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Sequentially Cohen–Macaulayness under Foxby equivalence

by
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Abstract

This paper studies the behavior of the subcategory of sequentially Cohen-Macaulay modules under Foxby equivalence.

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1 Introduction

Let R be a commutative Noetherian ring with identity, and let C be a semidualizing module of R. We denote by $\mathscr{A}_C(R)$ and $\mathscr{B}_C(R)$ the Auslander and Bass classes of R with respect to C, respectively. These classes are known as Foxby classes. Foxby equivalence states that the functors $C \otimes_R -$ and $\operatorname{Hom}_R(C, -)$ provide inverse equivalences between the Auslander and Bass classes. This fact is illustrated by the following diagram:

$$\mathscr{A}_C(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R). \tag{1}$$

Let I be a proper ideal of R, and let M be a finitely generated R-module. We say M is *Cohen-Macaulay with respect to* I if either $M = IM$ or $M \neq IM$ and $\operatorname{grade}(I, M) = \operatorname{cd}(I, M)$. Let n be a non-negative integer. We denote by $\mathrm{CM}_I^n(R)$ the full subcategory of Cohen-Macaulay R-modules M with respect to I such that $\operatorname{cd}(I, M) = n$. Corollary 2.8 (cf. [8, Theorem 6.3]) shows that the equivalence (1) restricts to an equivalence:

$$\mathscr{A}_C(R) \bigcap \mathrm{CM}_I^n(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R) \bigcap \mathrm{CM}_I^n(R). \tag{2}$$

Thus, the class of Cohen-Macaulay R-modules with respect to I behaves well with respect to Foxby equivalence. Next, we will consider the class of sequentially Cohen-Macaulay R-modules with respect to I as a generalization of the class of Cohen-Macaulay R-modules with respect to I. Let I be a proper ideal of R, and let M be a finitely generated R-module. A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M by submodules M_i is called a *Cohen–Macaulay filtration with respect to* I if each quotient M_i/M_{i-1} is Cohen–Macaulay with respect to I and $0 \leq \operatorname{cd}(I, M_1/M_0) < \operatorname{cd}(I, M_2/M_1) < \cdots < \operatorname{cd}(I, M_r/M_{r-1})$. If

M admits a Cohen–Macaulay filtration with respect to I, then we say M is *sequentially Cohen–Macaulay with respect to* I. Let $(R,\mathfrak{m})$ be a local ring. Then, one observes that M is sequentially Cohen–Macaulay if and only if M is sequentially Cohen–Macaulay with respect to $\mathfrak{m}$. This notion has been central in many papers in the literature since the late 1990s. Here are some of the key references, ([3], [5], [7], [14], [15]). For any non-negative integer n, we denote by $\mathrm{sCM}_I^n(R)$ the full subcategory of all sequentially Cohen-Macaulay modules M with respect to I such that $\mathrm{cd}\,(I,M)=n$. We may ask whether Equivalence (2) extends to the following equivalence:

Question 1.1. *Let C be a semidualizing module of R. Is there an equivalence of categories:*

$$\mathscr{A}_C\,(R)\bigcap sCM_I^n(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C\,(R)\bigcap sCM_I^n(R)?$$

There is a positive answer to Question 1.1 if M is a finitely generated R-module with $\mathrm{cd}(I,M)\leq 1$, as shown in Remark 3.2. However, we provide an answer to Question 1.1 in Proposition 3.6 and Proposition 3.8 by an extra assumption.

Let $\mathscr{F}\colon\ 0=M_0\subsetneq M_1\subsetneq\cdots\subsetneq M_r=M$ be the dimension filtration of M with respect to I, and $M_i\in\mathscr{A}_C(R)$ for $i=1,\ldots,r$, then the following conditions are equivalent:

(a) M is sequentially Cohen–Macaulay with respect to I;

(b) $C\otimes_R M$ is sequentially Cohen–Macaulay with respect to I.

See Proposition 3.6. Corollary 3.7 presents the ordinary case of this result when $(R,\mathfrak{m})$ is a local ring. In addition, let $\mathscr{G}\colon\ 0=N_0\subsetneq N_1\subsetneq\cdots\subsetneq N_r=N$ be the dimension filtration of N with respect to I, and $N_i\in\mathscr{B}_C(R)$ for $i=1,\ldots,r$, then the following conditions are equivalent:

(a) N is sequentially Cohen–Macaulay with respect to I;

(b) $\mathrm{Hom}_R(C,N)$ is sequentially Cohen–Macaulay with respect to I.

See Proposition 3.8. The ordinary case of this result is provided by Corollary 3.9 when $(R,\mathfrak{m})$ is a local ring.

In our final section, we consider modules with maximal depth as a generalization of sequentially Cohen-Macaulay modules, see [13] and [16]. Let $(R,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. We say M has maximal depth if there is an associated prime $\mathfrak{p}$ of M such that $\operatorname{depth} M=\dim R/\mathfrak{p}$. Let $\mathscr{M}\mathscr{D}\,(R)$ denote the full subcategory of R-modules with maximal depth. In terms of categories, there is an equivalence:

$$\mathscr{A}_C\,(R)\cap\mathscr{M}\mathscr{D}\,(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C\,(R)\cap\mathscr{M}\mathscr{D}\,(R)\,,$$

as shown in Proposition 4.1. Finally, we consider unmixed R-modules with respect to I as a generalization of Cohen–Macaulay R-modules with respect to I. As before, let I be a proper ideal of R and M a finitely generated R-module. We say M is *unmixed with respect to* I if $\mathrm{cd}(I,M)=\mathrm{cd}(I,R/\mathfrak{p})$ for all $\mathfrak{p}\in\mathrm{Ass}_R\,M$. This concept behaves well with

Foxby equivalence. Indeed, for any non-negative integer n, let $\mathscr{U}_I^n(R)$ denote the full subcategory of unmixed R-modules with respect to I such that $\operatorname{cd}(I, M) = n$. Then there is an equivalence of categories:

$$\mathscr{A}_C(R) \cap \mathscr{U}_I^n(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R) \cap \mathscr{U}_I^n(R),$$

as proven in Proposition 4.1.

2 Background on Foxby classes and sequentially Cohen–Macaulay

This section provides some background on Foxby classes and sequentially Cohen–Macaulay modules used throughout the paper.

Definition 2.1. The R-module C is *semidualizing* if it satisfies the following conditions:

(a) C is finitely generated,

(b) the homothety map $\chi_C^R : R \to \operatorname{Hom}_R(C, C)$ is an isomorphism, and

(c) $\operatorname{Ext}_R^i(C, C) = 0$ for all $i \geq 1$.

Fact 2.2. Let C be a semidualizing R-module and M a finitely generated R-module. Then

(a) $\operatorname{Supp}_R C \otimes_R M = \operatorname{Supp}_R M = \operatorname{Supp}_R \operatorname{Hom}_R(C, M)$ and hence,

(b) $C \otimes_R M \neq 0 \iff M \neq 0 \iff \operatorname{Hom}_R(C, M) \neq 0$.

See [1, Lemma 3.1].

The classes defined next are known as Foxby classes.

Definition 2.3. Let C be a finitely generated R-module. The *Auslander class* $\mathscr{A}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(a) the natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism, and

(b) $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, C \otimes_R M)$ for all $i \geq 1$.

The *Bass class* $\mathscr{B}_C(R)$ is the class of all R-modules N satisfying the following conditions:

(a) the evaluation map $\xi_C^N : C \otimes_R \operatorname{Hom}_R(C, N) \to N$ is an isomorphism, and

(b) $\operatorname{Ext}_R^i(C, N) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, N))$ for all $i \geq 1$.

It is worth noting that $R \in \mathscr{A}_C(R)$ and $C \in \mathscr{B}_C(R)$. Moreover, there is an equivalence of categories called Foxby equivalence:

$$\mathscr{A}_C(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R),$$

as shown in [17, Theorem 3.2.1]. We also require the following property of Foxby classes.

Fact 2.4. Let C be a semidualizing R-module, and consider an exact sequence of R-module homomorphisms $0 \to M_1 \to M_2 \to M_3 \to 0$.

(a) If two of the M_i's are in $\mathscr{A}_C(R)$, then so is the third.

(b) If two of the M_i's are in $\mathscr{B}_C(R)$, then so is the third.

See, [17, Proposition 3.1.7].

Let I be an ideal of R and M an R-module. We denote by $\operatorname{cd}(I, M)$ the *cohomological dimension of M with respect to I* which is the largest integer i for which $\mathrm{H}^i_I(M) \neq 0$.

Fact 2.5. The following statements hold:

(a) $\operatorname{cd}(I, M) = \max\{\operatorname{cd}(I, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}_R M\}$, see [4, Corollary 4.6].

(b) The exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated R-modules yields $\operatorname{cd}(I, M) = \max\{\operatorname{cd}(I, M'), \operatorname{cd}(I, M'')\}$, see [4, Proposition 4.4].

Fact 2.2(a) together with Fact 2.5(a) yield

$$\operatorname{cd}(I, C \otimes_R M) = \operatorname{cd}(I, M) = \operatorname{cd}(I, \operatorname{Hom}_R(C, M)). \tag{3}$$

In particular, $\operatorname{cd}(I, R) = \operatorname{cd}(I, C)$.

The following fact is derived from [1, Lemma 3.2].

Fact 2.6. Let C be a semidualizing module of R and I be an ideal of R. Assume that M and N are two finitely generated R-modules with $M \in \mathscr{A}_C(R)$ and $N \in \mathscr{B}_C(R)$. Then

(a) $\operatorname{Ass}_R M = \operatorname{Ass}_R C \otimes_R M$.

(b) $\operatorname{grade}(I, M) = \operatorname{grade}(I, C \otimes_R M)$.

(c) $\operatorname{Ass}_R N = \operatorname{Ass}_R \operatorname{Hom}_R(C, N)$.

(d) $\operatorname{grade}(I, N) = \operatorname{grade}(I, \operatorname{Hom}_R(C, N))$.

In particular, we have $\operatorname{Ass}_R R = \operatorname{Ass}_R C$ and $\operatorname{grade}(I, R) = \operatorname{grade}(I, C)$.

Definition 2.7. *Let I be a proper ideal of R and M a finitely generated R-module. We say M is Cohen–Macaulay with respect to I if either $M = IM$ or $M \neq IM$ and* $\operatorname{grade}(I, M) = \operatorname{cd}(I, M)$*, as introduced in [10].*

Let n be a non-negative integer. Let $\mathrm{CM}^n_I(R)$ denote the full subcategory of Cohen-Macaulay R-modules M with respect to I with $\operatorname{cd}(I, M) = n$. In view of (3) and Fact 2.6, if $M \in \mathscr{A}_C(R)$, then M is Cohen–Macaulay with respect to I if and only if $C \otimes_R M$ is Cohen–Macaulay with respect to I. Moreover, if $N \in \mathscr{B}_C(R)$, then N is Cohen–Macaulay with respect to I if and only if $\operatorname{Hom}_R(C, N)$ is Cohen–Macaulay with respect to I. In particular, R is Cohen–Macaulay with respect to I if and only if C is Cohen–Macaulay with respect to I. These observations, together with Foxby equivalence, yield the following equivalence of categories:

Corollary 2.8. *Assume that C is a semidualizing module of R. Then the following categories are equivalent:*

$$\mathscr{A}_C(R) \bigcap CM_I^n(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R) \bigcap CM_I^n(R).$$

This result is also shown in [8, Theorem 6.3].

Let I be an ideal of R and M a non-zero finitely generated R-module. There is a unique largest submodule N of M for which $\mathrm{cd}(I,N) < \mathrm{cd}(I,M)$. To see this, let $\sum$ be the set of all submodules K of M such that $\mathrm{cd}(I,K) < \mathrm{cd}(I,M)$. As M is a Noetherian R-module, $\sum$ has a maximal element with respect to inclusion, say N. Let T be an arbitrary element in $\sum$. Fact 2.5(b) implies that $\mathrm{cd}(I,T+N) < \mathrm{cd}(I,M)$; hence, the maximality of N yields $T \subseteq N$.

Definition 2.9. A filtration $\mathscr{D}$: $0 = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_r = M$ of M by submodules D_i is called the *dimension filtration of M with respect to I* if D_{i-1} is the largest submodule of D_i for which $\mathrm{cd}(I,D_{i-1}) < \mathrm{cd}(I,D_i)$ for all $i = 1,\ldots,r$.

We recall the following fact which will be used in the sequel.

Fact 2.10. Let $\mathscr{D}$ be the dimension filtration of M with respect to I. Then

$$\mathrm{Ass}_R\, D_i/D_{i-1} = \{\mathfrak{p} \in \mathrm{Ass}_R\, M : \mathrm{cd}(I,R/\mathfrak{p}) = \mathrm{cd}(I,D_i)\}.$$

This fact is proved in the same way as the proof of [12, Lemma 1.5] by replacing the ring R and the general ideal I with the polynomial ring S and the ideal Q, respectively.

Let I be a proper ideal of R and M a finitely generated R-module. A finite filtration $\mathcal{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M by submodules M_i is called a *Cohen–Macaulay filtration with respect to I* if each quotient M_i/M_{i-1} is Cohen–Macaulay with respect to I and $0 \le \mathrm{cd}(I,M_1/M_0) < \mathrm{cd}(I,M_2/M_1) < \cdots < \mathrm{cd}(I,M_r/M_{r-1})$. If M admits a Cohen–Macaulay filtration with respect to I, then we say M is *sequentially Cohen–Macaulay with respect to I*. Note that if M is sequentially Cohen–Macaulay with respect to I, then the filtration $\mathscr{F}$ is uniquely determined and it is just the dimension filtration of M with respect to I, that is, $\mathscr{F} = \mathscr{D}$. The proof of this fact is the same as the proof of [11, Proposition 2.9] by replacing the ring R and the general ideal I with the polynomial ring S and the ideal Q, respectively. If $(R,\mathfrak{m})$ is a local ring and M a finitely generated R-module, then M is sequentially Cohen–Macaulay if and only if M is sequentially Cohen–Macaulay with respect to $\mathfrak{m}$.

3 Sequentially Cohen–Macaulayness

Let n be a non-negative integer, I an ideal of R, and M a finitely generated R-module. We denote by $\mathrm{sCM}_I^n(R)$ the full subcategory of all sequentially Cohen-Macaulay modules with respect to I such that $\mathrm{cd}\,(I,M) = n$. In this section, we address the following question:

Question 3.1. *Let C be a semidualizing module of R. Is there an equivalence of categories*

$$\mathscr{A}_C(R) \bigcap sCM_I^n(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R) \bigcap sCM_I^n(R)?$$

Remark 3.2. Let M be a finitely generated R-module with $\operatorname{cd}(I,M) \leq 1$. Then, M is sequentially Cohen–Macaulay with respect to I. To show this, we may assume that M is not Cohen–Macaulay with respect to I. Thus $\operatorname{grade}(I,M) = 0$ and $\operatorname{cd}(I,M) = 1$. The filtration $0 \subsetneq \mathrm{H}^0_I(M) \subsetneq M$, is a Cohen–Macaulay filtration with respect to I. Now, let C be a semidualizing R-module. Based on (3), $\operatorname{cd}(I, C \otimes_R M) = \operatorname{cd}(I,M)$. Thus, by using Foxby equivalence "$M \in \mathscr{A}_C(R)$ and $\operatorname{cd}(I,M) \leq 1$" is equivalent to say that "$C \otimes_R M \in \mathscr{B}_C(R)$ and $\operatorname{cd}(I, C \otimes_R M) \leq 1$". Consequently, Question 3.1 has a positive answer in this case.

We make an additional assumption in order to answer Question 3.1. First, we need to prove the following lemmas. Note that the converse of Lemma 3.3 also holds, see Remark 3.5.

Lemma 3.3. *Let C be a semidualizing R-module, and let $\mathscr{F}\colon 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be a filtration of M. Assume that $M_i \in \mathscr{A}_C(R)$ for $i = 1,\dots,r$. If $\mathscr{F}$ is the dimension filtration of M with respect to I, then $C \otimes_R \mathscr{F}$ is the dimension filtration of $C \otimes_R M$ with respect to I.*

Proof. Since C is a semidualizing R-module, $C \otimes_R M_i \neq 0$ for $i = 1,\dots,r$ by Fact 2.2(b). Moreover, as $M_i \in \mathscr{A}_C(R)$ for $i = 1,\dots,r$, the exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ yields $M_i/M_{i-1} \in \mathscr{A}_C(R)$ for $i = 1,\dots,r$ by Fact 2.4(a). Thus, we have the following exact sequence of R-modules

$$0 = \operatorname{Tor}^R_1(C, M_i/M_{i-1}) \to C \otimes_R M_{i-1} \to C \otimes_R M_i \to C \otimes_R (M_i/M_{i-1}) \to 0. \tag{4}$$

Consequently, we obtain the following filtration

$$C \otimes_R \mathscr{F} : 0 = C \otimes_R M_0 \subsetneq C \otimes_R M_1 \subsetneq \cdots \subsetneq C \otimes_R M_r = C \otimes_R M.$$

The strict inclusions follow from the fact that

$$\operatorname{cd}(I, C \otimes_R M_{i-1}) = \operatorname{cd}(I, M_{i-1}) < \operatorname{cd}(I, M_i) = \operatorname{cd}(I, C \otimes_R M_i).$$

Here, the first equality is by (3), and the second inequality follows from our assumption that $\mathscr{F}$ is the dimension filtration of M with respect to I. Moreover, from (4), we obtain

$$(C \otimes_R M_i)/(C \otimes_R M_{i-1}) \cong C \otimes_R (M_i/M_{i-1}), \tag{5}$$

for $i = 1,\dots,r$. To complete our proof, we need to show that $C \otimes_R M_{i-1}$ is the largest submodule of $C \otimes_R M_i$ such that $\operatorname{cd}(I, C \otimes_R M_{i-1}) < \operatorname{cd}(I, C \otimes_R M_i)$. Let L be the largest submodule of $C \otimes_R M_i$ for which $\operatorname{cd}(I,L) < \operatorname{cd}(I, C \otimes_R M_i)$. We want to show that $C \otimes_R M_{i-1} = L$. By Fact 2.5(b), we have

$$\operatorname{cd}(I, L + (C \otimes_R M_{i-1})) = \max\{\operatorname{cd}(I,L), \operatorname{cd}(I, C \otimes_R M_{i-1})\} < \operatorname{cd}(I, C \otimes_R M_i).$$

Thus, the maximality of L yields $L = L + (C \otimes_R M_{i-1})$. Hence, $C \otimes_R M_{i-1} \subseteq L \subsetneq C \otimes_R M_i$. On the contrary, suppose that $C \otimes_R M_{i-1} \neq L$. Observe that

$$\begin{aligned} \emptyset \neq \operatorname{Ass}_R L/(C \otimes_R M_{i-1}) &\subseteq \operatorname{Ass}_R (C \otimes_R M_i)/(C \otimes_R M_{i-1}) \\ &= \operatorname{Ass}_R C \otimes_R (M_i/M_{i-1}) \\ &= \operatorname{Ass}_R M_i/M_{i-1} \\ &= \{\mathfrak{p} \in \operatorname{Ass}_R M : \operatorname{cd}(I, R/\mathfrak{p}) = \operatorname{cd}(I, M_i)\}. \end{aligned}$$

The exact sequence

$$0 \to L/(C \otimes_R M_{i-1}) \to (C \otimes_R M_i)/(C \otimes_R M_{i-1})$$

yields the first step in this sequence. The second step is by (5). Fact 2.6(a) implies the third step. The last step follows from Fact 2.10. Thus, there exists $\mathfrak{q} \in \operatorname{Ass}_R L/(C \otimes_R M_{i-1})$ such that $\operatorname{cd}(I, R/\mathfrak{q}) = \operatorname{cd}(I, M_i)$. Observe that

$$\begin{aligned} \operatorname{cd}(I, C \otimes_R M_i) &= \operatorname{cd}(I, M_i) \\ &= \operatorname{cd}(I, R/\mathfrak{q}) \\ &\leq \operatorname{cd}(I, L/(C \otimes_R M_{i-1})) \\ &\leq \operatorname{cd}(I, L). \end{aligned}$$

The first step is by (3). The exact sequence

$$0 \to R/\mathfrak{q} \to L/(C \otimes_R M_{i-1})$$

yields $\operatorname{cd}(I, R/\mathfrak{q}) \leq \operatorname{cd}(I, L/(C \otimes_R M_{i-1}))$ by Fact 2.5(b). Thus, the third step follows. The exact sequence

$$L \to L/(C \otimes_R M_{i-1}) \to 0$$

yields $\operatorname{cd}(I, L/(C \otimes_R M_{i-1})) \leq \operatorname{cd}(I, L)$ again by Fact 2.5(b). So, the fourth step follows. Consequently, $\operatorname{cd}(I, C \otimes_R M_i) \leq \operatorname{cd}(I, L)$, a contradiction. Therefore, $C \otimes_R M_{i-1} = L$ and so the proof is complete. □

Lemma 3.4. *Let C be a semidualizing R-module, and let $\mathscr{G}\colon 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$ be a filtration of N. Assume that $N_i \in \mathscr{B}_C(R)$ for $i = 1, \ldots, r$. Then the following conditions are equivalent:*

(a) $\mathscr{G}$ is the dimension filtration of N with respect to I;

(b) $\operatorname{Hom}_R(C, \mathscr{G})$ is the dimension filtration of $\operatorname{Hom}_R(C, N)$ with respect to I.

Proof. $(a) \Rightarrow (b)$: Because C is a semidualizing R-module, $\operatorname{Hom}_R(C, N_i) \neq 0$ for $i = 1, \ldots, r$ by Fact 2.2(b). As $\operatorname{Hom}_R(C, -)$ is a left exact functor, the exact sequence $0 \to N_{i-1} \to N_i$ yields the exact sequence $0 \to \operatorname{Hom}_R(C, N_{i-1}) \to \operatorname{Hom}_R(C, N_i)$ for all i. Thus, we have the following filtration

$$\operatorname{Hom}_R(C, \mathscr{G}) : 0 = \operatorname{Hom}_R(C, N_0) \subsetneq \operatorname{Hom}_R(C, N_1) \subsetneq \cdots \subsetneq \operatorname{Hom}_R(C, N_r) = \operatorname{Hom}_R(C, N).$$

The following fact explains the strict inclusions.

$$\operatorname{cd}(I, \operatorname{Hom}_R(C, N_{i-1})) = \operatorname{cd}(I, N_{i-1}) < \operatorname{cd}(Q, N_i) = \operatorname{cd}(I, \operatorname{Hom}_R(C, N_i)).$$

Here, the first equality is by (3), and the second inequality follows from our assumption that $\mathscr{G}$ is the dimension filtration of N with respect to I. Next, we need to show that $\operatorname{Hom}_R(C, N_{i-1})$ is the largest submodule of $\operatorname{Hom}_R(C, N_i)$ for which $\operatorname{cd}(Q, \operatorname{Hom}_R(C, D_{i-1})) < \operatorname{cd}(Q, \operatorname{Hom}_R(C, D_i))$. Let L be the largest submodule of $\operatorname{Hom}_R(C, N_i)$ for which $\operatorname{cd}(I, L) < \operatorname{cd}(I, \operatorname{Hom}_R(C, N_i))$. We claim that $\operatorname{Hom}_R(C, N_{i-1}) = L$. Based on Fact 2.5(b),

$$\operatorname{cd}(I, L + \operatorname{Hom}_R(C, N_{i-1})) = \max\{\operatorname{cd}(I, L), \operatorname{cd}(I, \operatorname{Hom}_R(C, N_{i-1}))\} < \operatorname{cd}(I, N_i).$$

Thus, the maximality of L yields $L = L + \operatorname{Hom}_R(C, N_{i-1})$. Hence, $\operatorname{Hom}_R(C, N_{i-1}) \subseteq L \subsetneq \operatorname{Hom}_R(C, N_i)$. Now, on the contrary suppose $\operatorname{Hom}_R(C, N_{i-1}) \neq L$. Observe that

$$\begin{aligned}\emptyset \neq \operatorname{Ass}_R L/\operatorname{Hom}_R(C, N_{i-1}) \quad &\subseteq \operatorname{Ass}_R \operatorname{Hom}_R(C, N_i)/\operatorname{Hom}_R(C, N_{i-1}) \\ &= \operatorname{Ass}_R \operatorname{Hom}_R(C, N_i/N_{i-1}) \\ &= \operatorname{Ass}_R N_i/N_{i-1} \\ &= \{\mathfrak{p} \in \operatorname{Ass}_R N : \operatorname{cd}(I, R/\mathfrak{p}) = \operatorname{cd}(I, N_i)\}.\end{aligned}$$

The imbedding

$$0 \to L/\operatorname{Hom}_R(C, N_{i-1}) \to \operatorname{Hom}_R(C, N_i)/\operatorname{Hom}_R(C, N_{i-1})$$

yields the first step in this sequence. The second step is shown as follows. As $N_i \in \mathscr{B}_C(R)$, the exact sequence $0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0$ yields $N_i/N_{i-1} \in \mathscr{B}_C(R)$ by Fact 2.4(b). Thus, we have the following exact sequence of R-modules

$$0 \to \operatorname{Hom}_R(C, N_{i-1}) \to \operatorname{Hom}_R(C, N_i) \to \operatorname{Hom}_R(C, N_i/N_{i-1}) \to \operatorname{Ext}^1_R(C, N_{i-1}) = 0.$$

Consequently,

$$\operatorname{Hom}_R(C, N_i)/\operatorname{Hom}_R(C, N_{i-1}) \cong \operatorname{Hom}_R(C, N_i/N_{i-1}), \tag{6}$$

for $i = 1, \ldots, r$. Therefore, the second step follows. Fact 2.6(c) provides the third step. The last step is by Fact 2.10. Thus, there exists $\mathfrak{q} \in \operatorname{Ass}_R L/\operatorname{Hom}_R(C, N_{i-1})$ such that $\operatorname{cd}(I, R/\mathfrak{q}) = \operatorname{cd}(I, N_i)$. Observe that

$$\begin{aligned}\operatorname{cd}(I, \operatorname{Hom}_R(C, N_i)) \quad &= \operatorname{cd}(I, N_i) \\ &= \operatorname{cd}(I, R/\mathfrak{q}) \\ &\leq \operatorname{cd}(I, L/\operatorname{Hom}_R(C, N_{i-1})) \\ &\leq \operatorname{cd}(I, L).\end{aligned}$$

The first step is by (3). The exact sequence

$$0 \to R/\mathfrak{q} \to L/\operatorname{Hom}_R(C, N_{i-1})$$

yields $\operatorname{cd}(I, R/\mathfrak{q}) \leq \operatorname{cd}(I, L/\operatorname{Hom}_R(C, N_{i-1}))$ by Fact 2.5(b). Thus, the third step holds. The exact sequence

$$L \to L/\operatorname{Hom}_R(C, N_{i-1}) \to 0$$

yields $\operatorname{cd}(I, L/\operatorname{Hom}_R(C, N_{i-1})) \leq \operatorname{cd}(I, L)$ again by Fact 2.5(b). So, the fourth step follows. Consequently, $\operatorname{cd}(I, \operatorname{Hom}_R(C, N_i)) \leq \operatorname{cd}(I, L)$, a contradiction. Therefore, $\operatorname{Hom}_R(C, N_{i-1}) = L$ and so the proof is compelte.

$(b) \Rightarrow (a)$: Suppose $\operatorname{Hom}_R(C, \mathscr{G})$ is the dimension filtration of $\operatorname{Hom}_R(C, N)$ with respect to I. Thus, by Lemma 3.3, $C \otimes_R \operatorname{Hom}_R(C, \mathscr{G}) \cong \mathscr{G}$ is the dimension filtration of $C \otimes_R \operatorname{Hom}_R(C, N) \cong N$ with respect to I, as desired. ∎

Remark 3.5. Notice that the converse of Lemma 3.3 holds. In fact, suppose that $C \otimes_R \mathscr{F}$ is the dimension filtration of $C \otimes_R M$ with respect to I. Lemma 3.4 implies that $\operatorname{Hom}_R(C, C \otimes_R \mathscr{F}) \cong \mathscr{F}$ is the dimension filtration of $\operatorname{Hom}_R(C, C \otimes_R M) \cong M$ with respect to I, as desired.

In the following, we provide an answer to Question 3.1 under an extra assumption.

Proposition 3.6. *Let C be a semidualizing R-module, and let $\mathscr{F}\colon 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be the dimension filtration of M with respect to I. Assume that $M_i \in \mathscr{A}_C(R)$ for $i = 1, \ldots, r$. Then the following conditions are equivalent:*

(a) M is sequentially Cohen–Macaulay with respect to I;

(b) $C \otimes_R M$ is sequentially Cohen–Macaulay with respect to I.

Proof. $(a) \Rightarrow (b)$: We first assume that M is sequentially Cohen–Macaulay with respect to I. Thus, the dimension filtration $\mathscr{F}$ is a Cohen–Macaulay filtration of M with respect to I. By Lemma 3.3,

$$C \otimes_R \mathscr{F} : 0 = C \otimes_R M_0 \subsetneq C \otimes_R M_1 \subsetneq \cdots \subsetneq C \otimes_R M_r = C \otimes_R M$$

is the dimension filtration of $C \otimes M$ with respect to I. We claim that $C \otimes_R \mathscr{F}$ is a Cohen–Macaulay filtration with respect to I for $C \otimes_R M$. We first show that $(C \otimes_R M_i)/(C \otimes_R M_{i-1})$ is Cohen–Macaulay with respect to I for all i. Observe that

$$\begin{aligned}
\operatorname{grade}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) &= \operatorname{grade}(I, C \otimes_R (M_i/M_{i-1})) \\
&= \operatorname{grade}(I, M_i/M_{i-1}) \\
&= \operatorname{cd}(I, M_i/M_{i-1}) \\
&= \operatorname{cd}(I, C \otimes_R (M_i/M_{i-1})) \\
&= \operatorname{cd}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})).
\end{aligned}$$

The first step follows from (5). As $M_i/M_{i-1} \in \mathscr{A}_C(R)$, Fact 2.6(b) implies the second step. Our assumption explains the third step and the fourth step follows from (3). The last step is again by (5).

Next, we want to show that

$$\operatorname{cd}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) < \operatorname{cd}(I, (C \otimes_R M_{i+1})/(C \otimes_R M_i)) \quad \text{for all } i.$$

Observe that

$$\begin{aligned}
\operatorname{cd}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) &= \operatorname{cd}(I, C \otimes_R (M_i/M_{i-1})) \\
&= \operatorname{cd}(I, M_i/M_{i-1}) \\
&< \operatorname{cd}(I, M_{i+1}/M_i) \\
&= \operatorname{cd}(I, C \otimes_R (M_{i+1}/M_i)) \\
&= \operatorname{cd}(I, (C \otimes_R M_{i+1})/(C \otimes_R M_i)).
\end{aligned}$$

As, $\mathscr{F}$ is a Cohen–Macaulay filtration of M with respect to I, the third step follows, and the other steps are standard. Thus, $C \otimes_R \mathscr{F}$ is a Cohen-Macaulay filtration of $C \otimes_R M$ with respect to I, which implies that $C \otimes_R M$ is sequentially Cohen-Macaulay with respect to I. Therefore, we have shown that $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$: Suppose $C \otimes_R M$ is sequentially Cohen–Macaulay with respect to I. As $\mathscr{F}$ is the dimension filtration of M with respect to I, Lemma 3.3 says that $C \otimes_R \mathscr{F} : 0 =$

$C\otimes_R M_0 \subsetneq C\otimes_R M_1 \subsetneq \cdots \subsetneq C\otimes_R M_r = C\otimes_R M$ is the dimension filtration of $C\otimes_R M$ with respect to I. Hence, our assumption implies that $C \otimes_R \mathscr{F}$ is a Cohen–Macaulay filtration of $C \otimes_R M$ with respect to I as well. We claim that $\mathscr{F}$ is a Cohen–Macaulay filtration of M with respect to I. Observe that

$$\begin{aligned}
\operatorname{grade}(I, M_i/M_{i-1}) &= \operatorname{grade}(I, C \otimes_R (M_i/M_{i-1})) \\
&= \operatorname{grade}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) \\
&= \operatorname{cd}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) \\
&= \operatorname{cd}(I, C \otimes_R (M_i/M_{i-1})) \\
&= \operatorname{cd}(I, M_i/M_{i-1}).
\end{aligned}$$

These steps follow by a similar argument as in the first display.

Next, we want to show that $\operatorname{cd}(I, M_i/M_{i-1}) < \operatorname{cd}(I, M_{i+1}/M_i)$ for all i. Notice that

$$\begin{aligned}
\operatorname{cd}(I, M_i/M_{i-1}) &= \operatorname{cd}(I, (C \otimes_R M_i)/(C \otimes_R M_{i-1})) \\
&< \operatorname{cd}(I, (C \otimes_R M_{i+1})/(C \otimes_R M_i)) \\
&= \operatorname{cd}(I, M_{i+1}/M_i).
\end{aligned}$$

As, $C \otimes_R \mathscr{F}$ is a Cohen–Macaulay filtration of $C \otimes_R M$ with respect to I, the second step follows, and the other steps are standard. Therefore, the proof is complete. □

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a local ring, and C a semidualizing R-module. Let $\mathscr{F}$: $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be the dimension filtration of M. Assume that $M_i \in \mathscr{A}_C(R)$ for $i = 1, \ldots, r$. Then M is sequentially Cohen–Macaulay if and only if $C \otimes_R M$ is sequentially Cohen–Macaulay.*

In the following, we address Question 3.1 with an additional assumption.

Proposition 3.8. *Let C be a semidualizing R-module, and let $\mathscr{G}$: $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$ be the dimension filtration of N with respect to I. Assume that $N_i \in \mathscr{B}_C(R)$ for $i = 1, \ldots, r$. Then the following conditions are equivalent:*

(a) N is sequentially Cohen–Macaulay with respect to I;

(b) $\operatorname{Hom}_R(C, N)$ is sequentially Cohen–Macaulay with respect to I.

Proof. $(a) \Rightarrow (b)$: Suppose N is sequentially Cohen–Macaulay with respect to I. Thus, the dimension filtration $\mathscr{G}$ is a Cohen–Macaulay filtration of N with respect to I. By Lemma 3.4,

$$\operatorname{Hom}_R(C, \mathscr{G}) : 0 = \operatorname{Hom}_R(C, N_0) \subsetneq \operatorname{Hom}_R(C, N_1) \subsetneq \cdots \subsetneq \operatorname{Hom}_R(C, N_r) = \operatorname{Hom}_R(C, N),$$

is the dimension filtration of $\operatorname{Hom}_R(C, N)$ with respect to I. We claim that $\operatorname{Hom}_R(C, \mathscr{G})$ is a Cohen–Macaulay filtration of $\operatorname{Hom}_R(C, N)$ with respect to I. We set $L_i = \operatorname{Hom}_R(C, N_i)$

for all i. We first show that L_i/L_{i-1} is Cohen–Macaulay with respect to I for all i. Observe that

$$\begin{aligned}\operatorname{grade}(I, L_i/L_{i-1}) &= \operatorname{grade}(I, \operatorname{Hom}_R(C, N_i/N_{i-1}))\\ &= \operatorname{grade}(I, N_i/N_{i-1})\\ &= \operatorname{cd}(I, N_i/N_{i-1})\\ &= \operatorname{cd}(I, \operatorname{Hom}_R(C, N_i/N_{i-1}))\\ &= \operatorname{cd}(I, L_i/L_{i-1}).\end{aligned}$$

The first step follows from (6). As $N_i/N_{i-1} \in \mathscr{B}_C(R)$, Fact 2.6(d) provides the second step in this sequence. Our assumption explains the third step and the fourth step follows from (3). The last step is again by (6). Next, we want to show that $\operatorname{cd}(I, L_i/L_{i-1}) < \operatorname{cd}(I, L_{i+1}/L_i)$ for all i. Observe that

$$\begin{aligned}\operatorname{cd}(I, L_i/L_{i-1}) &= \operatorname{cd}(I, \operatorname{Hom}_R(C, N_i/N_{i-1}))\\ &= \operatorname{cd}(I, N_i/N_{i-1})\\ &< \operatorname{cd}(I, N_{i+1}/N_i)\\ &= \operatorname{cd}(I, \operatorname{Hom}_R(C, N_{i+1}/N_i))\\ &= \operatorname{cd}(I, L_{i+1}/L_i).\end{aligned}$$

Our assumption explains the third step, and the other steps are standard.

$(b) \Rightarrow (a)$: Suppose $\operatorname{Hom}_R(C, N)$ is sequentially Cohen–Macaulay with respect to I. As $\mathscr{G}$: $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$ is the dimension filtration of N with respect to I, it follows from Lemma 3.4 that

$$\operatorname{Hom}_R(C, \mathscr{G}) : 0 = \operatorname{Hom}_R(C, N_0) \subsetneq \operatorname{Hom}_R(C, N_1) \subsetneq \cdots \subsetneq \operatorname{Hom}_R(C, N_r) = \operatorname{Hom}_R(C, N),$$

is the dimension filtration of $\operatorname{Hom}_R(C, N)$ with respect to I. Hence, our assumption implies that $\operatorname{Hom}_R(C, \mathscr{G})$ is a Cohen–Macaulay filtration of $\operatorname{Hom}_R(C, N)$ with respect to I. We claim that $\mathscr{G}$: $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$ is a Cohen–Macaulay filtration of N with respect to I. As before, we set $L_i = \operatorname{Hom}_R(C, N_i)$ for all i. Observe that

$$\begin{aligned}\operatorname{grade}(I, N_i/N_{i-1}) &= \operatorname{grade}(I, \operatorname{Hom}_R(C, N_i/N_{i-1}))\\ &= \operatorname{grade}(I, L_i/L_{i-1})\\ &= \operatorname{cd}(I, L_i/L_{i-1})\\ &= \operatorname{cd}(I, \operatorname{Hom}_R(C, N_i/N_{i-1}))\\ &= \operatorname{cd}(I, N_i/N_{i-1}),\end{aligned}$$

and

$$\begin{aligned}\operatorname{cd}(I, N_i/N_{i-1}) &= \operatorname{cd}(I, L_i/L_{i-1})\\ &< \operatorname{cd}(I, L_{i+1}/L_i)\\ &= \operatorname{cd}(I, N_{i+1}/N_i),\end{aligned}$$

for all i. Therefore, the proof is complete. ∎

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a local ring, and C a semidualizing R-module. Let $\mathscr{G}\colon 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$ be the dimension filtration of N. Assume that $N_i \in \mathscr{B}_C(R)$ for $i = 1, \ldots, r$. Then N is sequentially Cohen–Macaulay if and only if $\operatorname{Hom}_R(C, N)$ is sequentially Cohen–Macaulay.*

4 Maximal depth and unmixedness

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. A basic fact in commutative algebra states that

$$\operatorname{depth}_R M \leq \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R M\},$$

see [2]. We define $\operatorname{mdepth}_R M = \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R M\}$. We say M has *maximal depth* if the equality holds, i.e., $\operatorname{depth}_R M = \operatorname{mdepth}_R M$. In other words, there is an associated prime $\mathfrak{p}$ of M such that $\operatorname{depth}_R M = \dim R/\mathfrak{p}$. The maximal depth property generalizes the concept of sequentially Cohen-Macaulayness. In fact, sequentially Cohen-Macaulay modules have maximal depth, see [13, Proposition 1.4], see also [16, Theorem 6.4.23] where the ring R is a polynomial ring. Let $\mathscr{M}\mathscr{D}\,(R)$ denote the full subcategory of all R-modules with maximal depth. Note that $\operatorname{sCM}_{\mathfrak{m}}^{\dim M}(R) \subseteq \mathscr{M}\mathscr{D}\,(R)$.

Proposition 4.1. *For any semidualizing module C of R, there exists an equivalence of categories:*

$$\mathscr{A}_C\,(R) \cap \mathscr{M}\mathscr{D}\,(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C\,(R) \cap \mathscr{M}\mathscr{D}\,(R)\,.$$

Proof. Consider the Foxby equivalence

$$\mathscr{A}_C\,(R) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C\,(R)\,.$$

For any R-module $L \in \mathscr{B}_C\,(R)$, we have $\operatorname{Hom}_R(C, L) \in \mathscr{A}_C\,(R)$, and $L \cong C \otimes_R \operatorname{Hom}_R(C, L)$. Hence, to complete the proof, it suffices to show that a finitely generated R-module $M \in \mathscr{A}_C\,(R) \cap \mathscr{M}\mathscr{D}\,(R)$ if and only if $C \otimes_R M \in \mathscr{B}_C\,(R) \cap \mathscr{M}\mathscr{D}\,(R)$. Based on Fact 2.6(a), we have

$$\begin{aligned} \operatorname{mdepth}_R M &= \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R M\} \\ &= \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}_R C \otimes_R M\} \\ &= \operatorname{mdepth}_R C \otimes_R M. \end{aligned}$$

On the other hand, Fact 2.6(b) implies that $\operatorname{depth}_R M = \operatorname{depth}_R C \otimes_R M$. Therefore, M has maximal depth if and only if $C \otimes_R M$ has maximal depth, as desired. □

Let I be a proper ideal of R and M a finitely generated R-module. We say M is *unmixed with respect to I* if $\operatorname{cd}(I, M) = \operatorname{cd}(I, R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}_R M$. Note that if I is contained in the Jacobson radical of R and M is Cohen–Macaulay with respect to I, then M is unmixed with respect to I (see [9, Proposition 2.11], also [6, Corollary 1.11]). For any non-negative integer n, let $\mathscr{U}_I^n\,(R)$ denote the full subcategory of unmixed R-modules with respect to I with $\operatorname{cd}\,(I, M) = n$. Note that $\operatorname{CM}_I^n(R) \subseteq \mathscr{U}_I^n\,(R)$.

Proposition 4.2. *Let C be a semidualizing module of R. Then there exists an equivalence of categories:*

$$\mathscr{A}_C(R) \cap \mathscr{U}_I^n(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathscr{B}_C(R) \cap \mathscr{U}_I^n(R).$$

Proof. Based on the proof of Proposition 4.1, we only need to show that a finitely generated R-module $M \in \mathscr{A}_C(R) \cap \mathscr{U}_I^n(R)$ if and only if $C \otimes_R M \in \mathscr{B}_C(R) \cap \mathscr{U}_I^n(R)$. Notice that $\mathrm{cd}(I, M) = \mathrm{cd}(I, C \otimes_R M)$ by (3), and $\mathrm{Ass}_R M = \mathrm{Ass}_R C \otimes_R M$ by Fact 2.6(a). Therefore, $M \in \mathscr{U}_I^n(R)$ if and only if $C \otimes_R M \in \mathscr{U}_I^n(R)$, as desired. $\square$

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