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On the equivalent properties of certain general local cohomology modules

by
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Abstract

Let R be a commutative Noetherian ring and M be a ZD-module. In this paper, we investigate the Artinianness of general local cohomology modulus with respect to a system of ideals Φ of R. For this aim, we introduce the concept of Φ-Laskerian R-modules and we show that if M is a Φ-Laskerian module of finite dimension d such that $\mathfrak{m}$-relative Goldie dimension of any quotient of M is finite for all $\mathfrak{m} \in \mathrm{Max}(R)$, then $H^d_\Phi(M)/IH^d_\Phi(M)$ is Artinian for all $I \in \Phi$. Furthermore, if R is semi-local, then $\mathrm{Supp}_R(H^{d-1}_\Phi(M)/IH^{d-1}_\Phi(M))$ is a finite set consisting of prime ideals $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} \leq 1$ for all $I \in \Phi$. Also, among other things, we provide a relationship between the vanishing and the finiteness of modules $H^i_\Phi(M)$ and we show that if $H^i_\Phi(M)$ is minimax for all $i \geq n \geq 1$, then $H^i_\Phi(M)$ is Artinian for all $i \geq n$.

Key Words: Local cohomology modules, system of ideals, weakly Laskerian modules, ZD-modules, Artinian modules.
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1 Introduction

Throughout this paper, R is a commutative Noetherian ring, I is an ideal of R and $V(I)$ is the set of all prime ideals of R containing I. The theory of local cohomology is an active area of research in commutative algebra, algebraic geometry and related fields. For an R-module M, the i-th local cohomology module of M with respect to I is defined as

$$H^i_I(M) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, M).$$

For more details about the local cohomology, we refer the reader to [9]. In last decades some generalizations of the theory of ordinary local cohomology modules was introduced. For example, the following definition was introduced by Bijan-Zadeh in [6]. Let Φ be a non-empty set of ideals of R. The set Φ is said to be a system of ideals of R if, whenever $I_1, I_2 \in \Phi$, then there is an ideal $J \in \Phi$ such that $J \subseteq I_1 I_2$. Let Φ be a system of ideals of R. For every R-module M, one can define

$$\Gamma_\Phi(M) = \{x \in M \mid Ix = 0 \ \text{ for some } \ I \in \Phi\}.$$

An R-module M is called Φ-torsion-free if $\Gamma_\Phi(M) = 0$ and it is Φ-torsion if $\Gamma_\Phi(M) = M$. It is easy to see that $\Gamma_\Phi(-)$ is an additive, covariant, R-linear left exact functor from the category of all R-modules to itself. For each $i \geq 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. This functor is called the *general local cohomology functor with respect to* Φ. Based on [8, Lemma 2.1], the functors $H^i_\Phi(-)$ and $\varinjlim_{I\in\Phi} H^i_I(-)$ are naturally equivalent. Moreover, when Φ consists of only the powers of an ideal I, the functor $H^i_\Phi(-)$ is just the ordinary local cohomology functor $H^i_I(-)$.

Some of the main problems in the local cohomology theory is to determine when a given local cohomology module is Artinian, finitely generated, zero and non-zero. Regarding this problems, there are several papers, for examples see [3, 7, 10, 17]. Let M be an R-module. Based on [10], M is said to be a *ZD-module* if for any submodule N of M, the set of zero-divisors of M/N is a finite union of prime ideals in $\mathrm{Ass}_R(M/N)$ and according to [20], M is called *minimax* if there is a finitely generated submodule F of M such that M/F is Artinian. It is shown in [10, Corollary 3.3] that for a ZD-module M of dimension d, the local cohomology module $H^d_I(M)$ is Artinian. Also, following [3, Theorem 2.3], if M is a finitely generated R-module such that $H^i_I(M)$ is a minimax R-module for all $i \geq n \geq 1$, then $H^i_I(M)$ is Artinian for all $i \geq n$. It will be a noticeable achievement, if we could extend these results to general local cohomology or to local cohomology of a larger class of modules. In this regard, recently in [17] the concept of Φ-minimax modules has been introduced and by means of that have been obtained some results about the Artinianness of general local cohomology modules.

In this paper, we are going to study the Artinianness of general local cohomology modules and improve the above results. In Section 2, we first introduce the concept of Φ-Laskerian modules and examine some of its basic properties that are needed in the rest of the paper. Then in Section 3, we show as the first main result, in Theorem 3.1 that for a Φ-Laskerian ZD-module M of finite dimension d, if $\mathfrak{m}$-relative Goldie dimension of any quotient of M is finite for all $\mathfrak{m} \in \mathrm{Max}(R)$, then $H^d_\Phi(M)/IH^d_\Phi(M)$ is Artinian for all $I \in \Phi$. Also, if R is semi-local, then $\mathrm{Supp}_R(H^{d-1}_\Phi(M)/IH^{d-1}_\Phi(M))$ is a finite set consisting of all prime ideals $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} \leq 1$ for all $I \in \Phi$. Also in Theorem 3.5 we prove that if M is a Φ-Laskerian ZD-module over a local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}$-relative Goldie dimension of any quotient of M is finite and $\mathrm{Supp}_R(H^i_\Phi(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$, then $H^i_\Phi(M)$ is Artinian for all $i < n$. These theorems show that the assertion in [17, Theorems 3.1 and 3.2] holds for a larger class of modules. Next, in Theorem 3.6 we deal with the relation between the vanishing and the finiteness of modules $H^i_\Phi(M)$ for any ZD-module M (not necessarily Φ-minimax) over any Noetherian ring R (not necessarily local). This result provides an improvement of [17, Theorem 3.3 and Corollaries 3.4-3.7]. As a consequence of this theorem we prove in Theorem 3.9 that for a ZD-module M, if $H^i_\Phi(M)$ is a minimax R-module for all $i \geq n$, then $H^i_\Phi(M)$ is Artinian for all $i \geq n$. This result leads to a generalization of [17, Theorem 3.8 and Corollary 3.9] and [3, Theorem 2.3].

Throughout this paper, Φ is a system of ideals of R, $\mathrm{Spec}(R)$ is the set of all prime ideals and $\mathrm{Max}(R)$ is the set of all maximal ideals of R. For any unexplained notation and terminology we refer the reader to [9] and [13].

2 Φ-Laskerian modules

Let M be an R-module and $E(M)$ denotes the injective envelope of M. Recall that the *Goldie dimension of* M is defined as the cardinal number of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules. We use $\mathrm{Gdim}M$ to denote Goldie dimension of M. For a prime ideal $\mathfrak{p}$, let $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\mu^0(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$ denotes the 0-th Bass number of M with respect to $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, M) > 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R(M)$. So, by definition of the Goldie dimension it follows that

$$\mathrm{Gdim}M = \sum_{\mathfrak{p}\in \mathrm{Ass}_R(M)} \mu^0(\mathfrak{p}, M).$$

It is shown in [20] that when R is a Noetherian ring, an R-module M is minimax if and only if for any submodule N of M, $\mathrm{Gdim}M/N < \infty$. In order to further study of local cohmology modules, some authors have introduced new and larger classes of modules by generalizing these concepts. For any ideal I of R, the *I-relative Goldie dimension of M*, which is introduced in [10], is defined as

$$\mathrm{Gdim}_I M := \sum_{\mathfrak{p}\in \mathrm{Ass}_R(M)\cap V(I)} \mu^0(\mathfrak{p}, M).$$

Also, an R-module M is said to be *I-minimax* if for any submodule N of M, $\mathrm{Gdim}_I M/N < \infty$ (see [4]).
Recently, in [17] the above notions was extended for any system of ideals Φ of R. Following [17], *Φ-relative Goldie dimension of M* (denoted by $\mathrm{Gdim}_\Phi M$) is defined as

$$\mathrm{Gdim}_\Phi M := \sum_{\mathfrak{p}\in \mathrm{Ass}_R(M)\cap \Omega} \mu^0(\mathfrak{p}, M),$$

where $\Omega = \bigcup_{I\in\Phi} V(I)$. Also, an R-module M is said to be Φ-minimax if for any submodule N of M, $\mathrm{Gdim}_\Phi M/N < \infty$. Note that $\mathrm{Gdim}_\Phi M = \mathrm{Gdim}\Gamma_\Phi(M)$ by [17, Corollary 2.3].

On the other hand, following [11] an R-module M is called *weakly Laskerian* if for any submodule N of M, the set $\mathrm{Ass}_R M/N$ is finite. The class of weakly Laskerian modules includes the class of minimax modules. This motivates us to introduce the concept of Φ-Laskerian modules as a generalization of the concepts weakly Laskerian modules and Φ-minimax modules.

Definition 2.1. *An R-module M is said to be Φ-Laskerian if the set of associated primes of the Φ-torsion submodule of any quotient module of M is finite; i.e. for any submodule N of M, the set $\mathrm{Ass}_R\Gamma_\Phi(M/N)$ is finite.*

Obviously, any weakly Laskerian module is Φ-Laskerian. So, any Noetherian and any Artinian R-module is Φ-Laskerian. Also, any Φ-minimax R-module is Φ-Laskerian. In particular, any minimax R-module is Φ-Laskerian. Clearly, if M is Φ-torsion, then M is Φ-Laskerian if and only if M is weakly Laskerian.

We claim that the class of Φ-Laskerian modules is strictly larger than the class of weakly Laskerian modules. To see this, consider the $\mathbb{Z}$-module $M = \bigoplus_{p\in\Lambda} \mathbb{Z}/p\mathbb{Z}$, where Λ is the set of all prime integers. It is easy to see that $\mathrm{Ass}_{\mathbb{Z}}(M) = \{p\mathbb{Z} \,|\, p \in \Lambda\}$. So, M has infinitely many associated prime ideals. Hence, M is not weakly Laskerian. Now, let $t \geq 1$ and $\{p_1, \ldots, p_t\}$ be a fixed subset of Λ. Set $\Phi := \{I \trianglelefteq \mathbb{Z} \,|\, \sqrt{I} = p_1 \cdots p_t\mathbb{Z}\}$. It is easy to see that Φ is a system of ideals of $\mathbb{Z}$ and $\mathrm{Ass}_{\mathbb{Z}}\Gamma_\Phi(M/N) \subseteq \{p_1\mathbb{Z}, \ldots, p_t\mathbb{Z}\}$ for any submodule N of M. Therefore, M is a Φ-Laskerian $\mathbb{Z}$-module.

Here, we state some important properties of the class of Φ-Laskerian modules.

Lemma 2.2. *If M is a Φ-Laskerian R-module, then $\Gamma_\Phi(M)$ is weakly Laskerian.*

Proof. For any submodule N of $\Gamma_\Phi(M)$, we have $\Gamma_\Phi(N) = N$ and

$$\mathrm{Ass}_R\left(\frac{\Gamma_\Phi(M)}{N}\right) \subseteq \mathrm{Ass}_R\Gamma_\Phi(M/N).$$

So, the assertion follows by assumption. □

Proposition 2.3. *(i) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then M is Φ-Laskerian if and only if M' and M'' are both Φ-Laskerian.*

(ii) The class of Φ-Laskerian modules is closed under taking submodules, quotients and extensions. In particular, any finite sum of Φ-Laskerian modules is Φ-Laskerian.

(iii) Let M and N be two R-modules. If M is finitely generated and N is Φ-Laskerian, then $Ext^i_R(M,N)$ and $Tot^R_i(M,N)$ are Φ-Laskerian for all $i \geq 0$.

Proof. In view of [1, Lemma 2.2] it is enough to prove (i). For do this, we may assume that M' is a submodule of M and $M'' = M/M'$. If M is Φ-Laskerian, it is easy to see that M' and M'' are Φ-Laskerian. Now, suppose that M' and M/M' are Φ-Laskerian and N is an arbitrary submodule of M. Then the exact sequence

$$0 \to \frac{M'+N}{N} \to \frac{M}{N} \to \frac{M}{M'+N} \to 0$$

induces the exact sequence

$$0 \to \Gamma_\Phi\left(\frac{M'}{M'\cap N}\right) \to \Gamma_\Phi\left(\frac{M}{N}\right) \to \Gamma_\Phi\left(\frac{M}{M'+N}\right).$$

Since $\frac{M}{M'+N} \cong \frac{M/M'}{(M'+N)/M'}$ and

$$\mathrm{Ass}_R\Gamma_\Phi(M/N) \subseteq \mathrm{Ass}_R\Gamma_\Phi(M'/M'\cap N) \cup \mathrm{Ass}_R\Gamma_\Phi(M/M'+N),$$

it may be concluded that the set $\mathrm{Ass}_R\Gamma_\Phi(M/N)$ is finite, and so M is Φ-Laskerian. □

Proposition 2.4. *Let Φ and ψ be two systems of ideals of R and M be a Φ-Laskerian R-module such that $\mathrm{Ass}_R(M) \subseteq \bigcup_{\mathfrak{b}\in\psi} V(\mathfrak{b})$. Then $H^i_\psi(M)$ is Φ-Laskerian for all $i \geq 0$.*

Proof. For $i = 0$, since $\Gamma_\psi(M)$ is a submodule of M, we see that $H^0_\psi(M)$ is Φ-Laskerian by Proposition 2.3. Also, by assumption and [17, Lemma 2.7], $\Gamma_\psi(M) = M$. So, in the light of [16, 1.4], we have $H^i_\Phi(M) \cong H^i_\Phi(M/\Gamma_\Phi(M)) = 0$ for all $i \geq 1$. □

Corollary 2.5. *If M is a Φ-Laskerian R-module such that $\mathrm{Ass}_R(M) \subseteq \Omega$, then $H^i_\Phi(M)$ is weakly Laskerian for all $i \geq 0$.*

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\Phi_\mathfrak{p}$ denote the set $\{IR_\mathfrak{p} | I \in \Phi\}$. The following theorem shows that the local-global principle is valid for Φ-Laskerian modules over semi-local rings.

Proposition 2.6. *Let R be a semi-local ring. Then the following statements are equivalent:*

(i) M is Φ-Laskerian.

(ii) $M_{\mathfrak{p}}$ is $\Phi_{\mathfrak{p}}$-Laskerian for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

(iii) $M_{\mathfrak{m}}$ is $\Phi_{\mathfrak{m}}$-Laskerian for all $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. It is clear that (i)⇒(ii)⇒(iii) holds over any Noetherian ring. So, we prove only (iii)⇒(i). Let N be an arbitrary submodule of M and suppose contrary that $\mathrm{Ass}_R(\Gamma_\Phi(M/N))$ is an infinite set. So, by assumption there exists $\mathfrak{m} \in \mathrm{Max}(R)$ and a countably infinite subset $\{\mathfrak{p}_r\}_{r=1}^{\infty}$ of $\mathrm{Ass}_R(\Gamma_\Phi(M/N))$ such that $\mathfrak{p}_i \subseteq \mathfrak{m}$ for all $r \geq 1$. Thus, $\mathfrak{p}_r R_{\mathfrak{m}} \in \mathrm{Ass}_R(\Gamma_{\Phi_{\mathfrak{m}}}(M_{\mathfrak{m}}/N_{\mathfrak{m}}))$ for all $r = 1, 2, \cdots$, a contradiction. □

3 Main results

In this section, we use the arguments of the previous section and we prove our main results. We shall need the following Remark and Lemmas later.

Lemma 3.1. *Let M be an R-module with $\dim M = d < \infty$. Then $\dim H_{\Phi}^{d-i}(M) \leq i$ for all i.*

Proof. Let $\mathfrak{p} \in \mathrm{Supp}_R(H_{\Phi}^{d-i}(M))$. Then $(H_{\Phi}^{d-i}(M))_{\mathfrak{p}} = H_{\Phi_{\mathfrak{p}}}^{d-i}(M_{\mathfrak{p}}) \neq 0$. Thus, it follows from [6, 2.7] that $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq d - i$ and so

$$\dim R/\mathfrak{p} \leq d - \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq i.$$

□

Remark 3.2. *Based on [10], an R-module M is said to be a ZD-module if for any submodule N of M, the set of zero-divisors of M/N is a finite union of prime ideals in $\mathrm{Ass}_R(M/N)$. It is easy to see that any minimax module and any weakly Laskerian module is a ZD-module. Following [12], if M is a ZD R-module, then $S^{-1}M$ is a ZD $S^{-1}R$-module for all multiplicatively closed subsets S of R. Moreover, the class of ZD-modules is closed under taking submodules, quotients and extensions.*

Lemma 3.3. *Let M be a Φ-torsion-free ZD-module and $I \in \Phi$. Then there exists $x \in I$ and the long exact sequence*

$$\cdots \to H_{\Phi}^{i-1}(M/xM) \to H_{\Phi}^{i}(M) \xrightarrow{\cdot x} H_{\Phi}^{i}(M) \to H_{\Phi}^{i}(M/xM) \to \cdots$$

for all $i \geq 0$ where $\dim M/xM < \dim M$.

Proof. Since $\Gamma_\Phi(M) \cong \varinjlim_{I \in \Phi} \Gamma_I(M)$, it follows from assumption that M is I-torsion-free for all $I \in \Phi$. Thus, by [10, Lemma 2.4] there is an element $x \in I$ which is regular on M. So, $\dim M/xM < \dim M$ and we have the short exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0$$

which induces the long exact sequence

$$\cdots \to H_\Phi^{i-1}(M/xM) \to H_\Phi^i(M) \overset{\cdot x}{\to} H_\Phi^i(M) \to H_\Phi^i(M/xM) \to \cdots$$

for all $i \geq 0$. □

Theorem 3.4. *Let M be a Φ-Laskerian ZD-module of finite dimension d. Then the following statements hold.*

(i) If $\mathfrak{m}$-relative Goldie dimension of any quotient of M is finite for all $\mathfrak{m} \in \mathrm{Max}(R)$, then $H_\Phi^d(M)/IH_\Phi^d(M)$ is Artinian for all $I \in \Phi$.

(ii) If R is semi-local, then $\mathrm{Supp}_R(H_\Phi^{d-1}(M)/IH_\Phi^{d-1}(M))$ is a finite set consisting of all prime ideals $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} \leq 1$ for all $I \in \Phi$.

Proof. (i) We proceed by induction on d. Let $d = 0$. By assumption, $\mathrm{Ass}_R(H_\Phi^0(M))$ is a finite set and consist of maximal ideals of R since $\dim H_\Phi^0(M) = 0$. Note that for all $\mathfrak{m} \in \mathrm{Ass}_R(H_\Phi^0(M))$ we have

$$\mu^0(\mathfrak{m}, H_\Phi^0(M)) = \dim_{k(\mathfrak{m})} \mathrm{Hom}_{R_\mathfrak{m}}(k(\mathfrak{m}), (H_\Phi^0(M))_\mathfrak{m}) \leq \mu^0(\mathfrak{m}, M) = \mathrm{Gdim}_\mathfrak{m} M < \infty$$

where $k(\mathfrak{m}) := R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Thus,

$$E(H_\Phi^0(M)) = \bigoplus_{\mathfrak{m} \in \mathrm{Ass}_R(H_\Phi^0(M))} E(R/\mathfrak{m})^{\mu^0(\mathfrak{m}, H_\Phi^0(M))}$$

implies that $E(H_\Phi^0(M))$ is Artinian. Hence, $H_\Phi^0(M)$ is Artinian.
Now, let $I \in \Phi$ and suppose that $d > 0$ and the claim has been proved for $d - 1$. It follows from [16, 1.4] that $H_\Phi^i(M) \cong H_\Phi^i(M/\Gamma_\Phi(M))$ for all $i \geq 1$. Since $M/\Gamma_\Phi(M)$ is a Φ-torsion-free Φ-Laskerian R-module, without loss of generality, we can assume that M is Φ-torsion-free. Hence, by Lemma 3.3 we have the long exact sequence

$$\cdots \to H_\Phi^{d-1}(M/xM) \to H_\Phi^d(M) \overset{\cdot x}{\to} H_\Phi^d(M) \to H_\Phi^d(M/xM) \to \cdots$$

where $x \in I$ and $\dim M/xM < \dim M$. So, it follows from [6, 2.7] that $H_\Phi^d(M/xM) = 0$. Now the above long exact sequence induces the following exact sequence:

$$\cdots \to H_\Phi^{d-1}(M/xM)/IH_\Phi^{d-1}(M/xM) \to H_\Phi^d(M)/IH_\Phi^d(M) \overset{\cdot x}{\to} H_\Phi^d(M)/IH_\Phi^d(M) \to 0.$$

Note that M/xM is a Φ-Laskerian ZD-module by Proposition 2.3 and Remark 3.2. So, the inductive hypothesis implies that $H_\Phi^{d-1}(M/xM)/IH_\Phi^{d-1}(M/xM)$ is Artinian. This leads to the Artinianness of $(0 :_{H_\Phi^d(M)/IH_\Phi^d(M)} x)$. Since $x \in I$, we deduce from [14, Theorem 1.3] that $H_\Phi^d(M)/IH_\Phi^d(M)$ is Artinian, as desired.

(ii) In view of Lemma 3.1, it is enough to show that $\mathrm{Supp}_R(H_\Phi^{d-1}(M)/IH_\Phi^{d-1}(M))$ is a finite set. For this aim, we use induction on d. We can assume that $d \geq 1$. If $d = 1$, then $\dim H_\Phi^0(M) \leq \dim M = 1$ and so

$$\mathrm{Supp}_R(H_\Phi^0(M)) \subseteq \mathrm{Max}(R) \cup \mathrm{Ass}_R(H_\Phi^0(M)).$$

Thus, the assertion follows from the assumption. Now, assume that $d > 1$ and we have obtained the result for any Φ-Laskerian ZD-module of dimension smaller than d. As we mentioned in the proof of part (i), we may assume that M is Φ-torsion-free. So, by Lemma 3.3 we have the long exact sequence

$$\cdots \to H_{\Phi}^{d-2}(M/xM) \xrightarrow{f} H_{\Phi}^{d-1}(M) \xrightarrow{\cdot x} H_{\Phi}^{d-1}(M) \to H_{\Phi}^{d-1}(M/xM) \to \cdots \tag{1}$$

where $\dim M/xM \leq d-1$. If $\dim M/xM < d-1$, then $H_{\Phi}^{d-1}(M/xM) = 0$ by [6, 2.7] and so $H_{\Phi}^{d-1}(M) = xH_{\Phi}^{d-1}(M)$. Hence, this case is obvious since $x \in I$. Now suppose that $\dim M/xM = d-1$. Hence, by the inductive hypothesis $\mathrm{Supp}_R(H_{\Phi}^{d-2}(M/xM)/IH_{\Phi}^{d-2}(M/xM))$ is a finite set. Therefore, considering the exact sequences

$$H_{\Phi}^{d-2}(M/xM) \to \mathrm{Im} f \to 0$$

and

$$0 \to \mathrm{Im} f \to H_{\Phi}^{d-1}(M) \to xH_{\Phi}^{d-1}(M) \to 0$$

and applying the functor $R/I \otimes -$ to them, we obtain that $\mathrm{Supp}_R(H_{\Phi}^{d-1}(M)/IH_{\Phi}^{d-1}(M))$ is a finite set. □

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring and M a Φ-Laskerian ZD-module such that $\mathfrak{m}$-relative Goldie dimension of any quotient of M is finite. Assume that n is a positive integer with $\mathrm{Supp}_R(H_{\Phi}^{i}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Then $H_{\Phi}^{i}(M)$ is Artinian for all $i < n$.*

Proof. We prove this by induction on n. When $n = 1$, by the same method as in the proof of Theorem 3.4, $H_{\Phi}^{0}(M)$ is Artinian. Now, assume that $n > 0$ and that the claim holds for $n-1$. By inductive hypothesis we need only to prove that $H_{\Phi}^{n-1}(M)$ is Artinian. Similar to the proof of Theorem 3.4, we can assume that M is Φ-torsion-free. Hence, by Lemma 3.3 we have the long exact sequence

$$\cdots \to H_{\Phi}^{i-1}(M) \to H_{\Phi}^{i-1}(M/xM) \to H_{\Phi}^{i}(M) \xrightarrow{\cdot x} H_{\Phi}^{i}(M) \to \cdots$$

where $x \in I \subseteq \mathfrak{m}$. Hence, $\mathrm{Supp}_R(H_{\Phi}^{i}(M/xM)) \subseteq \{\mathfrak{m}\}$ for all $i < n-1$ where M/xM is a Φ-Laskerian ZD-module by Proposition 2.3 and Remark 3.2. Therefore, $H_{\Phi}^{n-2}(M/xM)$ is Artinian by inductive hypothesis. Thus, $(0 :_{H_{\Phi}^{n-1}(M)} x)$ is Artinian. Since $x \in \mathfrak{m}$, it follows from the assumption that $\mathrm{Supp}_R(H_{\Phi}^{n-1}(M)) \subseteq V(\mathfrak{m}) \subseteq V(Rx)$. Therefore, $H_{\Phi}^{n-1}(M)$ is Rx-torsion and so is Artinian by [14, Theorem 1.3]. This completes the proof. □

We recall that a module M is called *coatomic*, if every proper submodule of M is contained in a maximal submodule of M. Coatomic modules have been studied by Zöschinger in [19]. It is clear that every finitely generated R-module is coatomic. In [17, Theorem 3.3] it was shown that if $(R, \mathfrak{m})$ is a local ring and M is a Φ-minimax ZD-module, then $H_{\Phi}^{i}(M) = 0$ for all $i \geq n$ for some $n \geq 1$ if and only if $H_{\Phi}^{i}(M)$ is finitely generated for all $i \geq n$. Also, this result was proved in [3, Theorem 3.9] for any finitely generated module over any commutative Noetherian ring. In the following theorem we prove these results for any ZD-module (not necessarily finitely generated or Φ-minimax) over any commutative Noetherian ring (not necessarily local). Also, the following Theorem provides a generalization of [18, Proposition 3.1] and [17, Corollaries 3.4-3.6].

Theorem 3.6. *Let M be a ZD-module. If n is a positive integer, then the following statements are equivalent:*

(i) $H^i_\Phi(M) = 0$ *for all* $i \geq n$.

(ii) $H^i_\Phi(M)$ *is finitely generated for all* $i \geq n$.

(iii) $H^i_\Phi(M)$ *is coatomic for all* $i \geq n$.

Proof. Implications (i) ⇒ (ii) ⇒ (iii) are trivial.

(iii) ⇒ (i): By [19, Section 1, Folgerung] we may assume that $(R, \mathfrak{m})$ is a local ring. We proceed by induction on $d := \dim M$. If $d = 0$, then $H^i_\Phi(M) = 0$ for all $i \geq 1$ by [6, 2.7]. Let $d > 0$. By the same method as in the proof of Theorem 3.4, without loss of generality, we assume that M is Φ-torsion-free. Hence, by Lemma 3.3, there exists $x \in \mathfrak{m}$ such that M/xM is a ZD-module with $\dim M/xM < \dim M$. Moreover, we obtain the long exact sequence

$$\cdots \to H^i_\Phi(M) \stackrel{\cdot x}{\to} H^i_\Phi(M) \to H^i_\Phi(M/xM) \to H^{i+1}_\Phi(M) \to \cdots$$

for all $i \geq 0$. Thus, by the assumption we conclude that $H^i_\Phi(M/xM)$ is coatomic for all $i \geq n$. Therefore, the inductive hypothesis implies that $H^i_\Phi(M/xM) = 0$ for all $i \geq n$. Now the above long exact sequence yields that $H^i_\Phi(M) = xH^i_\Phi(M)$ for all $i \geq n$. Note that in the light of [19, Satz 2.4], the coatomic modules satisfy Nakayama's Lemma. Thus, $H^i_\Phi(M) = 0$ for all $i \geq n$, as desired. □

The following result is an immediate consequence of Theorem 3.6.

Corollary 3.7. *Let M be a ZD-module. If* $\operatorname{cd}(\Phi, M) := \sup\{i | H^i_\Phi(M) \neq 0\} \geq 1$, *then* $H^{\operatorname{cd}(\Phi,M)}_\Phi(M)$ *is not coatomic. In particular, it is not finitely generated.*

Remark 3.8. *By definition, for a minimax R-module M, there exists an exact sequence*

$$0 \to F \to M \to A \to 0 \tag{2}$$

which F is finitely generated and A is Artinian. Therefore, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is finitely generated for each non-maximal prime ideal $\mathfrak{p}$. Moreover, if $\operatorname{Supp}_R(M) \subseteq \operatorname{Max}(R)$, *then M is Artinian.*

It was shown in [17, Theorem 3.8] that for a Φ-minimax ZD-module M and a positive integer n, if $H^i_\Phi(M)$ are minimax R-modules for all $i \geq n$, then they must be Artinian. In the following theorem we are going to show that the Φ-minimax condition on M in [17, Theorem 3.8] is superfluous and the assertion is valid for any ZD-module.

Theorem 3.9. *Let M be a ZD-module and n be a positive integer. Then the following statements are equivalent:*

(i) $H^i_\Phi(M)$ *is an Artinian R-module for all* $i \geq n$.

(ii) $H^i_\Phi(M)$ *is a minimax R-module for all* $i \geq n$.

Proof. We only need to prove that (ii)⇒(i). Let $\mathfrak{p} \in \operatorname{Spec}(R)\backslash \operatorname{Max}(R)$. Then by assumption and Remark 3.2 $M_\mathfrak{p}$ is a ZD-module. Furthermore, $(H^i_\Phi(M))_\mathfrak{p} \cong H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for all $i \geq n$ by assumption and Remark 3.8. Therefore, Theorem 3.6 implies that $(H^i_\Phi(M))_\mathfrak{p} = 0$ for all $i \geq n$. Thus, $\operatorname{Supp}_R(H^i_\Phi(M)) \subseteq \operatorname{Max}(R)$ which yields that $H^i_\Phi(M)$ is Artinian for all $i \geq n$ by Remark 3.8, as desired. □

By [3, Theorem 2.3] it is known that when M is finitely generated such that $H^i_I(M)$ are minimax R-modules for all $i \geq n$, then they must be Artinian. Since every weakly Laskerian module is a ZD-module, the following result immediately follows from Theorem 3.9 and is a generalization of [3, Theorem 2.3].

Corollary 3.10. *Let M be a weakly Laskerian R-module and n be a positive integer such that $H^i_I(M)$ is a minimax R-module for all $i \geq n$. Then $H^i_I(M)$ is Artinian for all $i \geq n$.*

Remark 3.11. *Following [15], an R-module M is said to be* FSF *if there is a finitely generated submodule N of M such that the support of M/N is finite. It has shown in [5, Theorem 3.3] that over a Noetherian ring R, an R-module M is weakly Laskerian if and only if it is FSF. Also, if $t \geq -1$ is an integer, based on [2], an R-module M is said to be $FD_{\leq t}$ if there is a finitely generated submodule N of M such that $\dim M/N \leq t$. Note that for every R-module T with finite support we have $\dim T \leq 1$. This implies that every weakly Laskerian R-module M is $FD_{\leq 1}$. So, there exists a finitely generated R-module F such that $M_\mathfrak{p} \cong F_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} > 1$.*

In what follows, we state some results for the general local cohomology modules in these classes.

Theorem 3.12. *Let R be a semi-local ring and M be a ZD-module. If n is a positive integer, then the following statements are equivalent:*

(i) $H^i_\Phi(M)$ *is an weakly Laskerian R-module for all $i \geq n$.*

(ii) $\operatorname{Supp}_R(H^i_\Phi(M))$ *is a finite set consisting of all prime ideals $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} \leq 1$ for all $i \geq n$.*

Proof. (ii)⇒(i) is obvious by Remark 3.11 and so it is enough to show (i)⇒(ii). Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} > 1$. Then $(H^i_\Phi(M))_\mathfrak{p} \cong H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for all $i \geq n$ by assumption and Remark 3.11. Since, $M_\mathfrak{p}$ is a ZD-module, it follows from Theorem 3.6 that $(H^i_\Phi(M))_\mathfrak{p} = 0$ for all $i \geq n$. Thus,

$$\operatorname{Supp}_R(H^i_\Phi(M)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim R/\mathfrak{p} \leq 1\}.$$

Therefore,

$$\operatorname{Supp}_R(H^i_\Phi(M)) \subseteq \operatorname{Max}(R) \cup \operatorname{Ass}_R(H^i_\Phi(M)).$$

By assumption, this completes the proof. □

Theorem 3.13. *Let M be a ZD-module and t be a non-negative integer. If n is a positive integer, then the following statements are equivalent:*

(i) $H^i_\Phi(M)$ *is an* $FD_{\leq t}$ *R-module for all* $i \geq n$.

(ii) $\dim(H^i_\Phi(M)) \leq t$.

Proof. We need only to prove (i)⇒(ii). Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} > t$. Then $(H^i_\Phi(M))_\mathfrak{p} \cong H^i_{\Phi_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for all $i \geq n$ by assumption. Since, $M_\mathfrak{p}$ is a ZD-module, it follows from Theorem 3.6 that $(H^i_\Phi(M))_\mathfrak{p} = 0$ for all $i \geq n$. Thus,

$$\operatorname{Supp}_R(H^i_\Phi(M)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim R/\mathfrak{p} \leq t\}.$$

This completes the proof. □

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