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Melin calculus on homogeneous Lie groups

by
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Abstract

We provide a self-contained approach to two of Głowacki's theorems on convolution operators on homogeneous groups, namely the continuity of the product symbol in suitable symbol spaces, and sufficient conditions for L^2-boundedness.

Key Words: Pseudodifferential operators, homogeneous Lie groups, symbolic calculus, L^2-continuity.
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Contents

1 Introduction

This paper is an expanded version of the notes that were used for the series of lectures given during the winter 2012–2013 at the Institute of Mathematics of the Romanian Academy and based on P. Głowacki's papers [2], [3] on Melin calculus for pseudodifferential operators on homogeneous Lie groups. The aim of these two papers was to extend some of the Melin's results [6] on pseudodifferential operators on graded Lie groups to pseudodifferential operators on general homogeneous groups. The main Głowacki's ideas were to use Hörmander's results on slowly varying metrics and to introduce an operator, which we called the reduction operator (see Proposition 6.7 for its definition). Using the reduction operator and induction, one can reduce the study of pseudodifferential operators to commutative groups.

During these lectures we detailed the proofs from Głowacki's papers, using, partially, [4]. Also we introduced the notion of admissible metric (Definition 6.6), which, we think, may help to clarify the statements and proofs. We have to mention that we were able to prove the theorem on the continuity of Melin's operator ([2, Prop. 5.1], [3, Thm. 5.1]) only in a weaker form (Proposition 6.17 in the present paper). But we were able to prove the main results from Głowacki's papers (the theorem on the composition of symbols and the theorem which asserts the L^2-continuity of the pseudodifferential operators) using Proposition 6.17.

The structure of this paper is as follows. In Section 2, the definition of slowly varying metrics, self-tempered metrics and weights in Hörmander's sense [4] is given. An important, for us, subclass of such metrics is described in Lemma 2.6 The spaces of symbols associated to slowly varying metrics are also defined and their properties needed in our notes are proved. Section 3 deals with metrics and weights on homogeneous spaces. The basic tools of Hörmander's theory of Weyl calculus for pseudodifferential calculus are presented in Section 4. In Section 5, one introduces the homogeneous groups and one specifies some notations used in Section 6. Section 6 is the main section of this paper and corresponds to the main sections of Głowacki's papers. Here one proves the main propositions, Proposition 6.7, Proposition 6.15, and Proposition 6.17, used in the last section in the proofs of the theorem on the composition of symbols (Theorem 7.1) and of the theorem which asserts the L^2-continuity of the pseudodifferential operators (Theorem 7.3).

As the present paper provides a self-contained approach to some results from Glowacki's papers, we freely reproduced some facts from these papers when we considered that no completion or correction was needed.

The list of references is minimal.

2 Slowly varying metrics, weights, symbols

Let X be a real n- dimensional vector space. A family of Euclidian norms on X, $g = (g_x)_{x\in X}$ is called a *varying metric* on X or, simply, a *metric* on X. Ocasionally, in this section, we fix an orthogonal basis $\{e_j\}_{j=1,\dots,n}$ in X and, in this case, we denote $x = \sum_{j=1}^n x_j e_j = (x_1, \dots, x_n)$.

Definition 2.1. A metric g on X is called *slowly varying* if there exist some positive constant $\gamma \in (0, 1]$ such that

$$\forall x, y \in X, \gamma \leq \frac{g_x}{g_y} \leq \frac{1}{\gamma} \text{ if } g_x(x-y) \leq \gamma. \tag{2.1}$$

Remark 2.2. A metric g is slowly varying if and only if there exists some positive constant $\gamma \in (0, 1]$ such that

$$\forall x, y \in X, \gamma \leq \left(\frac{g_x}{g_y}\right)^{\pm 1} \leq \frac{1}{\gamma} \text{ if } g_x(x-y) \leq \gamma. \tag{2.2}$$

Indeed, if (2.1) holds for some γ, then (2.2) holds also if we replace γ with γ^2.

Definition 2.3. Let g and G be two metrics on X. The metric g is *G-tempered* if there exist some positive constants C and M such that

$$\left(\frac{g_x}{g_y}\right)^{\pm 1} \leq C(1 + G_x(x-y))^M, \forall x, y \in X \tag{2.3}$$

and if $g_x \leq G_x, \forall x \in X$.

The metric g is called *self-tempered* if it is g-tempered.

Remark 2.4. A self-tempered metric g is slowly varying. Indeed, let us assume that (2.3) holds. If $\gamma \in (0,1]$ is such that $C(1+\gamma)^M \leq \frac{1}{\gamma}$, then

$$\frac{g_x}{g_y} \leq \frac{1}{\gamma} \text{ and } \frac{g_y}{g_x} \leq \frac{1}{\gamma} \text{ if } g_x(x-y) \leq \gamma.$$

Lemma 2.5. *If g is a self-tempered metric with the constants C and M, $C \geq 1$, then for every $x, y, z \in X$*

$$1 + g_x(x-y) \leq C(1 + g_y(x-y))^{M+1}, \tag{2.4}$$
$$1 + g_x(x-y) \leq C(1 + g_z(x-z))^{M+1}(1 + g_z(z-y)), \tag{2.5}$$
$$1 + g_x(x-y) \leq C^2(1 + g_x(x-z))^M(1 + g_y(z-y))^{M+1}. \tag{2.6}$$

Proof. First of all, we have

$$1 + g_x(x-y) \leq 1 + Cg_y(x-y)(1 + g_y(x-y))^M \leq C(1 + g_y(x-y))^{M+1}.$$

Then, from (2.4) and the definition of the self-tempered metric, we obtain (2.5):

$$\begin{aligned} 1 + g_x(x-y) &\leq 1 + g_x(x-z) + g_x(z-y) \\ &\leq C(1 + g_z(x-z))^{M+1} + Cg_z(z-y)(1 + g_z(x-z))^M \\ &\leq C(1 + g_z(x-z))^{M+1}(1 + g_z(z-y)). \end{aligned}$$

The inequality (2.6) is proved in a similar manner:

$$\begin{aligned} 1 + g_x(x-y) &\leq 1 + g_x(x-z) + g_x(z-y) \\ &\leq 1 + g_x(x-z) + Cg_z(z-y)(1 + g_x(x-z))^M \\ &\leq (1 + g_x(x-z))^M(1 + Cg_z(z-y)) \\ &\leq C(1 + g_x(x-z))^M(1 + g_z(z-y)) \\ &\leq C^2(1 + g_x(x-z))^M(1 + g_y(z-y))^{M+1}. \end{aligned}$$

□

Lemma 2.6. *Let*

$$g = (g_x)_{x \in X},\ g_x(z)^2 = \sum_{j=1}^{n} a_j(x)^2 z_j^2, \forall x, z \in X, a_j : X \to (0, \infty), \forall j \in \{1, \ldots, n\}.$$

Then the following assertions hold true:

(a) *The metric g is slowly varying if and only if there exists $\gamma \in (0,1]$ so that*

$$\forall x, y \in X, \gamma \leq \frac{a_j(y)}{a_j(x)} \leq \frac{1}{\gamma}, \forall j \in \{1, \ldots, n\} \text{ if } g_x(x-y) \leq \gamma.$$

(b) *If the metric g is G-tempered with constants C and M, then*

$$\left(\frac{a_j(y)}{a_j(x)}\right)^{\pm 1} \leq C(1 + G_x(x-y))^M, \forall j \in \{1, \ldots, n\}, \forall x, y \in X.$$

Proof. If we take $z = e_j, j = 1, \ldots, n$, then (b) and the "only if" part from (a) follow straigtforwardly by the definitions. The "if" part of Assertion (a) is quite obvious. □

Definition 2.7. A function $m : X \to (0, \infty)$ is called a *G-tempered weight with respect to the G-tempered metric g* if

$$\forall x, y \in X, \left(\frac{m_x}{m_y}\right)^{\pm 1} \leq C \text{ if } g_x(x-y) \leq \gamma \tag{2.7}$$

and

$$\left(\frac{m_x}{m_y}\right)^{\pm 1} \leq C(1 + G_x(x-y))^M, \forall x, y \in X. \tag{2.8}$$

If g is self-tempered and m is a g-tempered weight with respect to g, we shall say simply that m is a g-tempered weight.

Remark 2.8. Let m, n be G-tempered weights with respect to g and let $k \in \mathbb{R}$. Then m^k, mn , $m + n$ and $\max(m, n)$ are G-tempered weights with respect to g.

Example 2.9. If g is a G-tempered slowly varying metric on X, if G is self-tempered and if $x_0 \in X$, then $m : X \to \mathbb{R}_+$, $m(x) = 1 + g_x(x - x_0), \forall x \in X$ is a G-tempered weight with respect to g.

Indeed, if g satisfies (2.1) and $g_x(x-y) \leq \gamma$, since

$$g_x(x - x_0) \leq \frac{1}{\gamma} g_y(x - x_0)$$

and

$$g_y(x - x_0) \leq g_y(x-y) + g_y(y - x_0) \leq \frac{1}{\gamma} g_x(x-y) + g_y(y - x_0) \leq 1 + g_y(y - x_0),$$

we have

$$\frac{1 + g_x(x - x_0)}{1 + g_y(y - x_0)} = \frac{1 + g_x(x - x_0)}{1 + g_y(x - x_0)} \cdot \frac{1 + g_y(x - x_0)}{1 + g_y(y - x_0)} \leq \frac{1 + \frac{1}{\gamma} g_y(x - x_0)}{1 + g_y(x - x_0)} \cdot \frac{2 + g_y(y - x_0)}{1 + g_y(y - x_0)} \leq \frac{2}{\gamma}.$$

Also, let as assume that (2.3) holds, without any loss of generality, with some constants $C \geq 1$ and $M > 0$. Then

$$\begin{aligned}\frac{1+g_x(x-x_0)}{1+g_y(y-x_0)} &= \frac{1+g_x(x-x_0)}{1+g_y(x-x_0)} \cdot \frac{1+g_y(x-x_0)}{1+g_y(y-x_0)} \\ &\leq \frac{1+C(1+G_x(x-y))^M g_y(x-x_0)}{1+g_y(x-x_0)} \cdot \frac{1+g_y(y-x_0)+G_y(x-y)}{1+g_y(y-x_0)} \\ &\leq C(1+G_x(x-y))^M(1+G_y(x-y)) \leq \\ &\leq C(1+G_x(x-y))^M(1+C(1+G_x(x-y))^{M+1}) \\ &\leq 2C^2(1+G_x(x-y))^{2M+1}.\end{aligned}$$

Example 2.10. Let $g = (g_x)_{x\in X}$, $g_x(z)^2 = \sum_{j=1}^n a_j(x)^2 z_j^2, \forall x, z \in X$ be a slowly varying, G-tempered metric. Then, from Lemma 2.6, it follows that the functions $a_j, j \in \{1,\dots,n\}$ are G-tempered weights with respect to g.

We shall define now the symbol classes we are working with ([4], [3]). If $f \in C^\infty(X)$ and if g is a metric on X, then

$$\begin{aligned}g_x(D^k f(x)) &= \sup_{y_j\in X, g_x(y_j)\leq 1, j=1,\dots,k} |D^k f(x)(y_1,\dots,y_k)| \\ &= \sup_{y_j\in X\setminus\{0\}, j=1,\dots,k} \frac{|D^k f(x)(y_1,\dots,y_k)|}{\prod_{j=1}^k g_x(y_j)}, \forall x \in X. \qquad (2.9)\end{aligned}$$

We denoted by $D^k f(x)$ the Fréchet derivative of order k of f.

Lemma 2.11. *Let $f, g \in C^\infty(X)$.*

(a) *We have*

$$g_x(D^k(fg)(x)) \leq \sum_{j=0}^k \binom{k}{j} g_x(D^j f(x)) g_x(D^{k-j} g(x)). \qquad (2.10)$$

(b) *For every $k \in \mathbb{N}^*$ there exists a positive constant C_k such that if $f(x) \neq 0$, then*

$$g_x(D^k(1/f)(x)) \leq C_k f(x)^k (g_x(Df)(x)) + \cdots + (g_x(D^k f(x)))^{1/k})^k \qquad (2.11)$$

if $u(x) \geq 1$ and

$$g_x(D^k(1/f)(x)) \leq C_k f(x)(g_x(Df)(x)) + \cdots + (g_x(D^k f)(x))^{1/k})^k \qquad (2.12)$$

if $u(x) < 1$.

Proof. (a) (2.10) follows from (2.9) and the Leibniz' rule:

$$D^k(fg)(x)(y_1,\dots,y_k) = \sum_{j=0}^k \sum_{\alpha\in F_{j,k}} D^j f(x)(y_\alpha) D^{k-j} g(x)(y_{C(\alpha)})$$

where $F_{j,k} = \{\alpha = (\alpha_1,\dots,\alpha_j); 1 \leq \alpha_1 < \cdots < \alpha_j \leq k\}$, $y_\alpha = (y_{\alpha_1},\dots,y_{\alpha_j})$, $C(\alpha)$ is the complement of α and card $F_{j,k} = \binom{k}{j}$.

(b) We shall prove (b) for $f(x) = 1$. The general result will follow by homogenization. Let $h = 1 - f$. Then, there exists a neighbourhood V of x such that $|h(y)| \leq 1/2, \forall y \in V$. Therefore $\frac{1}{f(y)} = \sum_{i=0}^{\infty} h^i(y), \forall y \in V$. Since $h(x) = 0$, $D^k h^i(x) = 0, \forall i > k$. Therefore, using (a), we obtain

$$\begin{aligned} g_x(D^k(1/f)(x)) &\leq \sum_{i=1}^{k} g_x(D^k h^i(x)) \\ &\leq C_k(g_x(Dh(x)) + (g_x(D^2 h)(x))^{1/2} + \cdots + (g_x(D^k h)(x))^{1/k})^k \\ &= C_k(g_x(Df(x)) + (g_x(D^2 f(x)))^{1/2} + \cdots + (g_x(D^k f)(x))^{1/k})^k. \end{aligned}$$

□

If m is a G-tempered weight with respect to the G-tempered metric g and $f \in C^\infty(X)$, then we put

$$|f|_{(k)}^m(g) = \sup_{x \in X} \frac{g_x(D^k f(x))}{m(x)}$$

and

$$|f|_k^m(g) = \sum_{j=0}^{k} |f|_{(j)}^m(g).$$

The space of symbols of order m with respect to g is

$$S(m, g) = \{a \in C^\infty(X); |a|_k^m(g) < \infty, \forall k \in \mathbb{N}\}.$$

Example 2.12. Let $g = (g_x)_{x \in X}$, $g_x(z)^2 = \sum_{j=1}^{n} a_j(x)^2 z_j^2, \forall x, z \in X$ be a slowly varying metric on X. If we fix a basis $\{e_1, \dots, e_n\}$ in X, then a function $f \in C^\infty(X)$ is in $S^m(X, g)$ if and only if for every $\alpha \in \mathbb{N}^n$ there exists a constant C_α so that

$$|\partial^\alpha f(x)| \leq C_\alpha m(x) a(x)^\alpha, \forall x \in X. \tag{2.13}$$

We have used the standard notations $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j, \forall j \in \{1, \dots, n\}$ and $a(x)^\alpha = a_1(x)^{\alpha_1} \dots a_n(x)^{\alpha_n}$.

Proof. We shall give the proof only for derivatives of order 1. Derivatives of higher order can be treated in a similar manner.

"⇒" For $i \in \{1, \dots, n\}$, $g_x(e_i) = a_i(x)$. If $f \in S^m(X, g)$, then

$$\infty > \sup_{x \in X} \frac{g_x(Df(x))}{m(x)} \geq \sup_{x \in X} \frac{|(Df(x)(e_i)|}{g_x(e_i)m(x)} = \sup_{x \in X} \frac{|\partial_i f(x)|}{a_i(x)m(x)}.$$

"⇐" If (2.13) holds, then

$$\begin{aligned} \sup_{x \in X} \frac{g_x(Df(x))}{m(x)} &= \sup_{x \in X} \sup_{y \in X \setminus \{0\}} \frac{|Df(x)(y)|}{g_x(y)m(x)} = \\ &= \sup_{x \in X} \sup_{y \in X \setminus \{0\}} \frac{|\sum_{i=1}^{n} y_i \partial_i f(x)|}{\left(\sum_{i=1}^{n} a_i(x)^2 y_i^2\right)^{1/2} m(x)} \end{aligned}$$

$$\leq \sup_{x\in X}\sup_{y\in X\setminus\{0\}} \frac{\left(\sum_{i=1}^n a_i(x)^2 y_i^2\right)^{1/2}\left(\sum_{i=1}^n a_i(x)^{-2}\partial_i f(x)^2\right)^{1/2}}{\left(\sum_{i=1}^n a_i(x)^2 y_i^2\right)^{1/2} m(x)} < \infty.$$

□

Remark 2.13. $S^m(X,g)$ with the family of norms $|\cdot|_k^m$ is a Fréchet space. If g is as in Example 2.12, then

$$\sup_{x\in X} |\partial^\alpha f(x)| m(x)^{-1} a(x)^{-\alpha}, \alpha \in \mathbb{N}^n$$

is an equivalent family of seminorms.

3 Metrics and weights on homogeneous euclidean spaces

A triple $(X,(X_1\ldots,X_R),(d_1,\ldots,d_R))$, where $X = X_1\oplus\cdots\oplus X_R$ is an euclidean vector space of dimension n, scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and $d_1,\ldots,d_R$ are real numbers, $1 = d_1 < \cdots < d_R$ is called a homogeneous euclidean space. We shall denote with n_k the dimension of X_k. Thus the variable $x\in X$ splits into $x = (x_1,\ldots,x_R)$. On X we introduce a family of dilations

$$\delta_t x = tx = (t^{d_1}x_1,\ldots,t^{d_R}x_R), \forall x\in X, \forall t>0.$$

This adhoc definition is justified by the fact that such a triple corresponds to the Lie algebra of a Lie homogeneous group (see Section 5).

For $x = (x_1,\ldots,x_R)\in X$ we put

$$|x| = \sum_{k=1}^R \|x_k\|^{1/d_k}.$$

$|\cdot|$ is a homogeneous norm ([1]), in the sense that

(a) $|x| = 0$ if and only if $x = 0$,

(b) $|-x| = |x|, \forall x\in X$, and

(c) $|tx| = t|x|, \forall x\in X, \forall t>0$.

More than that, since $(a+b)^\mu \leq a^\mu + b^\mu, \forall a,b\geq 0, \forall\mu\in(0,1]$, $|\cdot|$ satisfies also the triangle inequality $|x+y|\leq |x|+|y|, \forall x,y\in X$.

For $1\leq k\leq R$ we define

$$|x|_k = \sum_{j=k}^R \|x_j\|^{1/d_j}, \forall x\in X.$$

$|x|_k$ are homogeneous seminorms, in the sense that they satisfy (b) and (c) from above, and they satisfy also the triangle inequality. Let us remark that $|x|_1 = |x|$. We shall also put $|x|_{R+1} = 0, \forall x\in X$

$$q_k(x) = 1 + |x|_{k+1}, \forall x \in X, \forall k \in \{0, 1, \ldots, R\}$$

and

$$g_{k,\delta}(x) = \delta + |x|_{k+1}, \forall x \in X, \forall k \in \{0, 1, \ldots, R\}, \forall \delta > 0.$$

Let

$$g_x^\delta(z)^2 = \sum_{k=1}^{R} \frac{\|z_k\|^2}{g_{k,\delta}(x)^{2d_k}}, \forall x, z \in X, \forall \delta > 0.$$

We shall also use the notation $q_x = g_x^1$.

We shall prove now that the metrics $g_k(\cdot;\delta)$ are uniformly slowly varying and uniformly self-tempered with respect to $\delta > 0$.

Lemma 3.1. *For every $k \in \{0, \ldots, R\}$ and for every $\delta > 0$,*

$$\frac{1}{2} \leq \frac{g_{k,\delta}(x)}{g_{k,\delta}(y)} \leq 2 \text{ if } g_x^\delta(x-y) < \left(\frac{1}{2R}\right)^{d_R}. \tag{3.1}$$

Proof. If $g_x^\delta(x-y) < [1/(2R)]^{d_R}$, then

$$\|x_j - y_j\|^{1/d_j} \leq \frac{g_{j,\delta}(x)}{2R} \leq \frac{g_{k,\delta}(x)}{2R}, \forall j \in \{k+1, \ldots, R\}.$$

Therefore $|x-y|_{k+1} \leq g_{k,\delta}(x)/2$ and consequently

$$g_{k,\delta}(x) = \delta + |x|_{k+1} \leq \delta + |y|_{k+1} + |x-y|_{k+1} \leq g_{k,\delta}(y) + \frac{1}{2} g_{k,\delta}(x)$$

and

$$g_{k,\delta}(y) = \delta + |y|_{k+1} \leq \delta + |x|_{k+1} + |x-y|_{k+1} \leq \frac{3}{2} g_{k,\delta}(x)$$

which implies (3.1). ∎

Lemma 3.2. *There exists a constant $C > 0$ so that*

$$g_{k,\delta}(x) \leq C g_{k,\delta}(y)(1 + g_y^\delta(x-y)), \forall x, y \in X, \forall k \in \{0, 1, \ldots, R\}, \forall \delta > 0 \tag{3.2}$$

and

$$g_{k,\delta}(x) \leq C g_{k,\delta}(y)(1 + g_x^\delta(x-y))^{R-k}, \forall x, y \in X, \forall k \in \{0, 1, \ldots, R\}, \forall \delta > 0. \tag{3.3}$$

Proof. We shall prove first (3.2). We have

$$g_{k,\delta}(x) \leq g_{k,\delta}(y) + |x-y|_{k+1} = g_{k,\delta}(y)\left(1 + \frac{|x-y|_{k+1}}{g_{k,\delta}(y)}\right)$$
$$\leq g_{k,\delta}(y)\left(1 + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|^{1/d_j}}{g_{j,\delta}(y)}\right).$$

Using the inequality

$$\sum_{j=k+1}^{R} a_j \leq R - k + \sum_{j=k+1}^{R} a_j^{d_j}, \forall a_j > 0, \forall d_j \geq 1,$$

we obtain that

$$\sum_{j=k+1}^{R} \frac{\|x_j - y_j\|^{1/d_j}}{g_{j,\delta}(y)} \leq R + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|}{g_{j,\delta}(y)^{d_j}}.$$

Therefore

$$g_{k,\delta}(x) \leq C g_{k,\delta}(y)(1 + g_y^\delta(x-y))$$

for $C = 2(R+1)$ and (3.2) is proved.

We shall prove (3.3) by induction. For $k = R$ there is nothing to prove. We can take $C = 1$ in this case. Let us assume that (3.3) holds for $k+1$ with some constant $C > 1$. Then

$$\begin{aligned} g_{k,\delta}(x) &\leq g_{k,\delta}(y) + |x-y|_{k+1} = g_{k,\delta}(y)\left(1 + \frac{|x-y|_{k+1}}{g_{k,\delta}(y)}\right) \\ &\leq g_{k,\delta}(y)\left(1 + \frac{g_{k+1,\delta}(x)}{g_{k+1,\delta}(y)} \cdot \frac{|x-y|_{k+1}}{g_{k+1,\delta}(x)}\right). \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} g_{k,\delta}(x) &\leq C g_{k,\delta}(y)(1 + g_x^\delta(x-y))^{R-k-1}\left(1 + \frac{|x-y|_{k+1}}{g_{k+1,\delta}(x)}\right) \\ &\leq C g_{k,\delta}(y)(1 + g_x^\delta(x-y))^{R-k-1}\left(R + 1 + \sum_{j=k+1}^{R} \frac{\|x_j - y_j\|}{g_{j,\delta}(x)^{d_j}}\right) \\ &\leq C_1 g_{k,\delta}(y)(1 + g_x^\delta(x-y))^{R-k}, \end{aligned}$$

which proves that (3.3) holds with some new constant C_1. □

Corollary 3.3. *The following assertions hold:*

(a) *The metrics g_k^δ are uniformly slowly varying and uniformly self-tempered with respect to $\delta > 0$.*

(b) *$g_{k,\delta}$ are g_k^δ weights uniformly with respect to $\delta > 0$.*

(c) *$g_{k,\delta'}$ is a g_k^δ weight $\forall \delta, \delta' > 0$.*

Proof. All the assertions of the corollary follow from Lemmas 3.1, 3.2, and 2.6 if we put

$$(a_1, \ldots, a_{n_1}, \ldots, a_{n-n_R+1}, \ldots, a_n) = (g_{1,\delta}^{-1}, \ldots, g_{1,\delta}^{-1}, \ldots, g_{R,\delta}^{-d_R}, \ldots, g_{R,\delta}^{-d_R}).$$

□

4 Hörmander's lemmas

The following two results are essential tools in Hörmander's theory of Weyl calculus [4].

Proposition 4.1. *Let g be a slowly varying metric on X.*

(a) *If γ is the constant from formula* (2.2) *and $0 < \varepsilon < \gamma$, then there exists a sequence $(x_\nu)_\nu$ of points in X so that the balls*

$$B_\nu = B_\nu^g(x_\nu, r) = \{x \in X; g_{x_\nu}(x - x_\nu) < r\}$$

cover X and there exists some N so that card$\{\nu; x \in B_\nu\} \leq N, \forall x \in X, \forall \varepsilon \leq r \leq \gamma$.

(b) *For every $r \in (\varepsilon, \gamma)$, there exist $\phi_\nu \in C_0^\infty(B_\nu), \forall \nu \in \mathbb{N}^*$, so that $(\phi_\nu)_\nu$ is a bounded sequence in $S^1(X, g)$ and $\sum_\nu \phi_\nu(x) = 1, \forall x \in X$.*

(c) *If g is a self-tempered metric, then there exist two positive constants $\tilde{C}$ and $\tilde{M}$ which depend only on the constants C and M from* (2.3)*, on ε and on the dimension of X so that*

$$\sum_\nu (1 + d_\nu(x))^{-\tilde{M}} \leq \tilde{C}, \forall x \in X. \tag{4.1}$$

Here $d_\nu(x) = g_{x_\nu}(x - x_\nu)$.

Proof. (a) First of all, let us remark that if K is a compact set in X, if F is a totally ordered set of indices and if $(x_\nu)_{\nu \in F}$ is a family of points in K so that $g_{x_\mu}(x_\nu - x_\mu) \geq \varepsilon, \forall \nu > \mu$, then F is finite. Else, since K is a compact set, there exists a point $x \in K$ and a sequence $(x_{\nu_j})_j$ convergent to x so that $g_{x_{\nu_j}}(x_{\nu_k} - x_{\nu_j}) \geq \varepsilon, \forall k > j$. Let $\delta \in (0, \gamma)$. Then there exists $j_\delta \in \mathbb{N}$ so that $g_x(x_{\nu_j} - x) < \delta$ if $j \geq j_\delta$. Then

$$g_{x_{\nu_j}}(x_{\nu_k} - x_{\nu_j}) \leq g_{x_{\nu_j}}(x_{\nu_k} - x) + g_{x_{\nu_j}}(x_{\nu_j} - x) \leq \frac{1}{\gamma} g_x(x_{\nu_k} - x) + \frac{1}{\gamma} g_x(x_{\nu_k} - x) \leq \frac{2\delta}{\gamma}, \forall j, k \geq j_\delta.$$

If we take $\delta < \varepsilon\gamma/2$, we obtain a contradiction.

Therefore, there exists a maximal sequence of points $(x_\nu)_\nu$ in X so that

$$g_{x_\mu}(x_\nu - x_\mu) \geq \varepsilon, \forall \nu > \mu. \tag{4.2}$$

This sequence has all the required properties. Indeed, the balls $(B_\nu)_\nu$ cover X when $r = \varepsilon$ since otherwise would be possible to add some point x to the sequence $(x_\nu)_\nu$ without violating (4.2).

Next, let $x \in X$ and $x \in B_\nu \cap B_\mu$. We can always assume that $\mu < \nu$. Then $g_x(x - x_\nu) \leq \frac{r}{\gamma} \leq 1$ and $g_x(x_\nu - x_\mu) \geq \gamma g_{x_\mu}(x_\nu - x_\mu) \geq \gamma\varepsilon$. Therefore $\{y \in X; g_x(y - x_\mu) < \gamma\varepsilon/2\} \cap \{y \in X; g_x(y - x_\nu) < \gamma\varepsilon/2\} = \emptyset$. There is a fixed upper bound for the number of disjoint open balls of a fixed radius which are included in a ball of radius 1 in a finite dimensional normed space. This bound depends only on the dimension of the space and on the radius. It does not depend on the norm. This remark ends the proof of Assertion (a).

(b) Let $\psi \in C_0^\infty(-r^2, r^2)$, $\psi(t) = 1, \forall t \in (-\varepsilon^2, \varepsilon^2)$, $\psi_\nu(x) = \psi(g_{x_\nu}(x - x_\nu)^2), \forall x \in X, \forall \nu \in \mathbb{N}^*$, $\phi_\nu(x) = \psi_\nu(x) / \left(\sum_\mu \psi_\mu(x)\right)$. It is clear that $(\phi_\nu)_\nu \in C_0^\infty(B_\nu), \forall \nu \in \mathbb{N}^*$ and $\sum_\nu \phi_\nu(x) = 1, \forall x \in X$. Let us prove that $(\phi_\nu)_\nu$ is a bounded sequence in $S^1(X, g)$.

We shall first prove that $(\psi_\nu)_\nu$ is a bounded sequence in $S^1(X,g)$. We have

$$g_x(D\psi_\nu(x)) = \sup_{y\neq 0}\frac{|D\psi_\nu(x)y|}{g_x(y)} = \sup_{y\neq 0}\frac{|D\psi_\nu(x)y|}{g_{x_\nu}(y)}\frac{g_{x_\nu}(y)}{g_x(y)} \leq \frac{1}{\gamma}\cdot\sup_{y\neq 0}\frac{|D\psi_\nu(x)y|}{g_{x_\nu}(y)}$$

since $g_{x_\nu}(x-x_\nu)<\gamma$ on suppψ_ν. Let $(a_{ij})_{i,j=1,\ldots,n}$ be the matrix of the quadratic form $(g_{x_\nu})^2$. Then

$$D\psi_\nu(x)y = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\psi(g_{x_\nu}(x-x_\nu)^2)y_j = 2\psi'(g_{x_\nu}(x-x_\nu)^2)\sum_{i,j=1}^{n}y_j a_{ij}(x_i-x_{\nu;i}).$$

Here $x=(x_1,\ldots,x_n), n=\dim X$.

Since

$$|\sum_{i,j=1}^{n}y_j a_{ij}(x_i-x_{\nu;i})| \leq g_{x_\nu}(y)g_{x_\nu}(x-x_\nu),$$

we have

$$\frac{|D\psi_\nu(x)y|}{g_{x_\nu}(y)} \leq \frac{2r}{\gamma}|\psi'(g_{x_\nu}(x-x_\nu)^2)| \leq \frac{2Cr}{\gamma}, \forall x\in X, \forall y\in X\setminus\{0\}, \forall\nu\in\mathbb{N}^*$$

for some positive constant C. Therefore $|\psi_\nu|^1_{(1)}(g)$ are uniformly bounded with respect to ν. One can prove in a similar manner that $|\psi_\nu|^1_{(k)}(g)$ are uniformly bounded with respect to ν, $\forall k\in\mathbb{N}$.

Taking into account the uniform boundedness of the sequence $(\psi_\nu)_\nu$ and the fact that there exists some N so that card$\{\nu; x\in B_\nu\}\leq N, \forall x\in X, \forall\varepsilon\leq r\leq\gamma$, we obtain that $\sum_\nu\psi_\nu\in S^1(X,g)$. Lemma 2.11 ends the proof of Assertion (b).

(c) Let $M_k = M_k(x) = \{\nu; d_\nu(x)<k\}, \forall x\in X, \forall k\in\mathbb{N}$. It is sufficient to prove that there exist some constants c and m, which depend only on the constants C and M from (2.3), on ε and on the dimension of X so that

$$\text{card}(M_k)\leq c(1+k)^m, \forall x\in X, \forall k\in\mathbb{N}. \tag{4.3}$$

Indeed, if (4.3) is true and if $\tilde{M}=m+2$, then

$$\sum_\nu(1+d_\nu(x))^{-\tilde{M}} = \sum_{k\geq 0}\sum_{\nu\in M_{k+1}\setminus M_k}(1+d_\nu(x))^{-\tilde{M}} \leq \sum_{k\geq 0}\sum_{\nu\in M_{k+1}\setminus M_k}(1+k)^{-\tilde{M}}$$
$$\leq c\sum_{k\geq 0}(1+k)^{-\tilde{M}+m} = \tilde{C}<\infty.$$

We shall prove now (4.3). Let $\nu\in M_k$ and

$$V_\nu = \{z\in X; g_x(z-x_\nu)<r_k\}$$

where $r_k = r/(C(1+k)^M)$ and C and M are the constants from (2.3).

Then

$$V_\nu\subseteq B_\nu. \tag{4.4}$$

Indeed, if $z \in V_\nu$ and $\nu \in M_k$, then

$$g_{x_\nu}(z - x_\nu) \le C(1 + g_{x_\nu}(x - x_\nu)^M g_x(z - x_\nu) \le C(1+k)^M \cdot \frac{r}{C(1+k)^M} = r.$$

Also, for $z \in V_\nu$ and $\nu \in M_k$, we have

$$g_x(z-x) \le g_x(z-x_\nu)+g_x(x_\nu-x) < r_k+Cg_{x_\nu}(x_\nu-x)(1+g_{x_\nu}(x_\nu-x))^M \le r_k+C(1+k)^{M+1}.$$

Therefore

$$V_\nu \subseteq V = V(x,k) = \{z \in X; g_x(z-x) < R_k\}, \tag{4.5}$$

where $R_k = r_k + C(1+k)^{M+1}$.

Let $|V_\nu| = C_1(x)r_k^n, n = \dim X$ be the volume of V_ν. Using (4.4) and (4.5), we obtain

$$C_1(x)\mathrm{card}(M_k)r_k^n = \sum_{\nu \in M_k} |V_\nu| \le N\Big|\bigcup_{\nu\in M_k} V_\nu\Big| \le N|V| \le C_1(x)NR_k^n,$$

where N is the constant from Assertion (a).

Therefore

$$\mathrm{card}M_k \le Nr_k^{-n}R_k^n = N(1 + C(1+k)^{M+1}r_k^{-1})^n \le 2C^{2n}Nr^{-n}(1+k)^{(2M+1)n}.$$

□

Proposition 4.2. *Let $(X, \|\cdot\|)$ be an Euclidean normed space, $r_1 > r > 0$, $x_0 \in X$ and L an affine function so that $L(x) \neq 0, \forall x \in B(x_0, r_1)$. Then*

$$\|D^k\left(\frac{1}{L}\right)(x)\| \le \frac{k!r_1}{(r_1-r)^{k+1}|L(x_0)|}, \forall x \in B(x_0, r), \forall k \in \mathbb{N}. \tag{4.6}$$

Proof. We may assume, without loss of generality, that $x_0 = 0$ and that $L(0) = 1$. In this case, there exists $\xi \in X$ so that $L(x) = \langle \xi, x\rangle + 1, \forall x \in X$. Since $\langle \xi, x\rangle + 1 > 0, \forall x \in X, \|x\| < r_1$, it follows that $\|\xi\| \le 1/r_1$ and $L(x) \ge (r_1 - r)/r_1, \forall x \in B(0,r)$.

Since, for $x \in B(0,r)$, $L^{-1}(x) = \sum_{j\ge 0}(-1)^j\langle \xi, x\rangle^j$, we have

$$D(L^{-1})(x)y = \sum_{j\ge 1}(-1)^j j\langle \xi, x\rangle^{j-1}\langle \xi, y\rangle = -\langle \xi, y\rangle L(x)^{-2}.$$

We have used the fact that $-(1+a)^{-2} = \sum_{j\ge1}(-1)^j ja^{j-1}, \forall a \in \mathbb{R}, |a| < 1$.

Using the formula $D^k f(x)(y_1,\ldots,y_k) = D_x[D^{k-1}f(x)(y_1,\ldots,y_{k-1})]y_k$, one can prove by induction on k that

$$D^k(L^{-1})(x)(y_1,\ldots,y_k) = \langle \xi, y_1\rangle \cdots \langle \xi, y_k\rangle \cdot \frac{(-1)^k k!}{L(x))^k}, \forall x, y_1, \ldots, y_k \in X.$$

Therefore

$$\|D^k(L^{-1})(x)\| \le \frac{k!\|\xi\|^k}{|L(x)|^{k+1}} \le \frac{k!r_1}{(r_1-r)^{k+1}}, \forall x \in B(0,r).$$

□

5 Homogeneous groups

Let $(\mathfrak{g}, +, [\cdot,\cdot])$ be a finite dimensional Lie algebra of dimension n endowed with a scalar product $\langle\cdot,\cdot\rangle$. A family of dilations on $\mathfrak{g}$ is a family $(\delta_t)_{t>0}$ of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_t = \exp(A\log t)$, where A is a positive definite operator on $\mathfrak{g}$. Let $0 < d_1 < \cdots < d_R$ be the eigenvalues of A and

$$\mathfrak{g}_k = \left\{x \in \mathfrak{g}; \delta_t x = t^{d_k} x\right\}, \forall k \in \{1, \ldots, R\}.$$

Since $\delta_{t^\alpha} = \exp(\alpha A \log t)$, by adjusting α if necessary, we may assume that $d_1 = 1$.

Proposition 5.1. *If a Lie algebra $\mathfrak{g}$ admits a family of dilations, then $\mathfrak{g}$ is nilpotent.*

Proof. If $x \in \mathfrak{g}_j, y \in \mathfrak{g}_k$, then $\delta_t[x,y] = [\delta_t x, \delta_t y] = [t^{d_j}x, t^{d_k}y] = t^{d_j+d_k}[x,y]$. Hence $[\mathfrak{g}_j, \mathfrak{g}_k] = \{0\}$ if $d_j + d_k$ is not an eigenvalue of A and $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_l$ if $d_j + d_k = d_l$ for some eigenvalue d_l of A. Therefore, if we denote as usually, $\mathfrak{g}_{(1)} = \mathfrak{g}$, $\mathfrak{g}_{(j)} = [\mathfrak{g}, \mathfrak{g}_{(j-1)}]$, then $\mathfrak{g}_{(j)} \subseteq \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_R$. Consequenly, $\mathfrak{g}_{(j)} = \{0\}$ for $j \geq d_R$ and $\mathfrak{g}$ is nilpotent. □

A *homogeneous group* is a connected and simply connected nilpotent Lie group whose Lie algebra is endowed with a family of dilations. In these notes we shall consider that the Lie algebra $\mathfrak{g}$ itself is a Lie group with the multiplication given by the Campbell-Hausdorff-Baker formula

$$x \circ y = xy = x + y + r(x,y), \forall x, y \in \mathfrak{g},$$

where

$$r(x,y) = \frac{1}{2}[x,y] + \frac{1}{12}\left([x,[x,y]] + [y,[y,x]]\right) + \cdots$$

is the finite sum of terms of order at least 2 in the Campbell-Hausdorff-Baker series for $\mathfrak{g}$.

We shall also assume that $\mathfrak{g}$ is endowed with a fixed scalar product and we shall identify the dual vector space $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the scalar product.

Let us remark that according to the Campbell-Hausdorff-Baker formula, the inverse of a vector $x \in \mathfrak{g}$ with respect to the multiplication is $-x$. Therefore the Lebesgue measure is a bi-invariant Haar measure for the group $\mathfrak{g}$ and the convolution formula reads

$$f * g(x) = \int_{\mathfrak{g}} f(xy^{-1})g(y)\ \mathrm{d}y = \int_{\mathfrak{g}} f(x \circ (-y))g(y)\ \mathrm{d}y, \forall f, g \in \mathcal{S}(\mathfrak{g}).$$

The Lebesgue measure on $\mathfrak{g}$ will be normalized so that the inverse of the Fourier transform on the Schwartz space $\mathcal{S}(\mathfrak{g})$

$$\hat{f}(y) = \int_{\mathfrak{g}} e^{-i\langle x,y\rangle} f(x)\ \mathrm{d}x$$

is

$$\check{f}(x) = \int_{\mathfrak{g}} e^{i\langle x,y\rangle} f(y)\ \mathrm{d}y$$

and

$$\int_{\mathfrak{g}} |f(x)|^2\ \mathrm{d}x = \int_{\mathfrak{g}} |\hat{f}(y)|^2\ \mathrm{d}y, \forall f \in \mathcal{S}(\mathfrak{g}).$$

We shall also work with the notions of homogeneous degree of a multiindex and homogeneous degree of a polynomial function on $\mathfrak{g}$. If $\alpha = (\alpha_1, \ldots, \alpha_R), \alpha_k \in \mathbb{N}^{n_k}, n_k = \dim \mathfrak{g}_k, \forall k \in \{1, \ldots, R\}$ is a multiindex in $\mathbb{N}^n$, then we denote with $|\alpha|$ its usual length (the sum of all its n components). The homogeneous length of α is

$$d(\alpha) = \sum_{k=1}^{R} d_k |\alpha_k|.$$

If $x = (x_1, \ldots, x_R) = (x_{1;1}, \ldots, x_{1;n_1}, \ldots, x_{R;1}, \ldots, x_{R;n_R}) \in \mathfrak{g}$, then the homogeneous degree of x^α is $d(\alpha)$.

6 The Melin operator and the reduction operator

The *Melin operator* U on $\mathfrak{g}$ is defined by the formula

$$Uf(\mathbf{y}) = \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} \check{f}(\mathbf{x}) e^{-i\langle r(\mathbf{x}), \tilde{\mathbf{y}}\rangle} \, \mathrm{d}\mathbf{x}, \forall f \in C_0^\infty(\mathfrak{g}\times\mathfrak{g}),$$

where $\mathbf{x} = (x_1, x_2) \in \mathfrak{g}\times\mathfrak{g}$, $\mathbf{y} = (y_1, y_2) \in \mathfrak{g}\times\mathfrak{g}$, $r(\mathbf{x}) = r(x_1, x_2)$ and $\tilde{\mathbf{y}} = \frac{y_1+y_2}{2}$.

Remark 6.1. If $\mathfrak{g}$ is a commutative Lie algebra, then U is the identity operator.

Lemma 6.2. *For every $f, g \in C_0^\infty(\mathfrak{g})$*

$$\widehat{f * g}(y) = U(\hat{f} \otimes \hat{g})(y, y), \forall y \in \mathfrak{g}. \tag{6.1}$$

Proof. For every $f, g \in C_0^\infty(\mathfrak{g})$ we have

$$\begin{aligned}
\widehat{f * g}(y) &= \int_{\mathfrak{g}} e^{-i\langle y, z\rangle} f * g(z) \, \mathrm{d}z = \int_{\mathfrak{g}} e^{-i\langle y, z\rangle} \, \mathrm{d}z \int_{\mathfrak{g}} f(zu^{-1}) g(u) \, \mathrm{d}u \\
&= \int_{\mathfrak{g}} g(u) \, \mathrm{d}u \int_{\mathfrak{g}} e^{-i\langle y, z\rangle} f(zu^{-1}) \, \mathrm{d}z = \int_{\mathfrak{g}} g(u) \, \mathrm{d}u \int_{\mathfrak{g}} e^{-i\langle y, x_1 u\rangle} f(x_1) \, \mathrm{d}x_1 \\
&= \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle y, x_1 + x_2 + r(x_1, x_2)\rangle} f(x_1) g(x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \\
&= U(\hat{f} \otimes \hat{g})(y, y),
\end{aligned}$$

for every $y \in \mathfrak{g}$. ∎

Lemma 6.3. *For every $f \in C_0^\infty(\mathfrak{g}\times\mathfrak{g})$,*

$$\mathbf{D}^\alpha U f(\mathbf{y}) = \sum_{d(\beta)=d(\alpha)} c_{\beta\alpha} U(\mathbf{D}^\beta f)(\mathbf{y})$$

for some constants $c_{\beta\alpha} \in \mathbb{C}$.

Proof. Let us denote with $r_k(\mathbf{x})$ the sum of terms of homogeneous degree k from $r(\mathbf{x})$, for $k \in \{1, \ldots, R\}$. Then

$$\langle r(\mathbf{x}), y\rangle = \sum_{k=1}^{R} \langle r_k(\mathbf{x}), y\rangle, \forall \mathbf{x} \in \mathfrak{g} \times \mathfrak{g}, \forall y \in \mathfrak{g}.$$

Therefore

$$\begin{aligned}
\mathbf{D}^\alpha U f(\mathbf{y}) &= \iint_{\mathfrak{g}\times\mathfrak{g}} \check{f}(\mathbf{x}) \mathbf{D}^\alpha_{\mathbf{y}} \left(e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}), \tilde{\mathbf{y}}\rangle} \right) \, \mathrm{d}\mathbf{x} \\
&= \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}), \tilde{\mathbf{y}}\rangle} P_1(\mathbf{x}) \check{f}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle \mathbf{x}, \mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}), \tilde{\mathbf{y}}\rangle} (P(\mathbf{D}) f)\check{}(\mathbf{x}) \, \mathrm{d}\mathbf{x},
\end{aligned}$$

where P_1 and P are homogeneous polynomials of homogeneous degree $d(\alpha)$. □

Let

$$\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{R-1}.$$

The commutator

$$\mathfrak{g}' \times \mathfrak{g}' \ni (x_1, x_2) \mapsto [x_1, x_2]' \in \mathfrak{g}',$$

where $'$ stands for the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{g}'$, makes $\mathfrak{g}'$ into a Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{g}_R$. The group multiplication in $\mathfrak{g}'$ is

$$x_1 \circ' x_2 = x_1 + x_2 + r(x_1, x_2)', \forall x_1, x_2 \in \mathfrak{g}'.$$

Proposition 6.4. *Let $f \in C_0^\infty(\mathfrak{g} \times \mathfrak{g})$. Then*

$$U f(\mathbf{y}, \lambda) = U' \left(P_\lambda f(\cdot, \lambda)\right)(\mathbf{y}), \forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall \lambda \in \mathfrak{g}_R \times \mathfrak{g}_R, \tag{6.2}$$

where

$$P_\lambda f(\mathbf{y}) = \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \boldsymbol{x}, \boldsymbol{y}\rangle} \check{f}(\mathbf{x}) e^{-i\langle r(\mathbf{x}), \tilde{\lambda}\rangle} \, d\mathbf{x}, \forall f \in C_0^\infty(\mathfrak{g}' \times \mathfrak{g}')$$

is an integral operator on $C_0^\infty(\mathfrak{g}' \times \mathfrak{g}')$ invariant under abelian translations and U' is the Melin operator on $\mathfrak{g}'$.

Proof. Let us first remark that since $\mathfrak{g}_R$ is central,

$$r((x_1, \mu_1), (x_2, \mu_2)) = r(x_1, x_2), \forall (x_1, \mu_1), (x_2, \mu_2) \in \mathfrak{g}' \times \mathfrak{g}_R$$

and

$$\langle r((x_1, \mu_1), (x_2, \mu_2)), (\tilde{\mathbf{y}}, \tilde{\lambda})\rangle = \langle r(x_1, x_2), (\tilde{\mathbf{y}}, \tilde{\lambda})\rangle = \langle r(x_1, x_2)', \tilde{\mathbf{y}}\rangle + \langle r(x_1, x_2), \tilde{\lambda}\rangle$$

for all $(\mathbf{x}, \mu), (\mathbf{y}, \lambda) \in \mathfrak{g} \times \mathfrak{g}$.

Therefore

$$Uf(\mathbf{y},\lambda) = \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle \mu,\lambda\rangle} \check{f}(\mathbf{x},\mu) e^{-i\langle r(\mathbf{x})',\tilde{\mathbf{y}}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} \, \mathrm{d}\mathbf{x}\mathrm{d}\mu$$
$$= \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} \left(f(\check{\mathbf{x}},\lambda) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\right) e^{-i\langle r(\mathbf{x})',\tilde{\mathbf{y}}\rangle} \, \mathrm{d}\mathbf{x}$$

where we denoted with $f(\check{\mathbf{x}},\lambda)$ the partial inverse Fourier transform of f with respect to $\mathbf{y}$.

The proof of (6.2) is concluded by the equality

$$(P_\lambda f(\cdot,\lambda))\check{}(\mathbf{x}) = f(\check{\mathbf{x}},\lambda) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}.$$

If we denote with τ_z the translation with z, $(\tau_z g)(y) = g(y+z), \forall y,z \in \mathfrak{g}'\times\mathfrak{g}', \forall g \in C_0^\infty(\mathfrak{g}'\times\mathfrak{g}')$, then $(\tau_z g)\check{}(x) = e^{-i\langle x,z\rangle}\check{g}(x)$ and $(\tau_z(P_\lambda g))(y) = P_\lambda(\tau_z(g))(y)$. ∎

Remark 6.5. Since P_λ commutes with the translations, then it will commute also with derivatives.

If $\mathfrak{g}$ is a Lie algebra endowed with a family of dilations δ_t then on $X = \mathfrak{g}\times\mathfrak{g}$ we shall consider the family of dilations $\delta_t(\mathbf{x}) = \delta_t(x_1)\oplus\delta_t(x_2), \forall \mathbf{x} = (x_1,x_2)\in\mathfrak{g}\times\mathfrak{g}$.

Definition 6.6. We shall say that a self-tempered metric g on $\mathfrak{g}$ is *admissible* if it satisfies the conditions

$$g_x(z)^2 = \sum_{j=1}^{R} \frac{\|z_j\|^2}{g_j(x)^{2d_j}},$$
$$g_j(x) \geq g_{j+1}(x) \geq \delta, \forall x\in\mathfrak{g}, \forall j \in \{1,\dots,R-1\},$$
$$g_j(x) \geq g_j(0_{(j)}, x^{(j)}) \geq \delta + |x|_{j+1}, \forall x\in\mathfrak{g}, \forall j\in\{1,\dots,R-1\},$$

for some $\delta > 0$. Here $x^{(j)} = (x_{j+1},\dots,x_R)$.

Let us remark that the metrics g_k^δ introduced in Section 3 are admissible metrics.

We also define a metric $\mathbf{g}$ on $\mathfrak{g}\times\mathfrak{g}$ by the formula

$$\mathbf{g}_{\mathbf{x}}(\mathbf{z})^2 = (g\oplus g)_{\mathbf{x}}(\mathbf{z})^2 = \sum_{j=1}^{R}\frac{\|z_{1,j}\|^2}{g_j(x_1)^{2d_j}} + \sum_{j=1}^{R}\frac{\|z_{2,j}\|^2}{g_j(x_2)^{2d_j}}, \forall \mathbf{x} = (x_1,x_2), \mathbf{z} = (z_1,z_2)\in\mathfrak{g}\times\mathfrak{g}.$$

Then $\mathbf{g}$ is also self-tempered. For $\lambda = (\lambda_1,\lambda_2)\in\mathfrak{g}_R\times\mathfrak{g}_R$ we put

$$\mathbf{g}^\lambda_{\mathbf{x}}(\mathbf{z}) = \mathbf{g}_{\mathbf{x},\lambda}(\mathbf{z},0), \forall \mathbf{x},\mathbf{z}\in\mathfrak{g}'\times\mathfrak{g}'.$$

The metrics $\mathbf{g}^\lambda = (\mathbf{g}^\lambda_{\mathbf{x}})_{\mathbf{x}\in\mathfrak{g}'\times\mathfrak{g}'}$ are uniformly self-tempered and, consequently, uniformly slowly varying with respect to $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. Let C, M (in (2.3)) and γ (in (2.1)) be joint constants for all these metrics. We shall also use the following notations: $g_j^\lambda(x) = g_j(x,\lambda), \forall (x,\lambda)\in\mathfrak{g}$, $B_\nu = B_\nu^\lambda = B_\nu^\lambda(\mathbf{x}_\nu^\lambda, r)\subset\mathfrak{g}'\times\mathfrak{g}'$ for the covering of Proposition 4.1 for the metric $\mathbf{g}^\lambda, \lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$, $\mathbf{x}_\nu = \mathbf{x}_\nu^\lambda$ and $d_\nu^\lambda(\mathbf{y}) = \mathbf{g}^\lambda_{\mathbf{x}_\nu}(\mathbf{y}-\mathbf{x}_\nu), \forall \mathbf{y}\in\mathfrak{g}'\times\mathfrak{g}', \forall\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R, \forall \nu\in\mathbb{N}^*$.

For an admissible metric g we put

$$\tilde{g}_{R-1}(\lambda) = \max\left(\frac{g_{R-1}(0,\lambda_1)}{g_{R-1}(0,\lambda_2)}, \frac{g_{R-1}(0,\lambda_2)}{g_{R-1}(0,\lambda_1)}\right), \forall\lambda = (\lambda_1,\lambda_2) \in \mathfrak{g}_R \times \mathfrak{g}_R.$$

From Remark 2.8 and Example 2.10 it follows that $\tilde{g}_{R-1}$ is a $\mathbf{g}$-tempered weight.

Proposition 6.7. *Let* $\mathbf{g} = g \oplus g$, g *an admissible metric on* $\mathfrak{g}$. *Then* $\forall N \in \mathbb{N}$, $\exists C > 0$, $\exists k \in \mathbb{N}$, $\forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall\lambda \in \mathfrak{g}_R \times \mathfrak{g}_R, \forall\nu \in \mathbb{N}^*, \forall f \in C_0^\infty(B_\nu^\lambda)$ *so that*

$$|P_\lambda f(\mathbf{y})| \leq C|f|_k^{\tilde{g}_{R-1}(\lambda)^{N d_R}}(\mathbf{g}^\lambda)\left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N}. \tag{6.3}$$

Proof. For $f \in C_0^\infty(B_\nu^\lambda)$ we have

$$|P_\lambda f(\mathbf{y})| = \left|\iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle\mathbf{x},\mathbf{y}\rangle}\check{f}(\mathbf{x})e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\,\mathrm{d}\mathbf{x}\right| \leq \iint_{\mathfrak{g}'\times\mathfrak{g}'} |\check{f}(\mathbf{x})|\,\mathrm{d}\mathbf{x}.$$

Let

$$f_\nu^\lambda(\mathbf{y}) = f(\mathbf{y}_{(\nu)}^\lambda),$$

where

$$\mathbf{y}_{(\nu)}^\lambda = \left(g_1^\lambda(x_{\nu,1})^{d_1}y_{1,1},\ldots,g_{R-1}^\lambda(x_{\nu,1})^{d_1}y_{1,R-1},g_1^\lambda(x_{\nu,2})^{d_1}y_{2,1},\ldots,g_{R-1}^\lambda(x_{\nu,2})^{d_{R-1}}y_{2,R-1}\right).$$

Then

$$\iint_{\mathfrak{g}'\times\mathfrak{g}'} |\check{f}(\mathbf{x})|\,\mathrm{d}\mathbf{x} = \iint_{\mathfrak{g}'\times\mathfrak{g}'} |\check{f}_\nu^\lambda(\mathbf{x})|\,\mathrm{d}\mathbf{x}.$$

If $\mathbf{y} \in \mathrm{supp} f_\nu^\lambda$, then

$$\begin{aligned}
1 &\geq \gamma^2 > r^2 \\
&\geq \mathbf{g}_{\mathbf{x}_\nu}^\lambda\left(g_1^\lambda(x_{\nu,1})^{d_1}y_{1,1},\ldots,g_{R-1}^\lambda(x_{\nu,1})^{d_1}y_{1,R-1},g_1^\lambda(x_{\nu,2})^{d_1}y_{2,j},\ldots,g_{R-1}^\lambda(x_{\nu,2})^{d_{R-1}}y_{2,R-1}\right)^2 \\
&= \sum_{j=1}^{R-1}\frac{g_j^\lambda(x_{\nu,1})^{2d_j}\left\|y_{1,j} - \frac{x_{\nu,1,j}}{g_j^\lambda(x_{\nu,1})^{d_j}}\right\|^2}{g_j^\lambda(x_{\nu,1})^{2d_j}} + \sum_{j=1}^{R-1}\frac{g_j^\lambda(x_{\nu,2})^{2d_j}\left\|y_{2,j} - \frac{x_{\nu,2,j}}{g_j^\lambda(x_{\nu,2})^{d_j}}\right\|^2}{g_j^\lambda(x_{\nu,2})^{2d_j}} \\
&= \|\mathbf{y} - \tilde{\mathbf{x}}_\nu\|^2
\end{aligned}$$

for some $\tilde{\mathbf{x}}_\nu \in \mathfrak{g} \times \mathfrak{g}$.

Therefore $\mathrm{supp} f_\nu^\lambda$ is included in a ball of radius 1 with respect to the fixed euclidean norm on $\mathfrak{g} \times \mathfrak{g}$. Hence we obtain from Sobolev's lemma that there exists some positive constant C so that

$$\iint_{\mathfrak{g}'\times\mathfrak{g}'} |\check{f}_\nu^\lambda(\mathbf{x})|\,\mathrm{d}\mathbf{x} \leq \sup_{|\alpha|\leq 2n+1}\ \sup_{\mathbf{y}\in\mathfrak{g}'\times\mathfrak{g}'} |D^\alpha f_\nu^\lambda(\mathbf{y})|.$$

But, using Lemma 2.6, we have

$$\begin{aligned}
\left|D^\alpha f_\nu^\lambda(\mathbf{y})\right| &= \left|\prod_{j=1}^{R-1} g_j^\lambda(x_{\nu,1})^{|\alpha_{1,j}|d_j}(D^{\alpha_1}f)_\nu^\lambda(\mathbf{y}) \cdot \prod_{j=1}^{R-1} g_j^\lambda(x_{\nu,2})^{|\alpha_{2,j}|d_j}(D^{\alpha_2}f)_\nu^\lambda(\mathbf{y})\right| \\
&= \left|\prod_{j=1}^{R-1} g_j^\lambda(x_{\nu,1})^{|\alpha_{1,j}|d_j} D^{\alpha_1}f(\mathbf{y}_{(\nu)}^\lambda) \cdot \prod_{j=1}^{R-1} g_j^\lambda(x_{\nu,2})^{|\alpha_{2,j}|d_j} D^{\alpha_2}f(\mathbf{y}_{(\nu)}^\lambda)\right| \\
&\le \left(\frac{1}{\gamma}\right)^{|\alpha|} \prod_{j=1}^{R-1} g_j^\lambda(\mathbf{y}_{(\nu),1}^\lambda)^{|\alpha_{1,j}|d_j}\left|D^{\alpha_1}f(\mathbf{y}_\nu^\lambda)\right| \cdot \prod_{j=1}^{R-1} g_j^\lambda(\mathbf{y}_{(\nu),2}^\lambda)^{|\alpha_{2,j}|d_j}\left|D^{\alpha_2}f(\mathbf{y}_{(\nu)}^\lambda)\right| \\
&\le C_k|f|_k^1(\mathbf{g}^\lambda)
\end{aligned}$$

for $|\alpha| \le k$.

Therefore for $N = 0$, (6.3) holds with $k = n + 1$.

We shall prove (6.3) for $N \in \mathbb{N}$ by induction on N. So let us assume that (6.3) is true for some N and let us prove it for $N + 1$. Let $d_\nu^\lambda(\mathbf{y}) = a > 1$ (otherwise the estimate is a simple consequence of the estimate for $N = 0$).

Let $\xi \in (\mathfrak{g}' \times \mathfrak{g}')^*$ be a vector of unit length with respect to the norm dual to $\mathbf{g}_{\mathbf{x}_\nu}^\lambda$ so that $\xi(\mathbf{y} - \mathbf{x}_\nu) = a$. Then, for $r_1 \in (r, \gamma)$ we have

$$\xi(\mathbf{y}-\mathbf{x}) = \xi(\mathbf{y}-\mathbf{x}_\nu) - \xi(\mathbf{x}-\mathbf{x}_\nu) \ge a - |\xi(\mathbf{x}-\mathbf{x}_\nu)| \ge a - 1 > 0, \forall \mathbf{x} \in B_\nu^\lambda(\mathbf{x}_\nu^\lambda, r_1).$$

Let $L(\mathbf{x}) = \xi(\mathbf{x} - \mathbf{y}), \forall \mathbf{x} \in \mathfrak{g}' \times \mathfrak{g}'$. Then $L(\mathbf{x}_\nu) = -a$ and L does not vanish on $B_\nu^\lambda(\mathbf{x}_\nu^\lambda, r_1)$. Therefore, by Proposition 4.2, $\forall k \in \mathbb{N}$, there exists a positive constant $C_k = C_k(r, r_1)$ so that

$$\mathbf{g}_\mathbf{x}^\lambda\left(D^k\frac{1}{L}(\mathbf{x})\right) \le \gamma^{-k}\mathbf{g}_{\mathbf{x}_\nu}^\lambda\left(D^k\frac{1}{L}(\mathbf{x})\right) \le \frac{C_k}{a}, \forall \mathbf{x} \in B_\nu^\lambda(\mathbf{x}_\nu^\lambda, r). \tag{6.4}$$

Another inequality we shall need follows from the fact that $\xi \in (\mathfrak{g}' \times \mathfrak{g}')^*$ is a vector of unit length with respect to the norm dual to $\mathbf{g}_{\mathbf{x}_\nu}^\lambda$:

$$\begin{aligned}
1 &= \sum_{j=1}^{R-1} g_j^{\lambda_1}(x_{\nu,1})^{2d_j}\|\xi_{1,j}\|^2 + \sum_{j=1}^{R-1} g_j^{\lambda_1}(x_{\nu,2})^{2d_j}\|\xi_{2,j}\|^2 \\
&\ge \sum_{j=1}^{R-1} g_{R-1}(0,\lambda_1)^{2d_j}\|\xi_{1,j}\|^2 + \sum_{j=1}^{R-1} g_{R-1}(0,\lambda_2)^{2d_j}\|\xi_{2,j}\|^2.
\end{aligned} \tag{6.5}$$

Let $A\mathbf{x} = \langle \mathbf{x}, \xi \rangle, \forall \mathbf{x} \in \mathfrak{g}' \times \mathfrak{g}'$. Then, since $L(\mathbf{y}) = 0$, we have

$$\begin{aligned}
P_\lambda(Lf)(\mathbf{y}) =& [P_\lambda, L]f(\mathbf{y}) = [P_\lambda, A]f(\mathbf{y}) = \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} A f\check{}(\mathbf{x}) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, \mathrm{d}\mathbf{x} \\
&- \iint_{\mathfrak{g}'\times\mathfrak{g}'} \langle \mathbf{y}, \xi\rangle e^{-i\langle \mathbf{x},\mathbf{y}\rangle} \check{f}(\mathbf{x}) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, \mathrm{d}\mathbf{x} \\
=& -i \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} (\langle \xi, D - i\mathbf{y}\rangle \check{f})(\mathbf{x}) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, \mathrm{d}\mathbf{x} \\
=& - \iint_{\mathfrak{g}'\times\mathfrak{g}'} \langle \mathbf{y}, \xi\rangle e^{-i\langle \mathbf{x},\mathbf{y}\rangle} \check{f}(\mathbf{x}) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, \mathrm{d}\mathbf{x} \\
&+ i \iint_{\mathfrak{g}'\times\mathfrak{g}'} \check{f}(\mathbf{x}) \langle \xi, D_{\mathbf{x}}\rangle \left(e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} \right) \mathrm{d}\mathbf{x} \\
=& \iint_{\mathfrak{g}'\times\mathfrak{g}'} \check{f}(\mathbf{x}) \left(\langle \xi, D_{\mathbf{x}}\rangle (\langle r(\mathbf{x}), \tilde{\lambda}\rangle) \right) e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, \mathrm{d}\mathbf{x} \\
=& \sum_{j=1}^{R-1} \langle \xi_{1,j}, \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} \langle D_{x_{1,j}} r(\mathbf{x}), \tilde{\lambda}\rangle \check{f}(\mathbf{x})\, \mathrm{d}\mathbf{x}\rangle \\
&+ \sum_{j=1}^{R-1} \langle \xi_{2,j}, \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} \langle D_{x_{2,j}} r(\mathbf{x}), \tilde{\lambda}\rangle \check{f}(\mathbf{x})\, \mathrm{d}\mathbf{x}\rangle \\
=& \sum_{j=1}^{R-1} \langle \xi_{1,j}, \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} (\langle r_{1,j}(iD), \tilde{\lambda}\rangle f)\check{}(\mathbf{x})\, \mathrm{d}\mathbf{x}\rangle \\
&+ \sum_{j=1}^{R-1} \langle \xi_{2,j}, \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle} (\langle r_{2,j}(iD), \tilde{\lambda}\rangle f)\check{}(\mathbf{x})\, \mathrm{d}\mathbf{x}\rangle \\
=& \sum_{j=1}^{R-1} \langle \xi_{1,j}, P_\lambda(\langle r_{1,j}(iD), \tilde{\lambda}\rangle f)(\mathbf{y})\rangle + \sum_{j=1}^{R-1} \langle \xi_{2,j}, P_\lambda(\langle r_{2,j}(iD), \tilde{\lambda}\rangle f)(\mathbf{y})\rangle, \quad (6.6)
\end{aligned}$$

where $r_{i,j}(\mathbf{x}) = D_{x_{i,j}} r(\mathbf{x})$ for $i = 1, 2$ and $\forall j \in \{1, \ldots, R-1\}$ are homogeneous polynomials of homogeneous degree $d_R - d_j$. Let us stress that here D stands for the partial derivatives $D = \partial$.

Using the induction hypothesis, we obtain

$$\begin{aligned}
\left| P_\lambda(\langle r_{i,j}(iD), \tilde{\lambda}\rangle f)(\mathbf{y}) \right| \le& C \left| \langle r_{i,j}(iD), \tilde{\lambda}\rangle f \right|_k^{\tilde{g}_{R-1}(\lambda)^{N d_R}} (\mathbf{g}^\lambda) \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N} \\
\le& C \tilde{g}_{R-1}(\lambda)^{N d_R} \sup_{|\alpha| \le k, \mathbf{z} \in \mathfrak{g}'\times\mathfrak{g}'} \left| D^\alpha \langle r_{i,j}(iD), \tilde{\lambda}\rangle f(\mathbf{z}) \right| \\
&\times \left(\prod_{i_1=1}^{R-1} g_{i_1}^{\lambda_1}(z_1)^{d_{i_1} |\alpha_{1,i_1}|} \right) \left(\prod_{i_2=1}^{R-1} g_{i_2}^{\lambda_2}(z_2)^{d_{i_2} |\alpha_{2,i_2}|} \right) \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N} \\
\le& C \tilde{g}_{R-1}(\lambda)^{N d_R} \sup_{|\alpha| \le k, \mathbf{z} \in \mathfrak{g}'\times\mathfrak{g}', d(\beta) = d_R - d_j} \left| D^{\alpha+\beta} f(\mathbf{z}) \right| \|\tilde{\lambda}\|
\end{aligned}$$

$$\times \left(\prod_{i_1=1}^{R-1} g_{i_1}^{\lambda_1}(z_1)^{d_{i_1}(|\alpha_{1,i_1}|+|\beta_{1,i_1}|)} \right) \left(\prod_{i_2=1}^{R-1} g_{i_2}^{\lambda_2}(z_2)^{d_{i_2}(|\alpha_{2,i_2}|+|\beta_{2,i_2}|)} \right)$$
$$\times\, g_{R-1}(0,\lambda_1)^{-d(\beta_1)} g_{R-1}(0,\lambda_2)^{-d(\beta_2)} \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N}. \tag{6.7}$$

Formulas (6.4)–(6.7), the fact that g is an admissible metric, Lemma 2.11(a) and Remark 2.13 conclude the proof:

$$\begin{aligned}
|P_\lambda f(\mathbf{y})| &= \left| P_\lambda \left(L \cdot \frac{1}{L} f \right)(\mathbf{y}) \right| \\
&\le \sum_{j=1}^{R-1} \left| \langle \xi_{1,j}, P_\lambda(\langle r_{1,j}(iD), \tilde{\lambda} \rangle \left(\frac{1}{L} f \right))(\mathbf{y}) \rangle \right| \\
&\quad + \sum_{j=1}^{R-1} \left| \langle \xi_{2,j}, P_\lambda(\langle r_{2,j}(iD), \tilde{\lambda} \rangle \left(\frac{1}{L} f \right))(\mathbf{y}) \rangle \right| \\
&\le C \tilde{g}_{R-1}(\lambda)^{(N+1)d_R} \left| \frac{1}{L} f \right|_{k'}^{1} (\mathbf{g}_\lambda) \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N} \\
&\le C \tilde{g}_{R-1}(\lambda)^{(N+1)d_R} \frac{C_{k'}}{a} \cdot |f|_{k'}^{1} (\mathbf{g}_\lambda) \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N} \\
&\le C'_{k'} \tilde{g}_{R-1}(\lambda)^{(N+1)d_R} |f|_{k'}^{1} (\mathbf{g}_\lambda) \left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N-1}
\end{aligned}$$

for $k' \ge k(N) + d_R$. ∎

Remark 6.8. From the proof of Proposition 6.7 we can see that the conclusion of the proposition is still true if we replace $\tilde{g}_{R-1}$ with

$$\tilde{q}_{R-1}(\lambda) = \max \left(\frac{1 + \|\lambda_1\|}{1 + \|\lambda_2\|}, \frac{1 + \|\lambda_2\|}{1 + \|\lambda_1\|} \right), \forall \lambda = (\lambda_1, \lambda_2) \in \mathfrak{g}_R \times \mathfrak{g}_R.$$

Remark 6.9. If instead of the metric $g \oplus g$ with g admissible we have on $\mathfrak{g} \times \mathfrak{g}$ a metric of the form

$$\mathbf{g}_{\mathbf{x}}(\mathbf{z})^2 = \sum_{j=1}^{R} \frac{\|z_{1,j}\|^2}{g_{1,j}(\mathbf{x})^{2d_j}} + \sum_{j=1}^{R} \frac{\|z_{2,j}\|^2}{g_{2,j}(\mathbf{x})^{2d_j}}, \forall \mathbf{x} = (x_1, x_2), \mathbf{z} = (z_1, z_2) \in \mathfrak{g} \times \mathfrak{g},$$

with $g_{i,j}(x) \ge \delta + \|\mathbf{x}_R\|^{1/d_R}, \forall \mathbf{x} \in \mathfrak{g} \times \mathfrak{g}, \forall i \in \{1, 2\}, \forall j \in \{1, \ldots, R-1\}$, for some $\delta > 0$, then the conclusion of the Proposition 6.7 is still true if we replace $\tilde{g}_R$ with 1.

At this point we need to introduce the notion of double continuous mapping between spaces of symbols. Let $S^m(X, g)$ be a space of symbols on an euclidean space X, m a G-tempered weight with respect to a G-tempered slowly varying metric g. Besides the Fréchet topology on $S^m(X, g)$ (see Remark 2.13), we introduce the weak topology [5] of the C^∞ convergence on Fréchet bounded subsets.

Lemma 6.10. *The weak convergence is equivalent to the pointwise convergence on Fréchet bounded subsets of* $S^m(X, g)$.

Proof. We shall apply Arzela-Ascoli theorem. First of all, if $(f_j)_j$ is a bounded sequence in $S^m(X,g)$ and if K is a compact set in X, then $(D^k f_j)_j$ is a sequence of uniformly bounded functions on K.

Let us prove this assertion for $k=1$. The balls $(B^g(x,\gamma))_{x\in K}$ are an open covering of K. Therefore there exists a finite set $\{x_1,\dots,x_l\}\subset K$ so that $(B^g(x_i,\gamma))_{i\in\{1,\dots,l\}}$ is still a covering of K. Each of the metrics $g_{x_i}, i\in\{1,\dots,l\}$, is equivalent to the euclidean metric. Hence there exists some positive constant C_1 so that $g_{x_i}(y)\le C_1\|y\|, \forall i\in\{1,\dots,l\}, \forall y\in X$. An arbitrary point $x\in K$ belongs to some ball $B^g(x_i,\gamma)$. Therefore, from (2.1) we obtain

$$g_x(y)\le \frac{1}{\gamma}C_1\|y\|, \forall x\in K, \forall y\in X.$$

Now our assertion in case $k=1$ follows from the fact that m beeing a g weight is bounded on K and from the boundedness in $S^m(X,g)$ of the sequence $(f_j)_j$. Its proof for the other values of k is similar.

From Arzela-Ascoli theorem it follows that every sub-sequence of $(D^k f_j)_j$ contains a sub-sub-sequence uniformly convergent on K. If $f_j\to f$ pointwise, then, in case $k=0$, this limit is always equal to f. Therefore $f_j\to f$ uniformly on K. For $k>0$, in order to obtain the same conclusion we have to use also either the fact that $f_j\to f$ in the distribution sense, or the classical theorem of derivation of sequences of functions. ∎

Lemma 6.11. *Let m be a G-tempered weight with respect to a G-tempered slowly varying metric g, $f\in S^m(X,g)$ and $(\phi_\nu)_\nu$ the partition of unity from Proposition 4.1. Then $\sum_{\nu=1}^{j}\phi_\nu f\to f$ weakly in $S^m(X,g)$ when $j\to\infty$.*

Proof. This lemma is a straightforward consequence of the definitions, of Propositions 4.1(b) and of Lemma 2.11(a). ∎

Remark 6.12. Let $\mathbf{m}$ be a $\mathbf{G}$-tempered weight with respect to the slowly varying metric $\mathbf{g}$ on $\mathfrak{g}\times\mathfrak{g}$ and $\mathbf{m}^\lambda(\mathbf{x})=\mathbf{m}(\mathbf{x},\lambda), \forall\mathbf{x}\in\mathfrak{g}'\times\mathfrak{g}', \forall\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. Then $\mathbf{m}^\lambda$ is a $\mathbf{G}^\lambda$-tempered weight with respect to the slowly varying metric $\mathbf{g}^\lambda$, uniformly with respect to λ.

Remark 6.13. Let $\mathbf{g}$ be a slowly varying metric on $\mathfrak{g}\times\mathfrak{g}$ and $\mathbf{g}^\lambda(\mathbf{x})=\mathbf{g}(\mathbf{x},\lambda), \forall\mathbf{x}\in\mathfrak{g}'\times\mathfrak{g}', \forall\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. Then the partition of unity $(\phi_\nu^\lambda)_\nu$ in the conclusion of Proposition 4.1(b) can be selected so that the sequences $(\phi_\nu^\lambda)_\nu$ are uniformly bounded in $\mathcal{S}^1(\mathfrak{g}'\times\mathfrak{g}')$, with respect to $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. This assertion follows from the proof of Proposition 4.1(b), since the constant γ in (2.2) does not depend on λ.

Remark 6.14. Let $\mathbf{g}$ be a self-tempered metric on $\mathfrak{g}\times\mathfrak{g}$. Then the constants $\tilde{C}$ and $\tilde{M}$ in the conclusion of Proposition 4.1(c), corresponding to the metrics $\mathbf{g}^\lambda$ can be selected the same for all $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. This assertion follows from the proof of Proposition 4.1(c), since the constants γ in (2.2) and C and M in (2.3) do not depend on λ.

Proposition 6.15. *Let $\mathbf{g}=g\oplus g$, g an admissible metric on $\mathfrak{g}$ and $\mathbf{m}$ be a $\mathbf{g}$-tempered weight. Then for $N\in\mathbb{N}$, sufficiently large there exists a double continuous extension of P_λ to a map*

$$P_\lambda:\mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)\to\mathcal{S}^{\mathbf{m}^\lambda\tilde{g}_{R^{-1}}(\lambda)^{Nd_R}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$$

and the estimates are uniform with respect to $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$.

Proof. We shall apply Proposition 6.7. Let $(\phi_\nu^\lambda)_\nu$ be a partition of unity as in Remark 6.13. We shall denote with C_1 a constant which may depend on the constant γ in (2.2), on the constants C and M in (2.3), on $N \in \mathbb{N}$ and on r in Proposition 4.1, but does not depend on $\lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$ and on $\nu \in \mathbb{N}^*$. Then $\forall N \in \mathbb{N}$, $\forall f \in \mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{g}^\lambda)$

$$|P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| \le C_1 |\phi_\nu^\lambda f|_k^{\tilde{g}_{R-1}(\lambda)^{Nd_R}}(\mathbf{g}^\lambda)\left(1 + d_\nu^\lambda(\mathbf{y})\right)^{-N}, \forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall \lambda \in \mathfrak{g}_R \times \mathfrak{g}_R, \forall \nu \in \mathbb{N}^*.$$

Since $\mathbf{m}$ is a $\mathbf{g}$ tempered weight and ϕ_ν^λ are uniformly bounded in $\mathcal{S}^1(\mathfrak{g}' \times \mathfrak{g}')$, with respect to $\lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$ and to $\nu \in \mathbb{N}^*$, we have

$$\begin{aligned} \mathbf{m}_\lambda(\mathbf{y})^{-1}|P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| &\le C_1 \mathbf{m}_\lambda(\mathbf{x}_\nu)^{-1}(1 + d_\nu^\lambda(\mathbf{y}))^M |P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| \\ &\le C_1 \mathbf{m}_\lambda(\mathbf{x}_\nu)^{-1}|\phi_\nu^\lambda f|_k^{\tilde{g}_{R-1}(\lambda)^{Nd_R}}(\mathbf{g}^\lambda)(1 + d_\nu^\lambda(\mathbf{y}))^{M-N} \\ &\le C_1 |\phi_\nu^\lambda f|_k^{\mathbf{m}^\lambda \tilde{g}_{R-1}(\lambda)^{Nd_R}}(\mathbf{g}^\lambda)(1 + d_\nu^\lambda(\mathbf{y}))^{M-N} \\ &\le C_1 |f|_k^{\mathbf{m}^\lambda \tilde{g}_{R-1}(\lambda)^{Nd_R}}(\mathbf{g}^\lambda)(1 + d_\nu^\lambda(\mathbf{y}))^{M-N}. \end{aligned} \tag{6.8}$$

If N is sufficiently large so that

$$\sum_\nu (1 + d_\nu^\lambda(\mathbf{y}))^{M-N} \le C_0 < \infty, \forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall \lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$$

then, accordingly to (6.8),

$$\sum_\nu |P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| \le C_1 |f|_k^{\mathbf{m}^\lambda \tilde{g}_{R-1}(\lambda)^{Nd_R}}(\mathbf{g}^\lambda)\mathbf{m}_\lambda(\mathbf{y}), \forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall \lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$$

and $\forall f \in \mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{g}^\lambda)$.

Therefore the operator

$$f \mapsto \left(\mathbf{m}^\lambda\right)^{-1} \tilde{g}_{R-1}(\lambda)^{-Nd_R} \sum_\nu P_\lambda(\phi_\nu^\lambda f)$$

is an extension of $\left(\mathbf{m}^\lambda\right)^{-1} \tilde{g}_R(\lambda)^{-Nd_R} P_\lambda$ to a continuous operator defined on $\mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{g}^\lambda)$ and L^∞ valued. The estimates are uniform in λ.

The uniqueness of this extension follows from Lemma 6.11 if we prove that the linear form

$$\mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{g}^\lambda) \ni f \mapsto \sum_\nu P_\lambda(\Phi_\nu^\lambda f)(\mathbf{y}) \in \mathbb{C}$$

is weakly continuous $\forall \mathbf{y} \in \mathfrak{g}' \times \mathfrak{g}', \forall \lambda \in \mathfrak{g}_R \times \mathfrak{g}_R$.

Let $(f_j)_j$ be a bounded sequence in $\mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}' \times \mathfrak{g}', \mathbf{g}^\lambda)$ so that $f_j \to f$ in the C^∞ topology, $M_j = M_j(\mathbf{y}, \lambda)$ be sets as in the proof of Proposition 4.1(c) and let m be the constant in (4.3). Then

$$\begin{aligned} &\left(\mathbf{m}^\lambda(\mathbf{y})\right)^{-1} \tilde{g}_{R-1}(\lambda)^{-Nd_R} \sum_\nu |P_\lambda(\phi_\nu^\lambda(f_j - f))(\mathbf{y})| \\ &\qquad \le C_2 \sum_\nu |\phi_\nu^\lambda(f_j - f)|_k^{\mathbf{m}^\lambda}(\mathbf{g}^\lambda)\left(1 + d_\nu^\lambda(\mathbf{y})\right)^{M-N} \end{aligned}$$

$$\begin{aligned}
&\le C_2 \sum_{l\ge 0}\sum_{\nu\in M_{l+1}\setminus M_l} |\phi_\nu^\lambda(f_j-f)|_k^{\mathbf{m}^\lambda}(\mathbf{g}^\lambda)(1+l)^{M-N}\\
&\le C_2 \sum_{l=0}^{L}\sum_{\nu\in M_{l+1}\setminus M_l} |\phi_\nu^\lambda(f_j-f)|_k^{\mathbf{m}^\lambda}(\mathbf{g}^\lambda)\\
&\quad + C_2\sum_{l>L}(1+l)^{M+m-N}.
\end{aligned}$$

We denoted by C_2 a constant which may depend on λ and on the norms of the functions f_j also.

If we first select L sufficiently large and if we next remark that $\phi_\nu^\lambda f_j \to \phi_\nu^\lambda f, \forall \nu$, in the topology of $\mathcal{S}^{\mathbf{m}^\lambda}(\mathfrak{g}'\times\mathfrak{g}', \mathbf{g}^\lambda)$ we see that

$$\sum_\nu P_\lambda(\phi_\nu^\lambda f_j)(\mathbf{y}) = \sum_\nu P_\lambda(\phi_\nu^\lambda f)(\mathbf{y}).$$

Estimates for the derivatives of $P_\lambda f$ are obtained from the fact that P_λ and, consequently, its extension, commute with differentiation.

Finally, the weak convergence is now a consequence of Lemma 6.10. □

Remark 6.16. For the proof of Proposition 6.17 below we shall need a slightly different version of Proposition 6.15. Before stating this version we have to introduce some more notation. Starting fom this point, for $j\in\{1,\dots,R-1\}$ we put $\mathfrak{g}_{(j)}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_j$ (in the proof of Proposition 5.1, $\mathfrak{g}_{(j)}$ denoted a different object) , and $\mathfrak{g}^{(j)}=\mathfrak{g}_{j+1}\oplus\cdots\oplus\mathfrak{g}_R$. If $x=(x_1,\dots x_R)\in\mathfrak{g}$ and $j\in\{1,\dots,R-1\}$ then we put $x_{(j)}=(x_1,\dots x_j)$ and $x^{(j)}=(x_{j+1},\dots,x_R)$. Also, if g is a metric on $\mathfrak{g}$, then we define a metric $g^{\lambda^{(j)}}$ on $\mathfrak{g}_{(j)}$ by the formula

$$g^{\lambda^{(j)}}_{x_{(j)}}(z_{(j)}) = g_{(x_{(j)},\lambda^{(j)})}(z_{(j)},0^{(j)}), \forall x_{(j)}, z_{(j)}\in\mathfrak{g}_{(j)}, \forall\lambda^{(j)}\in\mathfrak{g}^{(j)}$$

and if g is an admissible metric, then $\forall\lambda^{(j)}=(\lambda_1^{(j)},\lambda_2^{(j)})\in\mathfrak{g}^{(j)}\times\mathfrak{g}^{(j)}, \forall j\in\{1,\dots,R-1\}$,

$$\tilde g_j(\lambda^{(j)}) = \max\left(\frac{g_j(0,\lambda_1^{(j)})}{g_j(0,\lambda_2^{(j)})}, \frac{g_j(0,\lambda_2^{(j)})}{g_j(0,\lambda_1^{(j)})}\right).$$

The functions $\mathfrak{g}_{(j+1)}\times\mathfrak{g}_{(j+1)}\ni(0_{(j)},\lambda_{j+1})\mapsto\tilde g_j(\lambda^{(j)})$ are $(g\oplus g)^{\lambda^{(j+1)}}$ tempered weights, uniformly with respect to $\lambda^{(j+1)}$ and $\mathfrak{g}\times\mathfrak{g}\ni(0_{(j)},\lambda^{(j)})\mapsto\tilde g_j(\lambda^{(j)})$ are $g\oplus g$ tempered weights. In this context, $(g\oplus g)^{\lambda^{(R)}}=g\oplus g$.

On $C_0^\infty(\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)})$, for $\lambda^{(j)}\in\mathfrak{g}^{(j)}\times\mathfrak{g}^{(j)}$, we define an operator $P_{\lambda^{(j)}}$ by the formula

$$P_{\lambda^{(j)}} f(\mathbf{y}_{(j)}) = \iint_{\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}} e^{-i\langle\mathbf{x}_{(j)},\mathbf{y}_{(j)}\rangle}\check f(\mathbf{x}_{(j)}) e^{-i\langle r_{j+1}(\mathbf{x}_{(j)}),\tilde\lambda_{j+1}\rangle}\,\mathrm{d}\mathbf{x}_{(j)}, \forall f\in C_0^\infty(\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}),$$

where $r_{j+1}(\mathbf{x}_{(j)})$ is the projection of $r(\mathbf{x}_{(j)})$ on $\mathfrak{g}_{(j+1)}$.

Finally, for $\mathbf{m}$ a $g\oplus g$ tempered weight, we put

$$\mathbf{m}^{\lambda^{(j)}}(\mathbf{x}_{(j)}) = \mathbf{m}((\mathbf{x}_{(j)},\lambda^{(j)})), \forall\mathbf{x}_{(j)}\in\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}, \forall\lambda^{(j)}\in\mathfrak{g}^{(j)}\times\mathfrak{g}^{(j)}.$$

Then, for N sufficiently large, $P_{\lambda^{(j)}}$ can be extended to a continuous operator

$$P_{\lambda^{(j)}} : S^{m^{\lambda^{(j)}}}(\mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)}, (g \oplus g)^{\lambda^{(j)}}) \to S^{m^{\lambda^{(j)}}(\tilde{g}_j(\lambda^{(j)}))^N}(\mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)}, (g \oplus g)^{\lambda^{(j)}})$$

and the estimates are uniform with respect to $\lambda^{(j)} \in \mathfrak{g}^{(j)} \times \mathfrak{g}^{(j)}$.

Proposition 6.17. *Let g be an admissible metric on $\mathfrak{g}$ and $\mathbf{m}$ a $g \oplus g$ tempered weight. Then the Melin operator U admits a unique weakly continuous extension*

$$U : S^{\mathbf{m}}(\mathfrak{g} \times \mathfrak{g}, g \oplus g) \to C^\infty(\mathfrak{g} \times \mathfrak{g})$$

so that $\forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{2n}$, there exist $N_\alpha \in \mathbb{N}$, $k_\alpha \in \mathbb{N}$ and $C_\alpha > 0$ such that

$$|\partial^\alpha U f(\mathbf{x})| \leq C_\alpha \mathbf{m}(\mathbf{x}) \prod_{j=1}^{R-1} \tilde{g}_j(\mathbf{x}^{(j)})^{N_\alpha} \prod_{i_1=1}^{R} g_{i_1}(x_1)^{-d_{i_1}|\alpha_{1,i_1}|} \prod_{i_2=1}^{R} g_{i_2}(x_2)^{-d_{i_2}|\alpha_{2,i_2}|} |f|^{\mathbf{m}}_{k_\alpha}(\mathbf{g}), \tag{6.9}$$

for all $\mathbf{x} \in \mathfrak{g} \times \mathfrak{g}$ and for all $f \in S^{\mathbf{m}}(\mathfrak{g} \times \mathfrak{g}, g \oplus g)$.

Proof. We shall prove (6.9) by induction on j. We shall denote with U_j the Melin operator defined on $C_0^\infty(\mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)})$.

If $j = 1$, then $\mathfrak{g}_{(1)} = \mathfrak{g}_1$ is an abelian algebra, $U_1 = I$ and there is nothing to prove. Let us assume that the assertion is true for j and let us prove it for $j+1$. For $f \in S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g}, g\oplus g)$ and $\lambda^{(j)} \in \mathfrak{g}^{(j)}$ let us put

$$f_{\lambda^{(j)}}(\mathbf{x}_{(j)}) = f(x_{(j)}, \lambda^{(j)}), \forall \mathbf{x}_{(j)} \in \mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)}.$$

Then

$$f_{\lambda^{(j)}} \in S^{\mathbf{m}^{\lambda^{(j)}}}(\mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)}, (g \oplus g)^{\lambda^{(j)}})$$

uniformly with respect to $\lambda^{(j)}$.

Therefore, accordingly to Remark 6.16, to the induction hypothesis and to the formula

$$U_{j+1} f(\mathbf{x}_{(j)}, \lambda^{(j)}) = U_j(P_{\lambda^{(j)}} f_{\lambda^{(j)}})(\mathbf{x}_{(j)}), \forall (\mathbf{x}_{(j)}, \lambda^{(j)}) \in \mathfrak{g} \times \mathfrak{g}$$

we obtain that $\forall \alpha_{(j)} = (\alpha_{1,(j)}, \alpha_{2,(j)}) \in \mathbb{N}^{\sum_{k=1}^{j} 2n_k}$, there exist $N_{\alpha_{(j)}}, k_{\alpha_{(j)}} \in \mathbb{N}$ and $C_{\alpha_{(j)}} > 0$ so that

$$\begin{aligned} |\partial^{\alpha_{(j)}} U_{j+1} f(\mathbf{x}_{(j)}, \lambda^{(j)})| &\leq C_{\alpha_{(j)}} \mathbf{m}^{\lambda^{(j)}}(\mathbf{x}_{(j)}) \prod_{k=1}^{j} \tilde{g}_k((\mathbf{x}_{(j)}, \lambda^{(j)})^{(k)})^{N_{\alpha_{(j)}}} \\ &\times \prod_{i_1=1}^{j+1} g_k(x_{1,(j)}, \lambda_1^{(j)})^{-d_{i_1}|\alpha_{1,i_1}|} \prod_{i_2=1}^{j+1} g_k(x_{2,(j)}, \lambda_2^{(j)})^{-d_{i_2}|\alpha_{2,i_2}|} |f|^{\mathbf{m}}_{k_{\alpha_{(j)}}}(\mathbf{g}), \end{aligned}$$

for all $\mathbf{x}_{(j)} \in \mathfrak{g}_{(j)} \times \mathfrak{g}_{(j)}$ and for all $f \in S^{\mathbf{m}^{\lambda^{(j+1)}}}(\mathfrak{g}_{(j+1)} \times \mathfrak{g}_{(j+1)}, (g \oplus g)^{\lambda^{(j+1)}})$. The estimates are uniform with respect to $\lambda^{(j)}$.

We have to estimate the derivatives of $U_{j+1} f(\cdot, \lambda^{(j)})$ with respect to λ_{j+1} also. In order to simplify the notations we shall assume that $\mathfrak{g}_{j+1}$ has dimension 1 and $\lambda_{j+1} =$

$(\lambda_{1,j+1}, \lambda_{2,j+1})$. We consider only the derivative of order 1 with respect to $\lambda_{1,j+1}$. Similar estimates for derivatives of greater order will follow by induction on the order of derivation. We have

$$\frac{\partial}{\partial_{\lambda_{1,j+1}}} U_{j+1} f(\mathbf{y}_{(j)}, \lambda^{(j)}) = \frac{\partial}{\partial_{\lambda_{1,j+1}}} U_j(P_{\lambda^{(j)}} f(\cdot, \lambda^{(j)}))(\mathbf{y}_{(j)}) = U_j(\frac{\partial}{\partial_{\lambda_{1,j+1}}} P_{\lambda^{(j)}} f(\cdot, \lambda^{(j)}))(\mathbf{y}_{(j)})$$

and

$$\begin{aligned}
&\frac{\partial}{\partial_{\lambda_{1,j+1}}} P_{\lambda^{(j)}} f(\cdot,\lambda^{(j)})(\mathbf{y}_{(j)}) \\
&\quad = \frac{\partial}{\partial_{\lambda_{1,j+1}}} \iint\limits_{\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}} e^{-i\langle \mathbf{x}_{(j)}, \mathbf{y}_{(j)}\rangle} f(\mathbf{x}_{(j)}\check{\ }, \lambda^{(j)}) e^{-i\langle r_{j+1}(\mathbf{x}_{(j)}), \tilde{\lambda}_{j+1}\rangle}\, \mathrm{d}\mathbf{x}_{(j)} \\
&\quad = \iint\limits_{\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}} e^{-i\langle \mathbf{x}_{(j)}, \mathbf{y}_{(j)}\rangle} \frac{\partial}{\partial_{\lambda_{1,j+1}}} f(\mathbf{x}_{(j)}\check{\ }, \lambda^{(j)}) e^{-i\langle r_{j+1}(\mathbf{x}_{(j)}), \tilde{\lambda}_{j+1}\rangle}\, \mathrm{d}\mathbf{x}_{(j)} \\
&\qquad - \frac{1}{2} \iint\limits_{\mathfrak{g}_{(j)}\times\mathfrak{g}_{(j)}} e^{-i\langle \mathbf{x}_{(j)}, \mathbf{y}_{(j)}\rangle} f(\mathbf{x}_{(j)}\check{\ }, \lambda^{(j)}) r^{j+1}(\mathbf{x}_{(j)}) e^{-i\langle r_{j+1}(\mathbf{x}_{(j)}), \tilde{\lambda}_{j+1}\rangle}\, \mathrm{d}\mathbf{x}_{(j)} \\
&\quad = P_{\lambda^{(j)}}(\frac{\partial}{\partial_{\lambda_{1,j+1}}} f)(\cdot, \lambda^{(j)})(\mathbf{y}_{(j)}) - \frac{1}{2} P_{\lambda^{(j)}}(r^{j+1}(-i\mathbf{D}_{(j)}) f(\cdot, \lambda^{(j)}))(\mathbf{y}_{(j)}) \\
&\quad = \varphi^f_{1,\lambda^{(j)}}(\mathbf{y}_{(j)}) + \varphi^f_{2,\lambda^{(j)}}(\mathbf{y}_{(j)}).
\end{aligned}$$

We denoted by $r^{j+1}(\mathbf{x}_{(j)})$ the sum of terms of homogeneous degree d_{j+1} from $r_{j+1}(\mathbf{x}_{(j)})$.

Now the operators

$$\frac{\partial}{\partial_{\lambda_{1,j+1}}} : S^{\mathbf{m}^{\lambda^{(j+1)}}}(\mathfrak{g}_{(j+1)} \times \mathfrak{g}_{(j+1)}, (g \oplus g)^{\lambda^{(j+1)}}) \to S^{\mathbf{m}^{\lambda^{(j+1)}} (g_{j+1}(\cdot, \lambda^{(j+1)}))^{-d_{j+1}}}(\mathfrak{g}_{(j+1)} \times \mathfrak{g}_{(j+1)}, (g \oplus g)^{\lambda^{(j+1)}})$$

are continuous and the estimates are uniform with respect to $\lambda^{(j+1)}$. Therefore, using again the induction hypothesis and Remark 6.16, we obtain that $\forall \alpha_{(j)} = (\alpha_{1,(j)}, \alpha_{2,(j)}) \in \mathbb{N}^{\sum_{k=1}^{j} 2n_k}$, there exist $N_{\alpha_{(j)}}, k'_{\alpha_{(j)}} \in \mathbb{N}$ and $C'_{\alpha_{(j)}} > 0$ so that

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial_{\lambda_{1,j+1}}} \partial^{\alpha_{(j)}} U_j \varphi^f_{1,\lambda^{(j)}} \right) (\mathbf{x}_{(j)}) \right| \leq & C'_{\alpha_{(j)}} \mathbf{m}^{\lambda^{(j)}}(\mathbf{x}_{(j)}) \prod_{k=1}^{j} \tilde{g}_k((\mathbf{x}_{(j)}, \lambda^{(j)})^{(k)})^{N_{\alpha_{(j)}}} \\
& \times \prod_{i_1=1}^{j+1} g_k(x_{1,(j)})^{-d_{i_1}|\alpha_{1,i_1}|} \prod_{i_2=1}^{j+1} g_k(x_{2,(j)})^{-d_{i_2}|\alpha_{2,i_2}|} \\
& \times (g_{j+1}(x_{1,(j)}, \lambda^{(j)}_1))^{-d_{j+1}} |f|^{\mathbf{m}}_{k'_{\alpha_{(j)}}}(\mathbf{g}),
\end{aligned}$$

and the estimates are uniform with respect to $\lambda^{(j)}$.

Since g is an admissible metric, we obtain, as in the final part of the proof of Proposition 6.7, that $\forall \alpha_{(j)} = (\alpha_{1,(j)}, \alpha_{2,(j)}) \in \mathbb{N}^{\sum_{k=1}^{j} 2n_k}$, there exist $N_{\alpha_{(j)}}, k"_{\alpha_{(j)}} \in \mathbb{N}$ and

$C"_{\alpha_{(j)}} > 0$ so that

$$\left|\left(\frac{\partial}{\partial_{\lambda_{1,j+1}}}\partial^{\alpha_{(j)}}U_j\varphi^f_{2,\lambda^{(j)}}\right)(\mathbf{x}_{(j)})\right| \leq C"_{\alpha_{(j)}}\mathbf{m}^{\lambda^{(j)}}(\mathbf{x}_{(j)})\prod_{k=1}^{j}\tilde{g}_k((\mathbf{x}_{(j)},\lambda^{(j)})^{(k)})^{N_{\alpha_{(j)}}}$$
$$\times\prod_{i_1=1}^{j+1}g_k(x_{1,(j)})^{-d_{i_1}|\alpha_{1,i_1}|}\prod_{i_2=1}^{j+1}g_k(x_{2,(j)})^{-d_{i_2}|\alpha_{2,i_2}|}$$
$$\times(g_{j+1}(x_{1,(j)},\lambda_1^{(j)}))^{-d_{j+1}}\tilde{g}_j(\lambda^{(j)})^{d_j}|f|^{\mathbf{m}}_{k"_{\alpha_{(j)}}}(\mathbf{g})$$

and the estimates are uniform with respect to $\lambda^{(j)}$.

The weak continuity follows from the induction hypothesis and from Proposition 6.15. ∎

7 Symbolic calculus and L^2- continuity

We shall prove first the continuity of the operation of composition of two symbols.

Theorem 7.1. *Let g be an admissible metric on the homogeneous Lie group $\mathfrak{g}$ and let m_1 and m_2 be two g-tempered weights. Then the product*

$$C_0^\infty(\mathfrak{g})\times C_0^\infty(\mathfrak{g})\ni(a,b)\mapsto a\#b=(\check{a}*\check{b})\hat{}\in\mathcal{S}(\mathfrak{g})$$

admits a unique double continuous extension

$$\#:S^{m_1}(\mathfrak{g},g)\times S^{m_2}(\mathfrak{g},g)\to S^{m_1m_2}(\mathfrak{g},g).$$

Proof. We have

$$(a\#b)(x)=U(\hat{\check{a}}*\hat{\check{b}})(x,x)=U(a\otimes b)(x,x),\forall x\in\mathfrak{g}.$$

For the estimation of the derivatives of $a\#b$ we shall apply Proposition 6.17. Let us remark that if $x_1=x_2\in\mathfrak{g}$, then $\tilde{g}_j(\mathbf{x}^{(j)})=1,\forall j\in\{1,\dots,R-1\}$. Therefore $\forall\alpha\in\mathbb{N}^n$ there exist $k_\alpha\in\mathbb{N}$ and $C_\alpha>0$ such that

$$|\partial^\alpha U(a\otimes b)(x,x)|\leq C_\alpha m_1(x)m_2(x)\prod_{i=1}^{R}g_i(x)^{-d_i|\alpha_i|}|a|^{m_1}_{k_\alpha}(g)|b|^{m_2}_{k_\alpha}(g),$$

for all $x\in\mathfrak{g},a\in S^{m_1}(\mathfrak{g},g)$ and $b\in S^{m_2}(\mathfrak{g},g)$. ∎

We pass now to the proof of the L^2-continuity of the pseudodifferential operators for the metric q on $\mathfrak{g}$. Remark that q is clearly an admissible metric. Let ϕ_ν be a partition of unity for q as in Proposition 4.1. We put $\Phi_{\mu\nu}(\mathbf{x})=\phi_\mu(x_1)\phi_\nu(x_2),\forall\mathbf{x}=(x_1,x_2)\in\mathfrak{g}\times\mathfrak{g}$. Since q is a self-tempered varying metric, by (2.5) we have

$$1+q_{x_\nu}(x_\mu-x_\nu)\leq C(1+q_y(x_\mu-y))^{M+1}(1+q_y(x_\nu-y)),\forall\nu,\mu\in\mathbb{N}^*,\forall y\in\mathfrak{g}. \qquad (7.1)$$

We shall use the notation $\mathbf{q}=q\oplus q$.

Lemma 7.2. *If*

$$f_{\mu\nu}(y) = U(\Phi_{\mu\nu} f)(y, y), \forall \nu, \mu \in \mathbb{N}^*, \forall y \in \mathfrak{g}, \forall f \in S^1(\mathfrak{g} \times \mathfrak{g}, \mathbf{q}).$$

then $\forall N \in \mathbb{N}, \exists k \in \mathbb{N}, \exists C > 0$ *so that*

$$\|\check{f}_{\mu\nu}\|_{L^1}(\mathfrak{g}) \leq C|f|_k^1(\mathbf{q})(1 + q_{x_\nu}(x_\mu - x_\nu))^{-N}, \forall \nu, \mu \in \mathbb{N}^*.$$

Proof. If m and n are two q-tempered weights, then $m \otimes n$ is a $\mathbf{q}$-tempered weight. Therefore, according to Example 2.9, the function

$$\mathbf{m}_{\mu\nu}(\mathbf{y}) = (1 + q_{y_1}(x_\mu - y_1))^{-N(M+1)}(1 + q_{y_2}(x_\nu - y_2))^{-N}, \forall \mathbf{y} = (y_1, y_2) \in \mathfrak{g} \times \mathfrak{g}$$

is a $\mathbf{q}$-tempered $\mathbf{q}$-weight. If $y_1 \in B_\mu$ and $y_2 \in B_\nu$, then

$$q_{x_\mu}(y_1 - x_\mu) < \gamma, q_{x_\nu}(y_2 - x_\nu) < \gamma$$

and therefore, by (2.1)

$$q_{y_1}(y_1 - x_\mu) < 1, q_{y_2}(y_2 - x_\mu) < 1.$$

Hence $\mathbf{m}_{\mu\nu}^{-1}$ is uniformly bounded on the support of $\Phi_{\mu\nu}$ and, consequently, there exists some constant $C > 0$ so that

$$|\Phi_{\mu\nu} f|_k^{\mathbf{m}_{\mu\nu}}(\mathbf{q}) \leq C|\Phi_{\mu\nu} f|_k^1(\mathbf{q}).$$

. Therefore, according to Proposition 4.1,

$$\Phi_{\mu\nu} f \in S^{\mathbf{m}_{\mu\nu}}(\mathfrak{g} \times \mathfrak{g}, \mathbf{q})$$

and the estimates are uniform with respect to μ and ν. More precisely, for all $k \in \mathbb{N}$ there exists a positive constant C_k so that

$$|\Phi_{\mu\nu} f|_k^{\mathbf{m}_{\mu\nu}}(\mathbf{q}) \leq C_k|f|_k^1(\mathbf{q}), \forall \mu, \nu \in \mathbb{N}^*, \forall f \in S^1(\mathfrak{g} \times \mathfrak{g}, \mathbf{q}).$$

Therefore, if we apply Proposition 6.17, we obtain that $\forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{2n}$, there exist $N_\alpha \in \mathbb{N}$, $k_\alpha \in \mathbb{N}$ and $C_\alpha > 0$ such that

$$\begin{aligned} |\partial^\alpha U(\Phi_{\mu\nu} f)(\mathbf{x})| \leq & C_\alpha \mathbf{m}_{\mu\nu}(\mathbf{x}) \prod_{j=1}^{R-1} \tilde{q}_j(\mathbf{x}^{(j)})^{N_\alpha} \prod_{i_1=1}^{R} q_{i_1}(x_1)^{-d_{i_1}|\alpha_{1,i_1}|} \\ & \times \prod_{i_2=1}^{R} q_{i_2}(x_2)^{-d_{i_2}|\alpha_{2,i_2}|}|f|_{k_\alpha}^1(\mathbf{q}), \end{aligned}$$

for all $\mathbf{x} \in \mathfrak{g} \times \mathfrak{g}$, for all $\mu, \nu \in \mathbb{N}^*$ and for all $f \in S^1(\mathfrak{g} \times \mathfrak{g}, q \oplus q)$.

As we already remarked, $\tilde{q}_j(y^{(j)}, y^{(j)}) = 0, \forall y \in \mathfrak{g}$. Hence for all $\alpha \in \mathbb{N}^n$ there exist $k_\alpha \in \mathbb{N}$ and $C_\alpha > 0$ so that

$$|\partial^\alpha f_{\mu\nu}(y)| \leq C_\alpha \mathbf{m}_{\mu\nu}(y, y) \prod_{i=1}^{R} q_i(y)^{-d_i|\alpha_i|}|f|_{k_\alpha}^1(\mathbf{q}) \leq C_\alpha \mathbf{m}_{\mu\nu}(y, y)|f|_{k_\alpha}^1(\mathbf{q})$$

for all $y \in \mathfrak{g}$, for all $\mu, \nu \in \mathbb{N}^*$ and for all $f \in S^1(\mathfrak{g} \times \mathfrak{g}, q \oplus q)$.

By (7.1)

$$\mathbf{m}_{\mu\nu}(y, y) \leq C(1 + q_{x_\nu}(x_\mu - x_\nu))^{-N}$$

for all $y \in \mathfrak{g}$, for all $\mu, \nu \in \mathbb{N}^*$. So, finally we obtain that for all $k \in \mathbb{N}$ there exist $k_1 \in \mathbb{N}$ and $C_k > 0$ so that

$$|\partial^\alpha f_{\mu\nu}(y)| \leq C_k |f|^1_{k_1}(\mathbf{q})(1 + q_{x_\nu}(x_\mu - x_\nu))^{-N}$$

for all $y \in \mathfrak{g}$ and $|\alpha| \leq k$. If k is large enough, the conclusion of the lemma follows from Sobolev inequality, as in the proof of Proposition 6.7. □

Theorem 7.3. *Let $a \in S^1(\mathfrak{g}, q)$. Then the linear operator $C_0^\infty(\mathfrak{g}) \ni f \mapsto Af = f * \check{a} \in L^2(\mathfrak{g})$ extends to a unique bounded mapping of $L^2(\mathfrak{g})$. More precisely, there exist $k \in \mathbb{N}$ and $C > 0$ so that*

$$\|Af\|_{L^2(\mathfrak{g})} \leq C|a|^1_k(q)\|f\|_{L^2(\mathfrak{g})}, \forall f \in C_0^\infty(\mathfrak{g}).$$

Proof. We shall apply Cotlar's lemma: if $A_1, \ldots, A_k$ are bounded operators in a Hilbert space $\mathcal{H}$ such that, for some constant M, $\sum_{\nu=1}^j \|A_\mu^* A_\nu\|^{1/2} \leq M$ and $\sum_{\nu=1}^j \|A_\mu A_\nu^*\|^{1/2} \leq M$, then $\|\sum_{\mu=1}^j A_\mu\| \leq M$.

Let

$$A_\nu f = f * (\phi_\nu a)\check{}, \forall f \in L^2(\mathfrak{g}).$$

The operators A_ν are bounded operators in $L^2(\mathfrak{g})$ since $\phi_\nu a \in C_0^\infty(\mathfrak{g})$ and $(\phi_\nu a)\check{} \in \mathcal{S}(\mathfrak{g}) \subset L^1(\mathfrak{g})$. The adjoint of A_ν is given by the formula $A_\nu^* f = f * (\phi_\nu \bar{a})\check{}$. Therefore, if we apply Lemma 6.2, we obtain that

$$A_\mu^* A_\nu f = f * ((\phi_\nu a)\check{} * (\phi_\mu \bar{a})\check{}) = f * (U(\phi_\nu a \otimes \phi_\mu \bar{a})_\Delta)\check{} = f * ((a \otimes \bar{a})_{\nu\mu})\check{}.$$

For a function h defined on $\mathfrak{g} \times \mathfrak{g}$ we put $h_\Delta(y) = h(y, y)$.

From Lemma 7.2 we obtain that $\forall N \in \mathbb{N}, \exists k \in \mathbb{N}, \exists C > 0$ so that

$$\|A_\mu^* A_\nu\| \leq C \left(|a|^1_k(\mathbf{q})\right)^2 (1 + q_{x_\nu}(x_\mu - x_\nu))^{-N}, \forall \nu, \mu \in \mathbb{N}^*.$$

Similar estimates hold for $A_\mu A_\nu^*$. So, choosing N sufficiently large, we obtain from Proposition 4.1 that

$$\sum_{\nu=1}^j \|A_\mu^* A_\nu\|^{1/2} \leq C|a|^1_k(\mathbf{q}) \text{ and } \sum_{\nu=1}^j \|A_\mu A_\nu^*\|^{1/2} \leq C|a|^1_k(\mathbf{q}), \forall j \in \mathbb{N} \quad (7.2)$$

for some $C > 0$ and some $k \in \mathbb{N}$.

On the other hand

$$a = \sum_\nu \phi_\nu a$$

in the sense of weak convergence in $S^1(\mathfrak{g}, q)$ so that, by Theorem 7.1,

$$Af = \sum_\nu A_\nu f, \forall f \in C_0^\infty(\mathfrak{g}) \quad (7.3)$$

in the sense of weak convergence in $S^1(\mathfrak{g}, q)$ of the Fourier transforms.

Cotlar's lemma, (7.2), and (7.3) conclude the proof. □

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