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On the biharmonic hypersurfaces with three distinct principal curvatures in space forms

by
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Abstract

In [19] the author proved that any hypersurface with at most three distinct principal curvatures in space forms has constant mean curvature. Recently, we found out that the proof given in [19] has a gap which we fill in the present paper. More specifically, in order to overcome this problem, we introduce a new method involving algebraic tools and Mathematica programming. We manage to find all cases that the original proof missed and show that all hypersurfaces of this type still have constant mean curvature.

Key Words: Biharmonic hypersurfaces, mean curvature, space forms.
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1 Introduction

Biharmonic maps $\varphi : M \to N$ between Riemannian manifolds are critical points of the bienergy functional and represent a natural generalization of the well-known harmonic maps. Their study was suggested in the mid-$60's$ by J. Eells and J.H. Sampson (see [16], [17]), but the first articles where biharmonic maps were systematically studied appeared in the mid-80's (see [27], [28]). In those articles, G.-Y. Jiang derived the first and the second variation formulas for the bienergy functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \ v_g,$$

where M is compact and $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$ is the tension field of φ.

The Euler-Lagrange equation for the bienergy is given by the vanishing of the bitension field, i.e.

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \operatorname{trace} R^N \left(d\varphi(\cdot), \tau(\varphi)\right) d\varphi(\cdot) = 0, \tag{1.1}$$

where Δ^φ is the rough Laplacian acting on the sections of $\varphi^{-1}(TN)$ and R^N is the curvature tensor field.

The non-linear fourth order elliptic equation $\tau_2(\varphi) = 0$ is called the biharmonic equation. Since any harmonic map is biharmonic, we are interested in the study of the biharmonic maps which are not harmonic, called proper-biharmonic.

When $\varphi : M \to N$ is an isometric immersion or, simply, when M is a submanifold of N, we say that M is biharmonic if the immersion φ is also a biharmonic map. In this case, the biharmonic equation splits into the tangent and the normal parts, the latter being elliptic.

Independently of G.-Y. Jiang, B.Y. Chen introduced in [8] the notion of biharmonic submanifolds in Euclidean spaces $\mathbb{R}^n$, and this notion can be easily recovered from (1.1) when the ambient manifold is flat and the map is an isometric immersion.

In spaces of non-positive sectional curvature, with only one exception (see [34]), we have only non-existence results, i.e. any biharmonic submanifold must be harmonic (minimal); for example, see [22], [29]. In particular, the following conjecture is still valid in its full generality (see [9]):

Chen's conjecture. *Any biharmonic submanifold in the Euclidean space is minimal.*

On the other hand, in spaces of positive sectional curvature, especially in Euclidean spheres, many examples and classification results had been obtained (see, for example, [11], [18], [21], [23], [31], [33]). Motivated by the known examples and results, the following conjecture has been proposed (see [2]):

Conjecture (C_1). *Any proper-biharmonic submanifold in the Euclidean sphere has constant mean curvature.*

The above conjecture was stated for submanifolds in the unit Euclidean sphere $\mathbb{S}^n$ because there we have examples of proper-biharmonic submanifolds, but it can be considered for submanifolds in any space form $N^n(c)$, i.e. space of constant sectional curvature c.

In the particular case of proper-biharmonic hypersurfaces in space forms, assuming some extra hypothesis, there have been obtained several results which confirm the Conjecture (C_1) (see, for example, [33]).

One way to tackle the Conjecture (C_1) for hypersurfaces in space forms is to divide the study according to the number ℓ of distinct principal curvatures. When $\ell = 1$ everywhere, we obtain in a standard way that the hypersurface has constant mean curvature, i.e. it is CMC (see, for example, [15]). When, at any point, ℓ is at most 2, (C_1) was proved in [2], [14].

When ℓ is at most 3 and $m = 3$, the result was obtained in [3], [13], [24]. Then, when $m \geq 4$, the Conjecture (C_1) was proved by Y. Fu in [19].

In our paper we show that there is a gap at the end of the proof in [19]. The author claimed that a certain polynomial, which cannot be explicitly determined, is a non-zero polynomial. Apparently surprising, computing this polynomial in some particular cases with Mathematica®, we find that there is at least one case when, actually, it becomes the zero polynomial. This occurs when the hypersurface M^m in $N^{m+1}(c)$ has dimension $m = 7$, the multiplicities of the three distinct principal curvatures are 1, 3, 3 and when $c \neq 0$. In fact, this special case has been announced by Proposition 2. In this situation, when the polynomial is the zero polynomial, we do not obtain the desired contradiction and thus the proof in [19] is not complete.

The above mentioned polynomial has high degree and we cannot determine explicit expressions for its coefficients. Even using Mathematica, it is very difficult to handle it. Because of that, we must change the strategy and consider the resultant of two new polynomials of degree 6 and 8 as a better alternative to the original polynomial. The resultant has degree 40 and it can be fully handled by Mathematica. We show that the only case when the resultant is the zero polynomial is the one mentioned above. For this special case, which is a singular case for our analysis, we manage to prove that the hypersurface M^7 is still CMC.

The Conjecture (C_1) is an important issue for biharmonic hypersurfaces in spheres because it would imply that any proper-biharmonic hypersurface in Euclidean spheres has constant mean curvature and constant scalar curvature. This fact is related to the following version of the Chern's Conjecture:

Generalized Chern's Conjecture. *Any hypersurface with constant mean and scalar curvatures in the Euclidean sphere is isoparametric.*

Further, if Conjecture (C_1) and the Generalized Chern's Conjecture are proved then, using the classification of proper-biharmonic isoparametric hyperspheres in spheres obtained in [25], [26], we can reach the full classification of proper-biharmonic hypersurfaces in Euclidean spheres, as it was conjectured in [2]:

Conjecture (C_2). *Let M^m be a proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$. Then M is either an open part of the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$ of radius $1/\sqrt{2}$ or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

We mention that, when $m \geq 4$ and M has three distinct principal curvatures, the Generalized Chern's Conjecture was proved in [12]. Therefore, as the Generalized Chern's Conjecture is of local nature, Conjecture (C_2) is proved for $m \geq 4$ and M with at most three distinct principal curvatures (see Corollary 1).

2 Conventions

In this paper, all manifolds are assumed to be connected and oriented. In general, the metrics are indicated by $\langle \cdot, \cdot \rangle$ or, simply, not explicitly mentioned. The Levi-Civita connection of the Riemannian manifold M is denoted by ∇.

The rough Laplacian defined on the set of all sections in the pull-back bundle $\varphi^{-1}(TN)$ is given by

$$\Delta^{\varphi} = -\operatorname{trace}\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

and the curvature tensor field on N is

$$R^N(U,V)W = \left[\nabla^N_U, \nabla^N_V\right]W - \nabla^N_{[U,V]}W.$$

For a hypersurface M^m in N^{m+1} we denote the mean curvature function by $f = (\operatorname{trace} A)/m$, where $A = A_\eta$ is the shape operator of M and η is a unit section in the normal bundle.

3 Preliminaries

We briefly recall that when M is a hypersurface in a space form we have the following characterization of the biharmonicity (for $c = 0$ see [8] and for any c see [4], [10], [27]).

Theorem 1. *Let M^m be a hypersurface in a space form $N^{m+1}(c)$. Then M is biharmonic if and only if*

$$\begin{cases} \text{(i)} & \Delta f + \left(|A|^2 - mc\right) f = 0, \\ \text{(ii)} & 2A(\operatorname{grad} f) + mf \operatorname{grad} f = 0. \end{cases} \tag{3.1}$$

We recall the result obtained by Y. Fu concerning the proper-biharmonic hypersurfaces with three distinct principal curvatures in space forms.

Theorem 2 ([19])**.** *Let M^m be a proper-biharmonic hypersurface in $N^{m+1}(c)$, $m \geq 4$, with at most three distinct principal curvatures. Then M^m is CMC.*

In order to give a direct application of Theorem 2, we first recall the recent result obtained in [12].

Theorem 3 ([12]). *Let M^m be a hypersurface with constant mean and scalar curvatures in $\mathbb{S}^{m+1}$, $m \geq 4$. Assume that M has three distinct principal curvatures at any point. Then M is isoparametric.*

Remark 1. *The above result is a generalization to the non-compact case of that obtained in [7].*

Now, using the classification of proper-biharmonic isoparametric hypersurfaces obtained in [25], [26] we can conclude

Corollary 1. *Let M^m be a proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$, $m \geq 4$, with at most three distinct principal curvatures. Then M is either an open part of the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$ of radius $1/\sqrt{2}$ or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Before giving a slightly different proof of Theorem 2, we will recall the fundamental equations of hypersufaces in space forms.

- The Gauss Equation:

$$R(X,Y)Z = c\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right) + \langle A(Y),Z\rangle A(X) - \langle A(X),Z\rangle A(Y) \tag{3.2}$$

  for any $X, Y, Z \in C(TM)$.

- The Codazzi Equation:

$$\left(\nabla_X A\right)(Y) = \left(\nabla_Y A\right)(X) \tag{3.3}$$

  for any $X, Y \in C(TM)$.

4 Proof of the Theorem 2

The result in [19] is a generalization to ℓ at most three of a result in [2], where ℓ is the number of distinct principal curvatures. Even if in both cases there were used the fundamental equations for hypersurfaces, the proof in [19] is much more elaborated because, when $\ell \leq 3$ at any point, it is more difficult to obtain the desired polynomials involving the mean curvature function. We will use similar notations as in [2] for an easier comparison with our proof.

For a better understanding and for making our paper self-contained, we will present here all steps of the proof of Theorem 2. The proofs for the intermediate results presented in Lemmas 1, 2, 3, 4, 5, 6 and 7 are already given in [19] and therefore, we will not include them.

For an arbitrary hypersurface $\varphi : M^m \to N^{m+1}(c)$, we denote by

$$\overline{k}_1 \geq \overline{k}_2 \geq \ldots \geq \overline{k}_m$$

its principal curvatures. The functions $\{\overline{k}_i\}_{i\in\overline{1,m}}$ are continuous on M, for any $i \in \overline{1,m}$, but not necessarily smooth everywhere. The set of all points at which the number of distinct principal curvatures is locally constant is an open and dense subset of M. We denote by M_A this set. On a non-empty connected component of M_A, which is open in M_A, thus in M, the number of distinct principal curvatures is constant. Therefore, on that connected component, the multiplicities of the distinct principal curvatures are constant and so the $\overline{k}_i$'s are smooth and A is smoothly locally diagonalizable (see [32], [35], [36]).

We will show that $\operatorname{grad} f = 0$ on every connected component of M_A and thus, from density, $\operatorname{grad} f = 0$ on M, i.e. f is constant.

We choose an arbitrary connected component of M_A. Because M has at most three distinct principal curvatures, on this component we have: either each of its points is umbilical, or each of its points has exactly two distinct principal curvatures, or each of its points has exactly three distinct principal curvatures. For simplicity, we denote by M the chosen connected component.

If M is umbilical or if M has exactly two distinct principal curvatures at any point, then the result is already proved (see [2]).

We suppose now that M has exactly three distinct principal curvatures at any point. In this case, A is (locally) diagonalizable with respect to an orthonormal frame field $\{E_1, \ldots, E_m\}$, thus $A(E_i) = \overline{k}_i E_i$, for any $i \in \overline{1, m}$.

Assume, by way of contradiction, that $\operatorname{grad} f \neq 0$ and, at the end of the proof, we will get a contradiction. If necessary, we can restrict ourselves to an open subset, also denoted (for simplicity) by M, and we can assume that $\operatorname{grad} f \neq 0$ at any point of M. Using a similar argument, we can suppose that $f = |H| > 0$ on M.

We denote by $\mathcal{D}$ the distribution orthogonal to that determined by $\operatorname{grad} f$. It is known that $\mathcal{D}$ is completely integrable (see [24], [30]).

Next, for any $p \in M$ we denote by

$$\begin{aligned}
k_1(p) &= \overline{k}_1(p) = \ldots = \overline{k}_{m_1}(p)\\
k_2(p) &= \overline{k}_{m_1+1}(p) = \ldots = \overline{k}_{m_1+m_2}(p)\\
k_3(p) &= \overline{k}_{m_1+m_2+1}(p) = \ldots = \overline{k}_{m}(p)
\end{aligned}$$

the distinct principal curvatures at the point p, $m_1 + m_2 + m_3 = m$. The multiplicities m_1, m_2, m_3 are constant functions on M.

From the tangent part of the biharmonic equation (3.1)(ii) we can suppose that

$$k_1 = -\frac{m}{2} f \quad \text{and} \quad E_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

on M.

In the first part of the proof we will try to get as much information as we can only from the tangent part of the biharmonic equation.

We note that

$$E_1(k_1) = -\frac{m}{2} E_1(f) = -\frac{m}{2} |\operatorname{grad} f| \neq 0$$

at any point of M.

We mention that all the following formulas hold on M, if it is not stated otherwise.

From the definition of E_1 we also get that for any $i \in \overline{2, m}$,

$$E_i(f) = 0.$$

We define $\omega_j^k : C(TM) \to \mathbb{R}$ such that $\nabla_X E_j = \omega_j^k(X)E_k$. It is easy to prove that ω_j^k is a one-form and that it has the property $\omega_i^j = -\omega_j^i$, for any $i, j \in \overline{1,m}$.

From Codazzi equation (3.3) we obtain for any $i, j \in \overline{1,m}$

$$E_i(\overline{k}_j)E_j + \sum_{\ell=1}^{m}(\overline{k}_j - \overline{k}_\ell)\omega_j^\ell(E_i)E_\ell = E_j(\overline{k}_i)E_i + \sum_{\ell=1}^{m}(\overline{k}_i - \overline{k}_\ell)\omega_i^\ell(E_j)E_\ell.$$

Considering the fact that $\{E_i\}_{i\in\overline{1,m}}$ is an orthonormal frame field, we get

$$E_i(\overline{k}_j) = (\overline{k}_i - \overline{k}_j)\omega_i^j(E_j), \tag{4.1}$$

$$(\overline{k}_j - \overline{k}_\ell)\omega_j^\ell(E_i) = (\overline{k}_i - \overline{k}_\ell)\omega_i^\ell(E_j) \tag{4.2}$$

for any mutually distinct $i, j, \ell \in \overline{1,m}$. In these relations we do not use the Einstein summation convention.

Next, we show that the multiplicity of k_1 is $m_1 = 1$. We suppose that $m_1 \geq 2$. Let $i_0 \neq 1$ be such that $\overline{k}_{i_0} = k_1$. In (4.1), for $i = 1$ and $j = i_0$, we have

$$E_1(\overline{k}_{i_0}) = (\overline{k}_1 - \overline{k}_{i_0})\omega_1^{i_0}(E_{i_0})$$

which is equivalent to $E_1(k_1) = 0$, contradiction. Therefore, $m_1 = 1$.

We set $r = 1 + m_2$, so we have

$$k_2(p) = \overline{k}_2(p) = \ldots = \overline{k}_r(p)$$

and

$$k_3(p) = \overline{k}_{r+1}(p) = \ldots = \overline{k}_m(p).$$

The multiplicities of k_2 and k_3 are $r-1$ and $m-r$, respectively. We can assume that $m_2 \geq m_3$ and thus $(m+1)/2 \leq r < m$.

From the definition of the mean curvature function we have

$$mf = \operatorname{trace} A = -\frac{m}{2}f + (r-1)k_2 + (m-r)k_3,$$

thus

$$k_3 = \frac{3m}{2(m-r)}f - \frac{r-1}{m-r}k_2. \tag{4.3}$$

Using the fact that $k_2 \neq k_1$, $k_3 \neq k_1$ and $k_3 \neq k_2$, we obtain that

$$k_2 \neq -\frac{m}{2}f, \quad k_2 \neq \frac{m(m-r+3)}{2(r-1)}f \quad \text{and} \quad k_2 \neq \frac{3m}{2(m-1)}f \tag{4.4}$$

at any point of M.

Our goal is to find two polynomial equations in the variables f and k_2 which have common solutions. Therefore, their resultant must vanish. It turns out that the resultant is a polynomial equation in f and, by continuity, f must be constant which contradicts $\operatorname{grad} f \neq 0$.

We use the tangent part of the biharmonic equation to obtain the properties of the connection forms.

Lemma 1. *The connection forms ω_i^j have the following properties:*

$$\omega_j^1(E_i) = \omega_i^1(E_j), \quad \forall i, j \in \overline{2, m}, \tag{4.5}$$

$$\omega_i^1(E_1) = 0, \quad \forall i \in \overline{1, m}, \tag{4.6}$$

i.e. the integral curves of E_1 are geodesic;

$$E_i(E_1(f)) = E_i(E_1(E_1(f))) = 0, \quad \forall i \in \overline{2, m}, \tag{4.7}$$

$$\omega_j^1(E_i) = 0, \quad \forall i, j \in \overline{2, m},\ i \neq j, \tag{4.8}$$

$$\omega_i^1(E_i) = -\frac{E_1(\overline{k}_i)}{k_1 - \overline{k}_i}, \quad \forall i \in \overline{2, m}, \tag{4.9}$$

$$\omega_m^2(E_2) = \frac{E_m(k_2)}{k_3 - k_2}. \tag{4.10}$$

Using both the normal and the tangent parts of the biharmonic equation and the fact that $m \geq 4$, we show that the functions k_2 and k_3 are constant along the leaves of $\mathcal{D}$.

Lemma 2. *We have $E_i(k_2) = E_i(k_3) = 0$, for any $i \in \overline{2, m}$.*

Proof. When $m - r \geq 2$, it is well known that $E_i(k_2) = 0$, $\forall i \in \overline{2, r}$ and $E_i(k_3) = 0$, $\forall i \in \overline{r + 1, m}$. (see, for example, [6]). Since $E_i(f) = 0$, $\forall i \in \overline{2, m}$ the conclusion follows without using the normal part of the biharmonic equation.

When $m - r = 1$ we still get the same conclusion, but now using both parts of the biharmonic equation (see [19]). □

Remark 2. *The distinct principal curvatures k_1, k_2 and k_3 are constant along the leaves of $\mathcal{D}$. Moreover, from Lemma 2 we obtain $E_i(E_1(k_2)) = E_i(E_1(k_3)) = 0$, $\forall i \in \overline{2, m}$.*

We recall here that a submanifold for which the tangent part of the biharmonic equation vanishes is called biconservative (see [5]). We also recall that if f is an isoparametric function, then the level hypersufaces of f, i.e. the leaves of $\mathcal{D}$, form an isoparametric family (see [1]).

From the proofs of the previous lemmas, we note that the normal part of the biharmonic equation is used in the proof of Lemma 2 only when $m-r = 1$, thus we can give the following property of f when M is biconservative.

Proposition 1. *Let M^m be a biconservative hypersurface in $N^{m+1}(c)$, $m \geq 5$, with $\operatorname{grad} f \neq 0$ at any point of M. Assume that M has three distinct principal curvatures such that $m_2, m_3 \geq 2$. Then, the mean curvature function f is isoparametric.*

Using the fact that the function k_2 is constant along the leaves of $\mathcal{D}$ we get more information about the connection forms.

Lemma 3. *The connection forms ω_i^j satisfy:*

$$\omega_j^i(E_j) = 0, \quad \forall i \in \overline{r + 1, m} \quad \textit{and} \quad \forall j \in \overline{2, r}, \tag{4.11}$$

$$\omega_j^i(E_j) = 0, \quad \forall i \in \overline{2, r} \quad \textit{and} \quad \forall j \in \overline{r + 1, m}, \tag{4.12}$$

$$\omega_1^\ell(E_j) = 0, \quad \forall j, \ell \in \overline{2, r} \quad \textit{and} \quad \ell \neq j, \tag{4.13}$$
$$\omega_1^\ell(E_j) = 0, \quad \forall j, \ell \in \overline{r+1, m} \quad \textit{and} \quad j \neq \ell, \tag{4.14}$$
$$\omega_i^\ell(E_j) = 0, \quad \forall i \in \overline{2, r}, \quad \forall j, \ell \in \overline{r+1, m} \quad \textit{and} \quad j \neq \ell, \tag{4.15}$$
$$\omega_j^\ell(E_i) = 0, \quad \forall j \in \overline{r+1, m}, \quad \forall i, \ell \in \overline{2, r} \quad \textit{and} \quad i \neq \ell, \tag{4.16}$$
$$\omega_j^1(E_i) = 0, \quad \forall i \in \overline{2, r} \quad \textit{and} \quad \forall j \in \overline{r+1, m}, \tag{4.17}$$
$$\omega_i^1(E_j) = 0, \quad \forall i \in \overline{2, r} \quad \textit{and} \quad \forall j \in \overline{r+1, m}, \tag{4.18}$$
$$\omega_j^\ell(E_1) = 0, \quad \forall j \in \overline{2, r} \quad \textit{and} \quad \forall \ell \in \overline{r+1, m}. \tag{4.19}$$

We set
$$\Omega = \frac{E_1(k_2)}{k_1 - k_2} \quad \text{and} \quad \Theta = \frac{E_1(k_3)}{k_1 - k_3}. \tag{4.20}$$
We use the Gauss equation and the tangent part of the biharmonic equation to infer some relations that Ω and Θ must satisfy.

Lemma 4. *The following relations hold:*
$$E_1(\Omega) + \Omega^2 = -c - k_1 k_2, \tag{4.21}$$
$$E_1(\Theta) + \Theta^2 = -c - k_1 k_3, \tag{4.22}$$
$$\Omega\Theta = -c - k_2 k_3. \tag{4.23}$$

Remark 3. *Relations* (4.21) *and* (4.22) *involve first and second order derivatives of* k_2 *and* k_3*, respectively, with respect to* E_1*. Our objective is to get a relation in terms of* f *and* k_2*, without any derivative.*

Next, we will use the normal part of the biharmonic equation to get another relation concerning Ω and Θ.

Lemma 5. *The following relation holds*
$$\begin{aligned} &- E_1(E_1(f)) - ((r-1)\Omega + (m-r)\Theta)\, E_1(f) \\ &+ \left(k_1^2 + (r-1)k_2^2 + (m-r)k_3^2\right) f = mcf. \end{aligned} \tag{4.24}$$

As a consequence of (4.21), (4.22) and (4.24) we have

Lemma 6. *The following formula holds*
$$\begin{aligned} &\Big((4-r)\Omega + (r-m+3)\Theta\Big) E_1(f) + \frac{3m^2(m-r+6)}{4(m-r)} f^3 \\ &- \frac{3m(m+4r-2)}{2(m-r)} f^2 k_2 + \frac{3m(r-1)}{m-r} f k_2^2 - 3(m+1)cf = 0. \end{aligned} \tag{4.25}$$

We note that (4.25) vanishes identically when $m = 7$, $r = 4$ and $\delta = 0$, where
$$\delta = 28k_2^2 - 98fk_2 + 147f^2 - 32c.$$
In this special case, (4.24) is just a consequence of (4.21) and (4.22) and therefore we do not need to use the normal part of the biharmonic equation in Lemma 5. Also, since $m - r > 1$,

we do not need the normal part in Lemma 2. Moreover, the relations derived from (4.25) cannot provide new information. This special case will appear naturally in our analysis at the end of the proof.

Since $\delta = 0$ is equivalent to $14|A|^2 - 245f^2 - 96c = 0$, we can state:

Proposition 2. *Let M^7 be a biconservative hypersurface in $N^8(c)$, with* $\operatorname{grad} f \neq 0$ *at any point of M. Assume that M has three distinct principal curvatures of multiplicities $m_1 = 1$, $m_2 = 3$, $m_3 = 3$. Then the following relation*

$$14|A|^2 - 245f^2 - 96c = 0$$

cannot hold on any open subset of M.

The proof of Proposition 2 will be given at the end of this section.

Using both the tangent and the normal parts of the biharmonic equation, we derive more properties of the functions Ω and Θ.

Lemma 7. *The functions Ω and Θ satisfy*

$$\begin{aligned}
&(r-1)(4-r)(mf+2k_2)\Omega^2 + (r-m+3)(m(m-r+3)f - 2(r-1)k_2)\Theta^2 \\
=&\frac{9m^3(m-r+6)}{4(m-r)}f^3 + \frac{3m^2(r-1)(2r-2m-15)}{2(m-r)}f^2k_2 \\
&+ \frac{m(r-1)(m+11r-12+2mr-2r^2)}{m-r}fk_2^2 + \frac{2(r-1)^2(m-2r+1)}{m-r}k_2^3 \\
&- m(2mr+4m-2r^2+5r)cf - 2(r-1)(m-2r+1)ck_2
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
&\Big(\frac{9}{4}m^3(3m-2r+17)f^3 + \frac{3}{2}m^2(6r^2-43r+37+11m-11mr)f^2k_2 \\
&+ m(r-1)(26r+4mr+1-4m)fk_2^2 + m(m-r)(8r-5mr-13m-17)cf \\
&- 2(r-1)^2(7+2m)k_2^3 + 2(m-r)(r-1)(m+17)ck_2\Big)\Omega \\
&+ \Big(\frac{9}{2}m^3(2r-2m-3)f^3 + \frac{9}{2}m^2(7r-m+3-m^2+3mr-2r^2)f^2k_2 \\
&+ 2m(r-1)(4m-13r-18-2mr+2m^2)fk_2^2 \\
&+ m(m-r)(5mr-5m^2-7m-8r+42)cf \\
&+ 2(r-1)^2(7+2m)k_2^3 - 2(r-1)(m-r)(m+17)ck_2\Big)\Theta = 0.
\end{aligned} \tag{4.27}$$

Remark 4. *Until relation* (4.27) *all our computations coincide with the computations in [19].*

Finally, from Lemma 7 we can deduce the expression of our first polynomial equation in f and k_2 mentioned at the beginning of the proof.

First, we will denote by P and Q the coefficients of Ω and Θ, respectively, in (4.27) and by R the right-hand side of (4.26). We see that P, Q and R do not contain any derivative. Thus, we get

$$(r-1)(4-r)(mf+2k_2)\Omega^2 + (3+r-m)(m(m-r+3)f - 2(r-1)k_2)\Theta^2 = R, \tag{4.28}$$

$$P\Omega + Q\Theta = 0. \tag{4.29}$$

Multiplying (4.28) by PQ, using (4.23) and (4.29), we finally get a relation without any derivative

$$\begin{aligned}(3+r-m)\left(m(m-r+3)f-2(r-1)k_2\right)P^2\left(c+\frac{3m}{2(m-r)}fk_2-\frac{r-1}{m-r}k_2^2\right)\\ +(r-1)(4-r)(mf+2k_2)Q^2\left(c+\frac{3m}{2(m-r)}fk_2-\frac{r-1}{m-r}k_2^2\right)=PQR.\end{aligned} \tag{4.30}$$

Equation (4.30) can be written as

$$\sum_{i=0}^{9}a_{i,9-i}k_2^if^{9-i}+c\left(\sum_{i=0}^{7}a_{i,7-i}k_2^if^{7-i}+\sum_{i=0}^{5}a_{i,5-i}k_2^if^{5-i}+\sum_{i=0}^{3}a_{i,3-i}k_2^if^{3-i}\right)=0, \tag{4.31}$$

where the coefficients a_{ij} depend on m, r and c, thus they are constants.

We can easily show that for any m, r and c

$$\begin{aligned}a_{9,0}&=\frac{729m^9(2m-2r+3)(3m-2r+17)(m-r+6)}{32(m-r)}>0,\\ a_{0,9}&=0.\end{aligned}$$

Therefore, the left hand-side of (4.31) is a non-zero polynomial in f and k_2 obtained using both the normal and tangent parts of the biharmonic equation. Equation (4.31) represents the key relation of our proof.

If k_2 is constant on M then, from (4.31), we obtain a 9^{th}-degree polynomial in the variable f with constant coefficients, thus f is constant on M, contradiction.

We will assume that k_2 is not constant on M. Restricting M, if necessary, we can suppose that $\operatorname{grad} k_2 \neq 0$ and $k_2 \neq 0$ at any point of M. The fact that

$$\operatorname{grad} k_2 = E_1(k_2)E_1 \neq 0$$

implies that $E_1(k_2) \neq 0$ at any point of M. Therefore $\Omega \neq 0$ at any point of M.

Let $\gamma : I \to M$ be an integral curve of E_1, $\gamma = \gamma(t)$.

Lemma 8. *Along γ, the ratio k_2/f cannot be constant.*

Proof. We work along γ. Assume, by way of contradiction, that k_2/f can be a constant. Let $\alpha \in \mathbb{R}$ be a non-zero constant such that $k_2 = \alpha f$. We have

$$k_3 = \frac{3m-2\alpha(r-1)}{2(m-r)}f.$$

Thus,

$$k_1 = -\frac{m}{2}f, \quad k_2 = \alpha f, \quad k_3 = \beta f,$$

where

$$\beta = \frac{3m-2\alpha(r-1)}{2(m-r)}.$$

Using (4.4) we get that $\alpha \neq -m/2$, $\beta \neq -m/2$ and $\alpha \neq \beta$.

Relation (4.21), which was obtained from the tangent part of the biharmonic equation, can be rewritten as

$$fE_1(E_1(f)) = \frac{m+4\alpha}{m+2\alpha}\left(E_1(f)\right)^2 + \frac{m+2\alpha}{2\alpha}cf^2 - \frac{m(m+2\alpha)}{4}f^4. \tag{4.32}$$

We show that $\beta \neq 0$. We suppose, by way of contradiction, that $\beta = 0$. Then, $k_3 = 0$ and relation (4.22) is equivalent to $c = 0$. Further, replacing k_1, k_2 and k_3 in (4.24), which was obtained from the normal part of the biharmonic equation, we get

$$fE_1\left(E_1(f)\right) = -\alpha(r-1)f\left(E_1(f)\right)^2 + \frac{m^2+4(r-1)\alpha^2}{4}f^4. \tag{4.33}$$

When $c = 0$, relation (4.32) becomes

$$fE_1\left(E_1(f)\right) = \frac{m+4\alpha}{m+2\alpha}\left(E_1(f)\right)^2 - \frac{m(m+2\alpha)}{4}f^4. \tag{4.34}$$

From (4.33) and (4.34) we will obtain a contradiction. For that, we consider

$$w(t) = \left(E_1(f)\right)^2(\gamma(t)) = \left(\gamma'(t)(f)\right)^2 = \left((f\circ\gamma)'(t)\right)^2 = \left(f'(t)\right)^2.$$

Since $t \longmapsto f(t)$ is a diffeomorphism and $t = t(f)$, we have $w = w(t) = w\left(t(f)\right)$, $f \in f(I)$. We denote by $\overline{w}(f) = w(t(f))$ and obtain

$$\frac{d\overline{w}}{df}(f) = \frac{dw}{dt}\left(t(f)\right)\frac{dt}{df}(f) = 2f''(t) = 2\left(E_1\left(E_1(f)\right)\right)(\gamma(t)).$$

Relations (4.33) and (4.34) become

$$\begin{cases} \dfrac{1}{2}f\dfrac{d\overline{w}}{df} &= \dfrac{m^2+4(r-1)\alpha^2}{4}f^4 - \alpha(r-1)f\overline{w}, \\ \dfrac{1}{2}f\dfrac{d\overline{w}}{df} &= -\dfrac{m(m+2\alpha)}{4}f^4 + \dfrac{m+4\alpha}{m+2\alpha}\overline{w}. \end{cases}$$

Using these two equations we obtain that

$$\overline{w} = \frac{\left(m^2+4(r-1)\alpha^2+m(m+2\alpha)\right)(m+2\alpha)}{4} \times \frac{f^4}{(m+2\alpha)\alpha(r-1)f+m+4\alpha}.$$

Differentiating $\overline{w}$ with respect to f we get

$$\begin{aligned}\frac{d\overline{w}}{df} =& \frac{\left(m^2+4(r-1)\alpha^2+m(m+2\alpha)\right)(m+2\alpha)}{4} \\ &\times \frac{4\left((m+2\alpha)\alpha(r-1)f+m+4\alpha\right)f^3-(m+2\alpha)\alpha(r-1)f^4}{\left((m+2\alpha)\alpha(r-1)f+m+4\alpha\right)^2}.\end{aligned}$$

Substituting $d\overline{w}/df$ in the second equation of the system we get

$$\frac{m(m+2\alpha)}{4}f^4\left((m+2\alpha)\alpha(r-1)f+m+4\alpha\right)^2$$

$$\begin{aligned}
&+\frac{1}{8}f\left(m^2+r(r-1)\alpha^2+m(m+2\alpha)\right)(m+2\alpha)\\
&\times\left(3(m+2\alpha)\alpha(r-1)f^4+4(m+4\alpha)f^3\right)\\
&-\frac{(m+4\alpha)\left(m^2+4(r-1)\alpha^2+m(m+2\alpha)\right)}{4}f^4\\
&\times((m+2\alpha)\alpha(r-1)f+m+4\alpha)=0.
\end{aligned}$$

We obtain a 6^{th}-degree polynomial relation in f with the dominant term

$$\frac{1}{4}m(m+2\alpha)^3\alpha^2(r-1)^2.$$

Since $\alpha\neq -m/2$, we obtain a polynomial equation in the variable f, with constant coefficients, thus f is constant along γ, contradiction. Therefore, $\beta\neq 0$.

From (4.22) and (4.23), which were obtained from the tangent part of the biharmonic equation, and from the expressions of k_1, k_2 and k_3, we get

$$fE_1\left(E_1(f)\right)=\frac{m+4\beta}{m+2\beta}\left(E_1(f)\right)^2+\frac{m+2\beta}{2\beta}cf-\frac{m(m+2\beta)}{4}f^4 \tag{4.35}$$

and

$$\left(E_1(f)\right)^2=-\frac{(m+2\alpha)(m+2\beta)}{4\alpha\beta}cf^2-\frac{(m+2\alpha)(m+2\beta)}{4}f^4. \tag{4.36}$$

Combining (4.32), (4.35) and (4.36) we obtain

$$\frac{m(\beta-\alpha)}{\alpha\beta}cf^2+m(\beta-\alpha)f^4=0.$$

Since $\beta\neq\alpha$, the last relation is a polynomial equation with constant coefficients in the variable f, contradiction. □

As we mentioned before, the polynomial equation (4.31) plays an important role in our proof and we note that, when $c=0$, it becomes a 9^{th}-degree homogeneous polynomial equation. Therefore, we will split our analysis into two cases.

Case 1: $\mathbf{c=0}$.

This case was already proved in [20] in a similar way, but we keep it here for the sake of completeness.

Relation (4.31) becomes

$$\sum_{i=0}^{9}a_{i,9-i}k_2^if^{9-i}=0.$$

Using the fact that $f>0$, we can divide with f^9 and obtain

$$\sum_{i=0}^{9}a_{i,9-i}\left(\frac{k_2}{f}\right)^i=0.$$

Therefore, we get a polynomial equation in the variable $z=k_2/f$ with constant coefficients. We have seen that $a_{9,0}$ is not zero and does not depend on c, thus this polynomial is non-zero and this implies that z is constant, which contradicts Lemma 8.

Case 2: $\mathbf{c \neq 0}$.

Along γ we have

$$k_2'(t) = (E_1(k_2))\,(\gamma(t)) \neq 0, \forall t \in I,$$

where $k_2 := k_2 \circ \gamma$. In this case $t \longmapsto k_2(t)$ is a diffeomorphism and $t = t(k_2)$.

Next, we denote

$$f(t) = (f \circ \gamma)(t), \quad \tilde{f}(k_2) = f(t(k_2)) \quad \text{and} \quad \tilde{\gamma}(k_2) = \gamma(t(k_2)).$$

A direct consequence of Lemma 8 is

Remark 5. *Along $\tilde{\gamma}$, the ratio $k_2/\tilde{f}$ cannot be a constant.*

Lemma 9. *Along $\tilde{\gamma}$, $\tilde{Q} = Q \circ \tilde{\gamma}$ does not vanish.*

Proof. If $\widetilde{Q}(k_2) = 0$, for any k_2, then, using the fact that $\widetilde{\Omega} \neq 0$, from (4.29) we obtain $\widetilde{P} = 0$.

The relations $\widetilde{P}(k_2) = 0$ and $\widetilde{Q}(k_2) = 0$, for any k_2, can be thought of as two polynomial equations in k_2 with coefficients depending on the function $\tilde{f} = \tilde{f}(k_2)$. We arbitrarily set $k_2 = k_2^0$ and thus the coefficients of the above two equations become constants. Now, we consider two polynomial equations in the variable $z = k_2$ with the corresponding above constant coefficients. Clearly, $z = k_2^0$ is a common solution of the last two equations. Therefore, the resultant of these polynomials with constant coefficients has to be 0. The resultant, which is a real number, can be written as a polynomial relation in $\tilde{f} = \tilde{f}(k_2^0)$. Letting k_2^0 free, we get that $\tilde{f} = \tilde{f}(k_2)$ is a solution of a polynomial equation with constant coefficients. Using Mathematica (see Appendix A), it can be shown that, since $c \neq 0$, this polynomial is non-zero. The fact that $\tilde{f}$ is continuous implies that $\tilde{f}$ is a constant function, thus f is constant along $\tilde{\gamma}$, contradiction. ∎

We can express the derivative of $\tilde{f}$ with respect to k_2 as a rational relation in $\tilde{f}$ and k_2.

Lemma 10. *Along $\tilde{\gamma}$ the derivative of $\tilde{f}$ with respect to k_2 is*

$$\frac{d\tilde{f}}{dk_2} = \frac{2(r-1)}{3m} - \frac{2(m(m-r+3)\tilde{f} - 2(r-1)k_2)\widetilde{P}}{3m(m\tilde{f} + 2k_2)\widetilde{Q}}. \tag{4.37}$$

In light of Lemma 9, we can restrict $\tilde{\gamma}$, if necessary, and assume that $\widetilde{Q}(k_2) \neq 0$ for any k_2.

To get another key relation for our proof, we differentiate the first polynomial given by (4.31) with respect to k_2 along $\tilde{\gamma}$ and substituting the derivative of $\tilde{f}$ with respect to k_2 from Lemma 10, we obtain

$$\begin{aligned}\sum_{i=0}^{12} b_{i,12-i} k_2^i \tilde{f}^{12-i} + c \Bigg(&\sum_{i=0}^{10} b_{i,10-i} k_2^i \tilde{f}^{10-i} + \sum_{i=0}^{8} b_{i,8-i} k_2^i \tilde{f}^{8-i} \\ &+ \sum_{i=0}^{6} b_{i,6-i} k_2^i \tilde{f}^{6-i} + \sum_{i=0}^{4} b_{i,4-i} k_2^i \tilde{f}^{4-i} \Bigg) = 0, \end{aligned} \tag{4.38}$$

where the coefficients b_{ij} depend on m, r and c and are constants.

Relations (4.31) and (4.38) can be seen as two polynomial equations in k_2 with coefficients depending on the function $\tilde{f} = \tilde{f}(k_2)$. As in the proof of Lemma 9, we can compute the resultant of the two polynomials and, finally, we can obtain a polynomial in $\tilde{f}$ with constant coefficients. If this polynomial is non-zero, then we get a contradiction and we end the proof.

Since the derivative of $\tilde{f}$ with respect to k_2 is different from zero, for any k_2, we can change the point of view and (4.31) together with (4.38) can be thought of as two polynomial equations in $\tilde{f}$ with coefficients depending on the function $\tilde{k}_2 = \tilde{k}_2(\tilde{f})$. As we described above, we can compute the resultant for these new polynomials obtaining a polynomial in $\tilde{k}_2$ with constant coefficients. Again, if this polynomial is non-zero, then we get a contradiction.

Since the volume of computations is very big, we could not compute the resultant for generic c, m and r. Because of that, we made a programme in Mathematica which computes the resultant for any particular choice of c, m and r.

In the first situation, when the resultant is a polynomial in $\tilde{f}$, we obtain that, for any $c \neq 0$, $m \in \overline{4,30}$ and all possible values of r, the only case when the resultant is the zero polynomial is given by

$$m = 7 \quad \text{and} \quad r = 4$$

(see Appendix B).

In the second situation, when the resultant is a polynomial in k_2, the resultant (which should be a polynomial in k_2) is the zero polynomial for any $c \neq 0$, $m \in \overline{4,30}$ and for all possible values of r (see Appendix B).

In order to reduce the volume of computations and to compute the resultant for the generic case, we will reduce the degree of polynomials in (4.31) and (4.38). Thus, we divide (4.31) by $\tilde{f}^3$ and (4.38) by $\tilde{f}^4$, and denoting $\tilde{z} = k_2/\tilde{f}$, we obtain

$$\sum_{i=0}^{9} a_{i,9-i}\tilde{z}^i\tilde{f}^6 + c\left(\sum_{i=0}^{7} a_{i,7-i}\tilde{z}^i\tilde{f}^4 + \sum_{i=0}^{5} a_{i,5-i}\tilde{z}^i\tilde{f}^2 + \sum_{i=0}^{3} a_{i,3-i}\tilde{z}^i\right) = 0 \tag{4.39}$$

and

$$\begin{aligned}\sum_{i=0}^{12} b_{i,12-i}\tilde{z}^i\tilde{f}^8 + c\Bigg(&\sum_{i=0}^{10} b_{i,10-i}\tilde{z}^i\tilde{f}^6 + \sum_{i=0}^{8} b_{i,8-i}\tilde{z}^i\tilde{f}^4 \\ &+ \sum_{i=0}^{6} b_{i,6-i}\tilde{z}^i\tilde{f}^2 + \sum_{i=0}^{4} b_{i,4-i}\tilde{z}^i\Bigg) = 0.\end{aligned} \tag{4.40}$$

Using Mathematica we can compute the resultant of these polynomials in the general case and prove that it vanishes only when $m = 7$ and $r = 4$ (see Appendix C).

Subcase 2.1: $\mathbf{c \neq 0}$, $\mathbf{m = 7}$ and $\mathbf{r = 4}$.

In this case equations (4.31) and (4.38) become

$$\begin{aligned}&\frac{45927}{16}\left(32c - 147\tilde{f}^2 + 98\tilde{f}k_2 - 28k_2^2\right)\\ &\times\left(336c\tilde{f} - 1715\tilde{f}^3 - 32ck_2 + 1470\tilde{f}^2k_2 - 294\tilde{f}k_2^2 + 28k_2^3\right)\end{aligned}$$

$$\times\left(32c(7\tilde{f}+k_2)+7\left(147\tilde{f}^3-63\tilde{f}^2k_2-4k_2^3\right)\right)=0 \tag{4.41}$$

and

$$\begin{aligned}&\frac{1240029}{16}\left(7\tilde{f}-4k_2\right)\left(32c-147\tilde{f}^2+98\tilde{f}k_2-28k_2^2\right)\\&\times\Big[81920c^3\left(343\tilde{f}^2+7\tilde{f}k_2-2k_2^2\right)-7168c^2\Big(66542\tilde{f}^4\\&-12201\tilde{f}^3k_2+2653\tilde{f}^2k_2^2+476\tilde{f}k_2^3-68k_2^4\Big)\\&-784c\Big(2384193\tilde{f}^6-1172717\tilde{f}^5k_2+559384\tilde{f}^4k_2^2\\&-154252\tilde{f}^3k_2^3+40656\tilde{f}^2k_2^4-6384\tilde{f}k_2^5+608k_2^6\Big)\\&+2401\Big(6950895\tilde{f}^8-10169607\tilde{f}^7k_2+5436942\tilde{f}^6k_2^2\\&-1685894\tilde{f}^5k_2^3+421456\tilde{f}^4k_2^4-69608\tilde{f}^3k_2^5\\&+\ 10288\tilde{f}^2k_2^6-896\tilde{f}k_2^7+64k_2^8\Big)\Big]=0,\end{aligned} \tag{4.42}$$

respectively.

We see from (4.41) and (4.42) that the polynomial $\delta=28k_2^2-98\tilde{f}k_2+147\tilde{f}^2-32c$ is the only common factor. The conic $\delta=0$ is an ellipse when $c>0$ and an imaginary ellipse when $c<0$.

On the other hand, from (4.25) we obtain that $\delta=0$. Thus, as we already mentioned, this case is nothing but Proposition 2.

We have $\delta(k_2)=0$, for any k_2. In the following, we do not need to work with the variable k_2, so we come back to the first variable t. Differentiating the relation $\delta(t)=0$, we obtain

$$7(3f(t)-k_2(t))f'(t)+(4k_2(t)-7f(t))k_2'(t)=0. \tag{4.43}$$

We recall that

$$k_1=-\frac{7}{2}f \quad\text{and}\quad k_3=\frac{7}{2}f-k_2.$$

Using Lemma 8 and (4.43), we get that

$$k_2'(t)=\frac{7(k_2(t)-3f(t))}{4k_2(t)-7f(t)}f'(t)$$

and thus, we have

$$k_3'(t)=\frac{7(2k_2(t)-f(t))}{2(4k_2(t)-7f(t))}f'(t).$$

We can write $\delta=0$ as follows

$$7(4k_2-7f)^2=128c-245f^2. \tag{4.44}$$

Using (4.20), we obtain

$$\Omega=-\frac{14(k_2-3f)}{(7f+2k_2)(4k_2-7f)}f',$$

$$\Theta = \frac{7(2k-f)}{2(k_2-7f)(4k_2-7f)} f'.$$

Thus, (4.23) can be written as

$$98(k_2-3f)(2k_2-f)\left(f'\right)^2 = \left(7fk_2-2k_2^2+2c\right)\left(4k_2-7f\right)^2(7f+2k_2)(k_2-7f). \tag{4.45}$$

Using the fact that $\delta = 0$ and its equivalent form (4.44), we obtain

$$98^2\left(32c-105f^2\right)\left(f'\right)^2 = \left(147f^2-4c\right)\left(128c-245f^2\right)\left(-833f^2+32c\right). \tag{4.46}$$

Differentiating (4.46), we get

$$\begin{aligned} &-20580f\left(f'\right)^2+196\left(32c-105f^2\right)f'' \\ =&3f(128c-245f^2)\left(-833f^2+32c\right) \\ &-5f\left(147f^2-4c\right)\left(-833f^2+32c\right) \\ &-17f\left(147f^2-4c\right)\left(128c-245f^2\right). \end{aligned} \tag{4.47}$$

From (4.24), we obtain

$$f'' = \frac{1911f}{833f^2-32c}\left(f'\right)^2+\frac{1}{14}\left(245f^2-2c\right)f. \tag{4.48}$$

Substituting (4.48) in (4.47) and using (4.46) we obtain

$$\begin{aligned} &14386462720\times c^4f-356598824960\times c^3f^3-2331746708480\times c^2f^5 \\ &+42758681977200\times cf^7+151265495839500f^9=0. \end{aligned}$$

We get a 9^{th}-degree polynomial in the variable f with constant coefficients, which is a contradiction.

A

Here we present the Mathematica code for computing the resultant of the two polynomials that appear in the proof of Lemma 9.

First, we have to declare $\widetilde{P}$ and $\widetilde{Q}$. For simplicity, we denote them by P and Q, respectively. Also, $\tilde{f}$ is denoted by f and k_2 by k.

```
P = 9/4 m^3*(3 m - 2 r + 17) f^3 +
3/2 m^2*(6 r^2 - 43 r + 37 + 11 m - 11 m*r)*f^2*k +
m*(r - 1)*(26 r + 4 m*r + 1 - 4 m)* f*k^2 +
m*(m - r) (8 r - 5 m*r - 13 m - 17) c*f -
2 (r - 1)^2 (7 + 2 m)*k^3 + 2 (m - r)*(r - 1)*(m + 17)*c*k


Q = 9/2 m^3*(2 r - 2 m - 3)*f^3 + 9/2 m^2*(7 r - m + 3 - m^2 +
3 m*r - 2 r^2)*f^2*k + 2 m*(r - 1)*(4 m - 13 r - 18 - 2 m*r +
2 m^2)* f*k^2 + m*(m - r)*(5 m*r - 5 m^2 - 7 m - 8 r + 42)*c*f +
2 (r - 1)^2*(7 + 2 m)*k^3 - 2 (r - 1)*(m - r)*(m + 17)*c*k
```

Now, we compute the resultant of P and Q with respect to k and simplify it.

```
ResPQ[m_][r_][c_][f_] = Collect[Resultant[P, Q, k], f, FullSimplify]
```

We want to prove that this resultant is not the zero polynomial. We obtain a 9^{th}-degree polynomial in f, free of the constant term and with vanishing coefficients of f and f^2. We will look to the coefficient of f^3 since it is the first non-zero monomial.

```
CoefF3 = Coefficient[ResPQ[m][r][c][f], f, 3]
```

which yields

$$1474560 \times c^3(-1+m)m^3(5+m)(7+2m)^3\left(20-3m+m^2\right)^2(m-r)^3(-1+r)^6.$$

Since the integers r and m satisfy $1 < r < m$, none of the factors of this coefficient can be zero, except c^3.

If $c \neq 0$, clearly the resultant is a non-zero polynomial.

B

In addition of $\tilde{P}$ and $\tilde{Q}$ from the Appendix A, we have to declare $\tilde{R}$. Again, for simplicity, we will denote $R = \tilde{R}$.

```
R = (9 m^3*(m - r + 6))/(4 (m - r))*
f^3 + (3 m^2*(r - 1)*(2 r - 2 m - 15))/(2 (m - r))*f^2*
k + (m*(r - 1)*(m + 11 r - 12 + 2 m*r - 2 r^2))/(m - r)*f*
k^2 + (2 (r - 1)^2*(m - 2 r + 1))/(m - r)*k^3 -
m*(2 m*r + 4 m - 2 r^2 + 5 r)*c*f - 2 (r - 1)*(m - 2 r + 1)*c*k
```

We use relation (4.30) to obtain the polynomial given by (4.31).

```
Rel430 = (3 + r - m)*(m (m - r + 3)*f - 2 (r - 1) k)*
P^2*(c + (3 m)/(2 (m - r))*f*k - (r - 1)/(m - r)*k^2) + (r -
1) (4 - r) (m*f + 2 k)*
Q^2*(c + (3 m)/(2 (m - r))*f*k - (r - 1)/(m - r)*k^2) - P*Q*R
```

Next, we define a function H which is just (4.30) simplified.

```
H[m_][r_][c_][f_][k_] = Total[FullSimplify[MonomialList[Rel430,
{f, k}]]]
```

We input the derivative of $\tilde{f}$, now f, with respect to k_2 which is denoted by k.

```
DerF = (2 (r - 1))/(3 m) - (2 (m (m - r + 3)*f - 2 (r - 1)*k)*
P)/(3 m (m*f + 2 k)*Q)
```

Now, we declare the numerator and the denominator of this relation, respectively.

```
NumDerF = Numerator[Together[DerF]]
```

```
DenDerF = Denominator[Together[DerF]]
```

If we denote by $\overline{H}\left(k_2, \tilde{f}\right)$ the polynomial in relation (4.31), we have

$$\frac{d\overline{H}}{dk_2}\left(k_2, \tilde{f}(k_2)\right) = \frac{\partial\overline{H}}{\partial k_2}\left(k_2, \tilde{f}(k_2)\right) + \frac{\partial\overline{H}}{\partial\tilde{f}}\left(k_2, \tilde{f}(k_2)\right)\frac{d\tilde{f}}{dk_2}(k_2).$$

Actually, $\overline{H}$ is denoted by H in the code.

We multiply this relation with the denominator of the derivative of $\tilde{f}$ with respect to k_2 in order to obtain the polynomial in (4.38).

```
Rel438 = D[H[m][r][c][f][k], f]*NumDerF +
D[H[m][r][c][f][k], k]*DenDerF
```

We define the function K to be the polynomial from (4.38) after simplifications.

```
K[m_][r_][c_][f_][k_]=Total[FullSimplify[MonomialList[Rel438, {f, k}]]]
```

Now, we compute the resultant for H and K with respect to k, for all $c \in \{-1, 0, 1\}$, $m \in \overline{4, 30}$ and $r \in \overline{2, m-1}$. We know that, since the biharmonicity and minimality are invariant under homothetic transformations, we can assume that $c \in \{-1, 0, 1\}$.

```
For[cc = -1, cc < 2, cc++,
 For[mm = 4, mm < 31, mm++,
  For[rr = 2, rr < mm, rr++,
   res = Resultant[H[mm][rr][cc][f][k], K[mm][rr][cc][f][k], k];
   Print["The resultant with respect to k for m = ", mm, ", r = ",
   rr, ", c = ", cc, " is \n", res];
   If[res === 0, Print["Exception"], ];
  ]
 ]
]
```

From these computations we find out that the resultant is the zero polynomial only in the case $m = 7$ and $r = 4$.

Further, we compute the resultant of H and K with respect to f, for any $c \in \{-1, 0, 1\}$, $m \in \overline{4, 30}$ and $r \in \overline{2, m-1}$.

```
For[cc = -1, cc < 2, cc++,
 For[mm = 4, mm < 31, mm++,
  For[rr = 2, rr < mm, rr++,
   res = Resultant[H[mm][rr][cc][f][k], K[mm][rr][cc][f][k], f];
   Print["The resultant with respect to f for m = ", mm, ", r = ",
   rr, ", c = ", cc, " is \n", res];
   If[res === 0, Print["Exception"], ];
  ]
 ]
]
```

In this situation, the resultant is the zero polynomial for any c, m or r.

C

We need to find the coefficients of lower degree polynomials (4.39) and (4.40). To do this, we will create two matrices with the entries being the coefficients of the polynomials from (4.31) and (4.38), respectively.

```
CoefH = CoefficientList[H[m][r][c][f][k], {k, f}]
```

```
CoefK = CoefficientList[K[m][r][c][f][k], {k, f}]
```

These matrices are made such that the element from the position (i, j) is the coefficient of $k_2^{i-1}\tilde{f}^{j-1}$ from H and K, respectively (see Appendix B). We look for a formula that links the elements of these matrices with $a_{i,s-i}$ and $b_{i,\overline{s}-i}$ from (4.31) and (4.38), respectively, where $s \in \{3, 5, 7, 9\}$ and $\overline{s} \in \{4, 6, 8, 10, 12\}$.

We study the case of $\overline{H}$ form Appendix B, the other one being similar. For simplicity, let $(A_{ij})_{i,j\in\overline{1,10}}$ be a matrix given by `CoefH`, thus

$$\overline{H} = \sum_{i=1}^{10}\sum_{j=1}^{10} A_{ij} k_2^{i-1}\tilde{f}^{j-1}$$

and $s \in \{3, 5, 7, 9\}$.

If $(i-1) + (j-1) = s$, then $j = s - i + 2$ and $A_{ij} = A_{i,s-i+2} = a_{i-1,s-(i-1)}$.

If $(i-1) + (j-1) \neq s$, for any $s \in \{3, 5, 7, 9\}$, then $A_{ij} = 0$.

Therefore,

$$\begin{aligned}\overline{H} &= \sum_{i=1}^{10}\sum_{j=1}^{10} A_{ij} k_2^{i-1}\tilde{f}^{j-1}\\ &= \sum_{s\in\{3,5,7,9\}}\sum_{i=1}^{s+1} A_{i,s-i+2} k_2^{i-1}\tilde{f}^{s-i+1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s\in\{3,5,7,9\}} \sum_{i=1}^{s+1} a_{i-1,s-(i-1)} k_2^{i-1} \tilde{f}^{s-(i-1)} \\
&= \sum_{s\in\{3,5,7,9\}} \sum_{i=0}^{s} a_{i,s-i} k_2^{i} \tilde{f}^{s-i}.
\end{aligned}
$$

Thus,

$$\overline{H} = \sum_{i=1}^{10}\sum_{j=1}^{10} A_{ij} \left(\frac{k_2}{\tilde{f}}\right)^{i-1} \tilde{f}^{i+j-2}$$

and

$$\frac{1}{\tilde{f}^3}\overline{H} = \sum_{i=1}^{10}\sum_{j=1}^{10} A_{ij} \left(\frac{k_2}{\tilde{f}}\right)^{i-1} \tilde{f}^{i+j-5}.$$

Now, we write down a little code to obtain the polynomials in (4.39) and (4.40), respectively.

```
AuxH = 0; For[i = 1, i <= 9, i++,
 For[j = 1, j <= 10, j++,
 AuxH = AuxH + CoefH[[i]][[j]]*f^(i + j - 5)*z^(i - 1);
 ]
]
newH [m_][r_][c_][f_][z_] = AuxH


AuxK = 0; For[i = 1, i <= 12, i++,
 For[j = 1, j <= 13, j++,
 AuxK = AuxK + CoefK[[i]][[j]]*f^(i + j - 6)*z^(i - 1);
 ]
]
newK[m_][r_][c_][f_][z_] = AuxK
```

Using the same approach as in Appendix B, we can compute the resultants for $c \in \{-1,1\}$, $m \in \overline{4,30}$ and $r \in \overline{2,m-1}$ of `newH` and `newK` with respect to f and z. We get that both resultants vanish only when $m = 7$ and $r = 4$. Recall that, since $c \neq 0$ and the biharmonicity and harmonicity are invariant under homothetic transformations, we can assume $c \in \{-1,1\}$.

First, we will compute the resultant of these new polynomials with respect to f in the generic case.

```
res = Resultant[newH[m][r][c][f][z], newK[m][r][c][f][z], f]
```

The resultant consists of a constant multiplied by a squared polynomial, denoted by `resPoly`. Since we study in which case this resultant vanishes, we will consider only the polynomial `resPoly`.

```
resFinal[m_][r_][c_][z_] = Total[ParallelMap[FullSimplify,
 MonomialList[resPoly, z]]]
```

We obtain a 40^{th}-degree polynomial in the variable z with constant coefficients depending on c, m and r. The dominant coefficient of `resPoly` is

```
dominantCoef = Coefficient[resFinal[m][r][c][z], z, 40]
```

which yields

$$\begin{aligned}&-6917529027641081856\times c^{12}(-10+m)^3(-7+m)^3(-3+m)(-1+m)^2m^8\\&\times(5+m)(7+2m)^5(7-5m+m^2)(20-3m+m^2)^4(-196+23m+11m^2)\\&\times(-497+16m+49m^2)(1+m-2r)(m-r)^{12}(-1+r)^{28}.\end{aligned}$$

Since $m\geq 4$ and $r\in\overline{2,m-1}$ are integers, also using the command `IntegerQ`, it is easy to see that this coefficient vanishes if and only if

$$m=7\quad\text{or}\quad m=10\quad\text{or}\quad m=2r-1.$$

We have seen that this resultant does not vanish if $m=7$ and $r\neq 4$ or if $m=10$.

If $m=2r-1$, we substitute m in the resultant above

```
resSpecial = resFinal[2r-1][r][c][z]
```

We obtain a 39^{th}-degree polynomial and its dominant coefficient is

```
FullSimplify[Coefficient[resSpecial, z, 39]]
```

which yields

$$\begin{aligned}&-56668397794435742564352\times c^{12}(11-2r)^2(-4+r)^4(-2+r)(-1+r)^{42}\\&\times(2+r)(-1+2r)^9(5+4r)^5(13-14r+4r^2)(12-5r+2r^2)^4\\&\times(-116-41r+49r^2)(-2356-1035r+120r^2+112r^3).\end{aligned}$$

Since r is an integer and using the command `IntegerQ` in Mathematica, the only possibilities when this coefficient is zero are

$$r=2\quad\text{or}\quad r=4.$$

If $r=2$, then $m=3<4$.

If $r=4$, then $m=7$.

Therefore, the only case in which the resultant vanishes is $m=7$ and $r=4$.

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