap H^1(\omega)$ follows then from (5.30). Indeed from (5.37) $s^\star$ is unique in (5.30), then we get

$$\int_\omega \frac{h^3}{12\mu} \nabla(p^1 - p^2) \nabla\phi dx' = 0,$$

taking $\phi = p^1 - p^2$, and by Poincaré's inequality we deduce $\| p^1 - p^2 \|_{L^2(\omega)} = 0$. □

References

[1] A. Assemien, G. Bayada, *Inertial effects in the asymptotic behavior of a thin film flow.* Asymptotic Analysis 9 (1994) 177-208 North-Holland.

[2] G. Bayada, M. Boukrouche, *On a free boundary problem for Reynolds equation derived from the Stokes system with Tresca boundary conditions.* Journal of Mathematical Analysis and Applications 2003, 282, 212-231.

[3] M. Boukrouche, G. ukaszewicz, *Asymptotic analysis of solutions of a thin film lubrication problem with nonlinear boundary conditions.* International Journal of Engineering Science 2003; 41, 521-537.

[4] M. Boukrouche, R. El Mir, *Asymptotic analysis of a non-Newtonian fluid in a thin domain with Tresca law.* Nonlinear Analysis, Theory methods and Applications, 59 (2004), 85-105.

[5] G. Duvaut, 1980, *Équilibre d'un Solide Élastique avec Contact Unilatéral et Frottement de Coulomb.* Comptes Rendus de l'Académie des Sciences de Paris, 290, pp. 263–265.

[6] G. Duvaut, J.L. Lions, *Les Inéquations en Mécanique des Fluides*, Dunod, 1969.

[7] R. El Mir, *Phd thesis in preparation.* Laboratory of Mathematics of University of Saint-Etienne.

[8] V.Girault, P.A Raviart, *Finite Element Approximation of the Navier Stokes Equations*, Springer-Verlag, 1979.

[9] R. Pit, H. Hervet, L. Lger, 2001, *Direct experimental evidences for flow with slip at hexadecane solid interfaces.* La revue de Métallurgie-CIT/Science.

Received: 1 December, 2004

Laboratory of Mathematics of
University of Saint-Etienne,
23 Dr. Paul Michelon 42023
Saint-Etienne, France.
E-mail: `Mahdi.Boukrouche@univ-st-etienne.fr`
E-mail: `Rachid.Elmir@univ-st-etienne.fr`