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On a question of Ray and Chakraborty
by

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Abstract

Let $\overline{A}_\ell(n)$ denote the number of ℓ-regular overpartitions of n. Quite recently, Ray and Chakraborty investigated the arithmetic density properties on powers of primes satisfied by $\overline{A}_\ell(n)$. Utilizing an algorithm of Radu and Sellers, they proved a congruence modulo 7 for $\overline{A}_7(n)$. Moreover, they stated without proof a congruence modulo 5 for $\overline{A}_5(n)$ and a congruence modulo 11 for $\overline{A}_{11}(n)$. For these three congruences, they asked for an elementary proof. In this paper, we establish six congruence families for these three partition functions, three of which are the corresponding generalizations of three congruences considered by Ray and Chakraborty.

Key Words: Partitions, ℓ-regular overpartitions, congruences, dissections, internal congruences.
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1 Introduction

The purpose of this paper is to establish several congruence families for ℓ-regular overpartition functions $\overline{A}_\ell(n)$ with $\ell \in \{5, 7, 11\}$ by utilizing some q-series manipulations. This not only answers a recent question posed by Ray and Chakraborty [14], but also greatly generalizes some results of them.

A partition λ of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The numbers λ_i are called the parts of the partition λ. Let $p(n)$ denote the number of partitions of n with the convention that $p(0) = 1$. The generating function of $p(n)$, derived by Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where here and throughout the rest of this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

In 1919, Ramanujan [12] discovered the following three celebrated congruences for partition function $p(n)$, namely,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

For an integer $\ell \geq 2$, a partition is called ℓ-regular if all of parts are not divisible by ℓ. In order to prove a combinatorial proof of some classical q-series identities, Corteel and Lovejoy [6] introduced the notion of overpartitions. An overpartition of n is a partition of n where the first occurrence of each distinct part may be overlined. In 2003, Lovejoy [10] introduced the ℓ-regular overpartitions in order to provide certain overpartition analogues of combinatorial generalizations of the Rogers–Ramanujan identities. An ℓ-regular overpartition of n is an overpartition of n in which all parts are not divisible by ℓ. Let $\overline{A}_\ell(n)$ denote the number of ℓ-regular overpartitions of n. The generating function of $\overline{A}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n) q^n = \frac{(-q;q)_\infty (q^\ell;q^\ell)_\infty}{(q;q)_\infty (-q^\ell;q^\ell)_\infty}. \tag{1}$$

In 2015, Andrews [2] introduced the singular overpartition function $\overline{C}_{k,i}(n)$, which denotes the number of overpartitions of n in which all parts are not divisible by k and only parts congruent to $\pm i$ modulo k may be overlined. A simple calculation implies that $\overline{C}_{3,1}(n) = \overline{A}_3(n)$ holds for any $n \geq 0$. Since then, many scholars subsequently considered congruence properties enjoyed by $\overline{A}_\ell(n)$; see, for example, Barman and Ray [3, 13], and Shen [15].

In a recent paper, Ray and Chakraborty [14, Corollary 1.2] derived the following powerful result by utilizing the theory of modular forms.

Theorem 1.1 (Ray–Chakraborty)**.** *Let $p \geq 5$ be a prime number and let k be a positive integer. Then $\overline{A}_p(n)$ is almost always divisible by p^k, namely,*

$$\lim_{X \to \infty} \frac{\#\{0 < n \leq X : \overline{A}_p(n) \equiv 0 \pmod{p^k}\}}{X} = 1. \tag{2}$$

However, the theory of modular forms used to derive (2) is not constructive and it does not give explicit Ramanujan type congruences. Therefore, Ray and Chakraborty [14, Theorem 1.3] proved the following congruence modulo 7 for $\overline{A}_7(n)$ by utilizing an algorithm of Radu and Sellers [11]:

$$\overline{A}_7(16n+4) \equiv 0 \pmod{7}. \tag{3}$$

At the end of their paper, they stated without proof the following two congruences:

$$\overline{A}_5(81n+27) \equiv 0 \pmod{5}, \tag{4}$$
$$\overline{A}_{11}(64n+48) \equiv 0 \pmod{11}. \tag{5}$$

Ray and Chakraborty [14, p. 469] remarked that (4) and (5) can also be proved by utilizing the algorithm of Radu and Sellers. Therefore, they asked whether there exists an elementary proof of (3)–(5). In this paper, we not only provide such a proof, but also generalize (3)–(5) to the corresponding congruence family.

Theorem 1.2. *For any $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{A}_5\big(3^{4\alpha+3}(3n+1)\big) \equiv 0 \pmod{5}, \tag{6}$$
$$\overline{A}_5\big(3^{4\alpha+3}(3n+2)\big) \equiv 0 \pmod{5}, \tag{7}$$
$$\overline{A}_7\big(2^{3\alpha+2}(4n+1)\big) \equiv 0 \pmod{7}, \tag{8}$$
$$\overline{A}_{11}\big(2^{10\alpha+4}(4n+3)\big) \equiv 0 \pmod{11}. \tag{9}$$

Remark. Two remarks on Theorem 1.2 are necessary.

(i) By connecting $\overline{A}_\ell(n)$ with $r_4(n)$ and $r_8(n)$, Chern [5, Theorem 2.2] proved the following congruence families for $\overline{A}_5(n)$, namely, for any $\alpha \geq 0$ and $n \geq 0$,

$$\overline{A}_5\big(p^{4\alpha+3}(pn+i)\big) \equiv 0 \pmod 5,$$

where p is an odd prime, $1 \leq i \leq p-1$, and $r_k(n)$ denotes the number of representations of n by the sum of k squares. However, our proofs of (6) and (7) rely on an internal congruence satisfied by $\overline{A}_5(n)$ (see (51) below). Therefore, our proof is different from that of Chern. We will present the proofs of (6) and (7) for completeness.

(ii) Chern [5, Theorem 2.4] also proved that for any $\alpha \geq 0$, $n \geq 0$ and $1 \leq i \leq p-1$,

$$\overline{A}_7\big(p^{6\alpha+5}(pn+i)\big) \equiv 0 \pmod 7, \tag{10}$$

where p is an odd prime such that $p \neq 7$. The congruence family (8) can be viewed as a complement of (10).

Moreover, we also find two new congruence families modulo 11 enjoyed by $\overline{A}_{11}(n)$.

Theorem 1.3. *For any $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{A}_{11}\big(3^{10\alpha+9}(3n+1)\big) \equiv 0 \pmod{11}, \tag{11}$$

$$\overline{A}_{11}\big(3^{10\alpha+9}(3n+2)\big) \equiv 0 \pmod{11}. \tag{12}$$

The rest of this paper is organized as follows. In Section 2, we collect and prove some necessary identities, these are the main ingredients in proofs of Theorems 1.2 and 1.3. The proofs of Theorems 1.2 and 1.3 are presented in Section 3. We conclude this paper with two questions.

2 Some necessary identities

In order to prove Theorems 1.2 and 1.3, we first collect some necessary identities.

For notational convenience, we write

$$E(q^k) = (q^k; q^k)_\infty.$$

We need the following 2-dissections and 3-dissections.

Lemma 2.1.

$$E(q)^4 = \frac{E(q^4)^{10}}{E(q^2)^2E(q^8)^4} - 4q\frac{E(q^2)^2E(q^8)^4}{E(q^4)^2}, \tag{13}$$

$$\frac{1}{E(q)^4} = \frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}. \tag{14}$$

Proof. The identities (13) and (14) follow from Berndt's book [4, p. 40, Entry 25]; see also [16, Lemma 2.3]. ∎

Lemma 2.2.

$$\frac{E(q)^2}{E(q^2)} = \frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}, \tag{15}$$

$$\frac{E(q^2)}{E(q)^2} = \frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3} + 2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7} + 4q^2\frac{E(q^6)^2E(q^{18})^2}{E(q^3)^6}, \tag{16}$$

$$\frac{E(q)}{E(q^2)^2} = \frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}. \tag{17}$$

Proof. The identity (15) follows from Berndt's book [4, p. 49, Corollary (i)] and the following identity

$$\varphi(-q) = \sum_{n=-\infty}^{\infty}(-1)^nq^{n^2} = \frac{E(q)^2}{E(q^2)}.$$

The identity (16) was established by Hirschhorn and Sellers [8, Theorem 1.1]. The identity (17) was proved by Hirschhorn and Sellers [9, Lemma 2.2]. ∎

Next, the U_m-operator is defined by

$$U_m\left(\sum_{n=n_0}^{\infty}a(n)q^n\right) = \sum_{n=\lceil n_0/m\rceil}^{\infty}a(mn)q^n.$$

Based on (13)–(17), we derive the following lemma which plays a vital role in the proof of Theorem 1.2.

Lemma 2.3.

$$U_2\left(\frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}}\right) = \frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}} + 48q\frac{E(q^2)^6E(q^4)^4}{E(q)^4}, \tag{18}$$

$$U_2\left(q\frac{E(q^2)^6E(q^4)^4}{E(q)^4}\right) = 4q\frac{E(q^2)^6E(q^4)^4}{E(q)^4}, \tag{19}$$

$$U_2\left(\frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}\right) = \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 160q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 1280q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}, \tag{20}$$

$$U_2\left(q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}\right) = 12q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 64q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}, \tag{21}$$

$$U_2\left(q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}\right) = q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}, \tag{22}$$

$$U_3\left(q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)}\right) = 3q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)}. \tag{23}$$

Proof. The proofs of (18)–(22) are highly similar, thus we only present the proof of (20). From (14) we find that

$$U_2\left(\frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}\right) = \frac{E(q)^{50}}{E(q^2)^{20}}U_2\left(\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^5\right)$$
$$= \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 160q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 1280q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},$$

as desired.

Next we prove (23).

It follows from (15) and (17) that

$$U_3\left(q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)}\right) = \frac{E(q^2)^5}{E(q)}U_3\left(q\left(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)\right.$$
$$\left.\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)\right).$$

After simplification, we obtain (23).

We therefore complete the proof of Lemma 2.3. ∎

The following lemma is the main ingredient in the proof of Theorem 1.3.

Lemma 2.4. *Let*

$$\alpha = \frac{E(q)^{20}}{E(q^2)^{10}}, \qquad \beta = \frac{E(q^3)^{20}}{E(q^6)^{10}}, \qquad \gamma = q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3},$$
$$\delta = q\frac{E(q)^3E(q^3)^{11}}{E(q^2)^3E(q^6)}, \qquad \varepsilon = q^2\frac{E(q^3)^{10}E(q^6)^4}{E(q)^2E(q^2)^2}, \qquad \xi = q\frac{E(q)^3E(q^6)^9}{E(q^2)^3E(q^3)^9}.$$

Then

$$U_3(\alpha) = \beta + \delta\big(-960 + 13440\xi - 5120\xi^2\big), \tag{24}$$
$$U_3(\beta) = \alpha, \tag{25}$$
$$U_3(\gamma) = \delta\big(3 - 24\xi\big), \tag{26}$$
$$U_3(\gamma\xi) = \delta\big(-2 + 17\xi - 8\xi^2\big), \tag{27}$$
$$U_3(\gamma\xi^2) = \delta\big(1 - 19\xi + 7\xi^2\big), \tag{28}$$
$$U_3(\gamma\xi^3) = \varepsilon\big(30 - 156\xi + 57\xi^2\big), \tag{29}$$
$$U_3(\delta) = \gamma\big(3 - 48\xi + 192\xi^2\big), \tag{30}$$
$$U_3(\delta\xi) = \gamma\big(-6 + 69\xi - 168\xi^2\big), \tag{31}$$
$$U_3(\delta\xi^2) = \gamma\big(1 - 87\xi + 147\xi^2 - 8\xi^3\big), \tag{32}$$
$$U_3(\varepsilon) = \gamma\big(2 + 13\xi - 16\xi^2\big), \tag{33}$$
$$U_3(\varepsilon\xi) = \gamma\big(1 - 7\xi + 19\xi^2\big), \tag{34}$$
$$U_3(\varepsilon\xi^2) = \gamma\big(10\xi - 16\xi^2 + \xi^3\big). \tag{35}$$

Proof. The proofs of (24)–(35) are more trickier than that of (23), and thus, we shall present the details here.

Proof of (24). According to (15), we find that

$$\begin{aligned}U_3(\alpha) &= U_3\left(\left(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)^{10}\right)\\ &= \frac{E(q^3)^{20}}{E(q^6)^{10}} - 960q\frac{E(q)^3E(q^3)^{11}}{E(q^2)^3E(q^6)}\\ &\quad + 13440q^2\frac{E(q)^6E(q^3)^2E(q^6)^8}{E(q^2)^6} - 5120q^3\frac{E(q)^9E(q^6)^{17}}{E(q^2)^9E(q^3)^7}.\end{aligned}$$

Proof of (25). The identity (25) follows by the definitions of U_3-operator, α and β.

Proof of (26). With the help of (16) and (17), we obtain that

$$\begin{aligned}U_3(\gamma) = &\frac{E(q)^{15}}{E(q^2)^3}U_3\left(q\left(\frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3} + 2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7} + 4q^2\frac{E(q^6)^2E(q^{18})^2}{E(q^3)^6}\right)\right.\\ &\times\left.\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)\right)\\ = &3q\frac{E(q)^{11}E(q^3)^3E(q^6)^3}{E(q^2)^7}.\end{aligned}$$

Therefore, in order to prove (26), it suffices to prove that

$$Z_1(q) := 3q\frac{E(q)^3E(q^3)^{11}}{E(q^2)^3E(q^6)} - 24q^2\frac{E(q)^6E(q^3)^2E(q^6)^8}{E(q^2)^6} - 3q\frac{E(q)^{11}E(q^3)^3E(q^6)^3}{E(q^2)^7} = 0. \quad (36)$$

Now we recall Horschhorn's version of parameterized identities (see [7, Chap. 35, Eqs. (35.1.1)–(35.1.6)]), which idea comes from [1].

$$E(q) = s^{1/2}t^{1/24}(1-2qt)^{1/2}(1+qt)^{1/8}(1+2qt)^{1/6}(1+4qt)^{1/8}, \quad (37)$$
$$E(q^2) = s^{1/2}t^{1/12}(1-2qt)^{1/4}(1+qt)^{1/4}(1+2qt)^{1/12}(1+4qt)^{1/4}, \quad (38)$$
$$E(q^3) = s^{1/2}t^{1/8}(1-2qt)^{1/6}(1+qt)^{1/24}(1+2qt)^{1/2}(1+4qt)^{1/24}, \quad (39)$$
$$E(q^4) = s^{1/2}t^{1/6}(1-2qt)^{1/8}(1+qt)^{1/2}(1+2qt)^{1/24}(1+4qt)^{1/8}, \quad (40)$$
$$E(q^6) = s^{1/2}t^{1/4}(1-2qt)^{1/12}(1+qt)^{1/12}(1+2qt)^{1/4}(1+4qt)^{1/12}, \quad (41)$$
$$E(q^{12}) = s^{1/2}t^{1/2}(1-2qt)^{1/24}(1+qt)^{1/6}(1+2qt)^{1/8}(1+4qt)^{1/24}, \quad (42)$$

where

$$s := s(q) = \frac{E(q)^2E(q^4)^2E(q^6)^{15}}{E(q^2)^5E(q^3)^6E(q^{12})^6} \qquad \text{and} \qquad t := t(q) = \frac{E(q^2)^3E(q^3)^3E(q^{12})^6}{E(q)E(q^4)^2E(q^6)^9}.$$

Substituting (37)–(42) into the right-hand side of (36), upon simplification, we obtain that

$$Z_1(q) = 0,$$

from which we obtain (26).

Proof of (27). From (17) we find that

$$\begin{aligned} U_3(\gamma\xi) &= E(q)^6E(q^2)^6 \\ &\quad\times\left(q^2\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^2\right) \\ &= -2q\frac{E(q)^{11}E(q^3)^3E(q^6)^3}{E(q^2)^7}+q^2\frac{E(q)^{14}E(q^6)^{12}}{E(q^2)^{10}E(q^3)^6}. \end{aligned}$$

In order to prove (27), we consider the following function, given by

$$\begin{aligned} Z_2(q) := &-2q\frac{E(q)^3E(q^3)^{11}}{E(q^2)^3E(q^6)}+17q^2\frac{E(q)^6E(q^3)^2E(q^6)^8}{E(q^2)^6}-8q^3\frac{E(q)^9E(q^6)^{17}}{E(q^2)^9E(q^3)^7} \\ &+2q\frac{E(q)^{11}E(q^3)^3E(q^6)^3}{E(q^2)^7}-q^2\frac{E(q)^{14}E(q^6)^{12}}{E(q^2)^{10}E(q^3)^6}. \end{aligned} \tag{43}$$

Plugging (37)–(42) into the right-hand side of (43), we obtain that

$$Z_2(q)=0.$$

The identity (27) thus follows.

Proof of (28). It follows from (15) and (17) that

$$\begin{aligned} U_3(\gamma\xi^2) &= \frac{E(q^2)^{15}}{E(q)^3}U_3\left(q^3\left(\frac{E(q^9)^2}{E(q^{18})}-2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)\right. \\ &\quad\left.\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^3\right) \\ &= q\frac{E(q)^3E(q^3)^{11}}{E(q^2)^3E(q^6)}-19q^2\frac{E(q)^6E(q^3)^2E(q^6)^8}{E(q^2)^6}+7q^3\frac{E(q)^9E(q^6)^{17}}{E(q^2)^9E(q^3)^7}. \end{aligned}$$

This proves (28).

Proof of (29). According to (15) and (17), we derive that

$$\begin{aligned} U_3(\gamma\xi^3) &= \frac{E(q^2)^{24}}{E(q)^{12}}U_3\left(q^4\left(\frac{E(q^9)^2}{E(q^{18})}-2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)^2\right. \\ &\quad\left.\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^4\right) \\ &= 30q^2\frac{E(q^3)^{10}E(q^6)^4}{E(q)^2E(q^2)^2}-156q^3\frac{E(q)E(q^3)E(q^6)^{13}}{E(q^2)^5}+57q^4\frac{E(q)^4E(q^6)^{22}}{E(q^2)^8E(q^3)^8}, \end{aligned}$$

which is nothing but (29).

Proof of (30). From (15) and (17) we find that

$$\begin{aligned} U_3(\delta) &= \frac{E(q)^{11}}{E(q^2)} U_3\Bigg(q\bigg(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)} \bigg) \\ &\quad \times \bigg(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3} \bigg) \Bigg) \\ &= 3q\frac{E(q)^{15}E(q^6)^5}{E(q^2)^9E(q^3)}. \end{aligned}$$

We next consider the following function, given by

$$\begin{aligned} Z_3(q) := & \, 3q\frac{E(q)^{15}E(q^6)^5}{E(q^2)^9E(q^3)} - 3q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3} \\ & + 48q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4} - 192q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}. \end{aligned} \tag{44}$$

Substituting (37)–(42) into the right-hand side of (44), we conclude that

$$Z_3(q) = 0.$$

The identity (30) thus follows.

Proof of (31). In view of (15) and (17), we deduce that

$$\begin{aligned} U_3(\delta\xi) &= E(q)^2E(q^2)^8 U_3\Bigg(q^2\bigg(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)} \bigg)^2 \\ &\quad \times \bigg(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3} \bigg)^2 \Bigg) \\ &= -6q\frac{E(q)^7E(q^3)^7E(q^6)}{E(q^2)^5} + 21q^2\frac{E(q)^{10}E(q^6)^{10}}{E(q^2)^8E(q^3)^2}. \end{aligned}$$

At this time, we consider the following function, defined by

$$\begin{aligned} Z_4(q) := & -6q\frac{E(q)^7E(q^3)^7E(q^6)}{E(q^2)^5} + 21q^2\frac{E(q)^{10}E(q^6)^{10}}{E(q^2)^8E(q^3)^2} + 6q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3} \\ & - 69q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4} + 168q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}. \end{aligned} \tag{45}$$

Plugging (37)–(42) into the right-hand side of (45), we obtain that

$$Z_4(q) = 0,$$

from which we obtain (31).

Proof of (32). It follows from (15) and (17) that

$$\begin{aligned}U_3(\delta\xi^2) &= \frac{E(q^2)^{17}}{E(q)^7}U_3\Bigg(q^3\left(\frac{E(q^9)^2}{E(q^{18})}-2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)^3\\ &\quad\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^3\Bigg)\\ &= q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3}-87q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4}\\ &\quad+147q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}-8q^4\frac{E(q)^8E(q^6)^{24}}{E(q^2)^{10}E(q^3)^{12}},\end{aligned}$$

which is nothing but (32).

Proof of (33). According to (16) and (17), we deduce that

$$\begin{aligned}U_3(\varepsilon) &= E(q)^{10}E(q^2)^4\\ &\quad\times U_3\Bigg(q^2\left(\frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3}+2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7}+4q^2\frac{E(q^6)^2E(q^{18})^2}{E(q^3)^6}\right)^2\\ &\quad\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^2\Bigg)\\ &= 2q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3}+13q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4}-16q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}.\end{aligned}$$

This proves (33).

Proof of (34). It follows from (16) and (17) that

$$\begin{aligned}U_3(\varepsilon\xi) &= E(q)E(q^2)^{13}\\ &\quad\times U_3\Bigg(q^3\left(\frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3}+2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7}+4q^2\frac{E(q^6)^2E(q^{18})^2}{E(q^3)^6}\right)\\ &\quad\times\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^3\Bigg)\\ &= q\frac{E(q^3)^{15}}{E(q)E(q^2)E(q^6)^3}-7q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4}+19q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}.\end{aligned}$$

This is (34).

Proof of (35). By (17) we obtain that

$$\begin{aligned}U_3(\varepsilon\xi^2) &= \frac{E(q^2)^{22}}{E(q)^8}U_3\left(q^4\left(\frac{E(q^3)^2E(q^9)^3}{E(q^6)^6}-q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7}+q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}\right)^4\right)\\ &= 10q^2\frac{E(q)^2E(q^3)^6E(q^6)^6}{E(q^2)^4}-16q^3\frac{E(q)^5E(q^6)^{15}}{E(q^2)^7E(q^3)^3}+q^4\frac{E(q)^8E(q^6)^{24}}{E(q^2)^{10}E(q^3)^{12}},\end{aligned}$$

from which we obtain (35).

This completes the proof of Lemma 2.4. ∎

3 Proofs of Theorems 1.2 and 1.3

This section is devoted to the proofs of Theorems 1.2 and 1.3.

Now it is time to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from (1) and (15) that

$$\sum_{n=0}^{\infty}\overline{A}_5(n)q^n \equiv \frac{E(q)^8}{E(q^2)^4} = \left(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)^4 \pmod{5}. \tag{46}$$

Picking out all terms of the form q^{3n} in the right-hand side of (46), after simplification, we find that

$$\sum_{n=0}^{\infty}\overline{A}_5(3n)q^n \equiv \frac{E(q^3)^8}{E(q^6)^4} + 3q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)} \pmod{5}. \tag{47}$$

Applying the U_3-operator on the both sides of (47) and utilizing (23), we derive that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_5(9n)q^n &\equiv \frac{E(q)^8}{E(q^2)^4} + 9q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)} \pmod{5}\\ &= \left(\frac{E(q^9)^2}{E(q^{18})} - 2q\frac{E(q^3)E(q^{18})^2}{E(q^6)E(q^9)}\right)^4 + 9q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)},\end{aligned} \tag{48}$$

where we have used (15) in the last equation of (48). Applying the U_3-operator on the both sides of (48) and using (23), upon simplification, we obtain that

$$\sum_{n=0}^{\infty}\overline{A}_5(27n)q^n \equiv \frac{E(q^3)^8}{E(q^6)^4} - 5q\frac{E(q)^3E(q^6)^5}{E(q^2)^3E(q^3)} \equiv \frac{E(q^3)^8}{E(q^6)^4} \pmod{5}, \tag{49}$$

from which we further conclude that

$$\overline{A}_5(81n+27)q^n \equiv \overline{A}_5(81n+54) \equiv 0 \pmod{5} \tag{50}$$

and

$$\sum_{n=0}^{\infty}\overline{A}_5(81n)q^n \equiv \frac{E(q)^8}{E(q^2)^4} \pmod{5}.$$

Thanks to (46), we have

$$\overline{A}_5(81n) \equiv \overline{A}_5(n) \pmod{5}. \tag{51}$$

By induction, one readily finds that for any $\alpha \geq 0$,

$$\overline{A}_5\big(3^{4\alpha}n\big) \equiv \overline{A}_5(n) \pmod{5}. \tag{52}$$

The congruence families (6) and (7) follow from (50) and (51).

From (1) and (13), we find that

$$\sum_{n=0}^{\infty}\overline{A}_7(n)q^n \equiv \frac{\left(E(q)^4\right)^3}{E(q^2)^6} \pmod 7$$
$$= \frac{1}{E(q^2)^6}\left(\frac{E(q^4)^{10}}{E(q^2)^2E(q^8)^4} - 4q\frac{E(q^2)^2E(q^8)^4}{E(q^4)^2}\right)^3. \tag{53}$$

Taking all terms of the form q^{2n}, after simplification, we find that

$$\sum_{n=0}^{\infty}\overline{A}_7(2n)q^n \equiv \frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}} + 6q\frac{E(q^2)^6E(q^4)^4}{E(q)^4} \pmod{7}. \tag{54}$$

Applying the U_2-operator on both sides of (54) and using (18) and (19), we get that

$$\sum_{n=0}^{\infty}\overline{A}_7(4n)q^n \equiv \frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}} + 2q\frac{E(q^2)^6E(q^4)^4}{E(q)^4} \pmod{7}. \tag{55}$$

Thanks to (14), we have, modulo 7,

$$\sum_{n=0}^{\infty}\overline{A}_7(4n)q^n \equiv \frac{E(q^2)^{30}}{E(q^4)^{12}}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^3$$
$$+ 2qE(q^2)^6E(q^4)^4\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right). \tag{56}$$

Collecting all terms of the form q^{2n+1} in the right-hand side of (56), after simplification, we deduce that

$$\sum_{n=0}^{\infty}\overline{A}_7(8n+4)q^n \equiv q\frac{E(q^4)^{12}}{E(q^2)^6} \pmod{7},$$

from which we get that

$$\overline{A}_7(16n+4) \equiv 0 \pmod{7}. \tag{57}$$

Applying the U_2-operator on both sides of (55) and utilizing (18) and (19), we have

$$\sum_{n=0}^{\infty}\overline{A}_7(8n)q^n \equiv \frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}} \pmod{7}. \tag{58}$$

Applying the U_2-operator on both sides of (58) and using (18), we find that

$$\sum_{n=0}^{\infty}\overline{A}_7(16n)q^n \equiv \frac{E(q^2)^{30}}{E(q)^{12}E(q^4)^{12}} + 6q\frac{E(q^2)^6E(q^4)^4}{E(q)^4} \pmod{7}. \tag{59}$$

It follows from (54) and (59) that

$$\overline{A}_7(16n) \equiv \overline{A}_7(2n) \pmod{7}.$$

By induction, we find that for any $\alpha \geq 0$,

$$\overline{A}_7\big(2^{3\alpha+1}n\big) \equiv \overline{A}_7(2n) \pmod 7. \tag{60}$$

The congruence family (8) follows from (57) and (60).

According to (1) and (13),

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_{11}(n)q^n &\equiv \frac{\big(E(q^4)\big)^5}{E(q^2)^{10}} \pmod{11}\\ &= \frac{1}{E(q^2)^{10}}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^5.\end{aligned} \tag{61}$$

Picking all terms of the form q^{2n} in the right-hand side of (61), upon simplification, we find that, modulo 11,

$$\sum_{n=0}^{\infty}\overline{A}_{11}(2n)q^n \equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}+6q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}+4q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}. \tag{62}$$

Next, we utilize the U_2-operator and (20)–(22) repeatedly to deduce that, modulo 11,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_{11}(4n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}+5q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}+3q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\ \sum_{n=0}^{\infty}\overline{A}_{11}(8n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}+3q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}+5q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}\\ \sum_{n=0}^{\infty}\overline{A}_{11}(16n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}}+3q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4}+9q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}.\end{aligned} \tag{63}$$

Thanks to (14), we find that, modulo 11,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_{11}(16n)q^n &\equiv \frac{E(q^2)^{50}}{E(q^4)^{20}}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^5\\ &\quad+3q\frac{E(q^2)^{26}}{E(q^4)^4}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^3\\ &\quad+9q^2E(q^2)^2E(q^4)^{12}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right).\end{aligned} \tag{64}$$

Collecting all terms of the form q^{2n+1} in the right-hand side of (64), upon simplification, we deduce that, modulo 11,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_{11}(32n+16)q^n &\equiv \frac{E(q^2)^{38}}{E(q)^{16}E(q^4)^{12}}+6q\frac{E(q^2)^{14}E(q^4)^4}{E(q)^8}+q^2\frac{E(q^4)^{20}}{E(q^2)^{10}}\\ &= \frac{E(q^2)^{38}}{E(q^4)^{12}}\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^4+q^2\frac{E(q^4)^{20}}{E(q^2)^{10}}\\ &\quad+6qE(q^2)^{14}E(q^4)^4\left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4}+4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}\right)^2.\end{aligned} \tag{65}$$

Taking all terms of the form q^{2n+1} in the right-hand side of (65), we conclude that

$$\sum_{n=0}^{\infty} \overline{A}_{11}(64n+48)q^n \equiv 0 \pmod{11},$$

or, equivalently,

$$\overline{A}_{11}(64n+48) \equiv 0 \pmod{11}. \tag{66}$$

According to the U_2-operator, (20)–(22) and (63), we obtain that, modulo 11,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{A}_{11}(32n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 7q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 9q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(64n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(128n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 7q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 4q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(256n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 6q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(512n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 2q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 3q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(1024n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}},\\
\sum_{n=0}^{\infty} \overline{A}_{11}(2048n)q^n &\equiv \frac{E(q^2)^{50}}{E(q)^{20}E(q^4)^{20}} + 6q\frac{E(q^2)^{26}}{E(q)^{12}E(q^4)^4} + 4q^2\frac{E(q^2)^2E(q^4)^{12}}{E(q)^4}.
\end{aligned} \tag{67}$$

Combining (62) and (67) yields that

$$\overline{A}_{11}(2048n) \equiv \overline{A}_{11}(2n) \pmod{11}.$$

By induction, we find that for any $\alpha \geq 0$,

$$\overline{A}_{11}\big(2^{10\alpha+1}n\big) \equiv \overline{A}_{11}(2n) \pmod{11}. \tag{68}$$

The congruence family (9) follows from (66) and (68).

This completes the proof of Theorem 1.2. ∎

Finally, we turn to prove Theorem 1.3.

Proof of Theorem 1.3. From (1) and (24), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{11}(n)q^n \equiv \alpha \pmod{11}, \tag{69}$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3n)q^n \equiv \beta + \delta\big(8 + 9\xi + 6\xi^2\big) \pmod{11}. \tag{70}$$

Next, we apply the U_3-operator and utilize (24)–(35) repeatedly, after simplification, we obtain that

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^2n)q^n \equiv \alpha + \gamma(9+\xi+4\xi^2+7\xi^3) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^3n)q^n \equiv \beta + \delta(4+9\xi+4\xi^2) + \varepsilon(1+8\xi+3\xi^2) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^4n)q^n \equiv \alpha + \gamma(5+2\xi+9\xi^2+4\xi^3) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^5n)q^n \equiv \beta + \delta(6+5\xi+9\xi^2) + \varepsilon(10+3\xi+8\xi^2) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^6n)q^n \equiv \alpha + \gamma(9+2\xi+7\xi^2+2\xi^3) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^7n)q^n \equiv \beta + \delta(5+2\xi+6\xi^2) + \varepsilon(5+7\xi+4\xi^2) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^8n)q^n \equiv \alpha + \gamma(4+4\xi+10\xi^2) \pmod{11},$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^9n)q^n \equiv \beta \pmod{11}, \tag{71}$$

$$\sum_{n=0}^{\infty} \overline{A}_{11}(3^{10}n)q^n \equiv \alpha \pmod{11}. \tag{72}$$

It follows immediately from (69) and (72) that

$$\overline{A}_{11}(3^{10}n) \equiv \overline{A}_{11}(n) \pmod{11}.$$

By induction, we deduce that for any $\alpha \geq 0$,

$$\overline{A}_{11}(3^{10\alpha}n) \equiv \overline{A}_{11}(n) \pmod{11}. \tag{73}$$

Moreover, from (71) we have

$$\overline{A}_{11}(3^9(3n+1)) \equiv \overline{A}_{11}(3^9(3n+2)) \equiv 0 \pmod{11}. \tag{74}$$

The congruence families (11) and (12) follow from (73) and (74) immediately. ∎

4 Concluding remarks

We conclude this paper with two questions.

First, a natural question is whether there exist some similar congruence families modulo 11 enjoyed by $A_{11}(n)$ with another prime $p \geq 5$.

Second, motivated by (2) and (6)–(12), it is natural to ask whether there exist some similar congruence families modulo p enjoyed by $\overline{A}_p(n)$, where $p \geq 13$ is a prime number.

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