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A supercongruence related to Ramanujan-type formula for $1/\pi$

by

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Abstract

We prove a Ramanujan-type supercongruence involving the Almkvist–Zudilin numbers, which confirms a conjecture of Z.-H. Sun and is corresponding to Ramanujan-type formula for $1/\pi$ due to Chan and Verrill:

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-27)^k} \gamma_k = \frac{3\sqrt{3}}{\pi}.$$

Here γ_k are the Almkvist–Zudilin numbers.

Key Words: Supercongruences, Almkvist–Zudilin numbers, harmonic numbers.
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1 Introduction

In 1914, Ramanujan [15] discovered 17 infinite series representations of $1/\pi$, such as

$$\sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} = \frac{2}{\pi},$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$. In 1997, Van Hamme [25] investigated supercongruences on partial sums of Ramanujan's infinite series for $1/\pi$ and proposed 13 interesting supercongruence conjectures, which opened up the study of supercongruences related to infinite series for $1/\pi$. We refer to [23] for more recent developments on Van Hamme's supercongruences.

Supercongruences for partial sums of infinite series for $1/\pi$ are sometimes called Ramanujan-type supercongruences. Although all of Van Hamme's 13 supercongruence conjectures have been proved by many mathematicians through various methods, Ramanujan-type supercongruences still attract many experts' attention (see, for instance, [6, 7, 11, 13, 22, 28]).

The Almkvist–Zudilin numbers (see [1] and [17, A125143]) are defined as

$$\gamma_n = \sum_{j=0}^{n} (-1)^{n-j} \frac{3^{n-3j}(3j)!}{(j!)^3} \binom{n}{3j} \binom{n+j}{j},$$

which appears to be first recorded by Zagier [27] as integral solutions to Apéry-like recurrence equations.

Chan and Verrill [4] established several new Ramanujan-type series for $1/\pi$ in terms of Almkvist–Zudilin numbers, such as

$$\sum_{k=0}^{\infty} \frac{4k+1}{81^k}\gamma_k = \frac{3\sqrt{3}}{2\pi}, \tag{1}$$

and

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-27)^k}\gamma_k = \frac{3\sqrt{3}}{\pi}. \tag{2}$$

Zudilin [28, (33)] conjectured that (1) possesses the following nice p-adic analogue:

$$\sum_{k=0}^{p-1} \frac{4k+1}{81^k}\gamma_k \equiv \left(\frac{-3}{p}\right) p \pmod{p^3},$$

which was recently confirmed by the author [12]. Here and in what follows, $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. We remark that congruence properties for the Almkvist–Zudilin numbers have been widely studied by Amdeberhan and Tauraso [2], Chan, Cooper and Sica [3], and Z.-H. Sun [18, 20, 21].

The motivation of the paper is to establish a p-adic analogue of (2), which was originally conjectured by Z.-H. Sun [18, Conjecture 6.8].

Theorem 1.1. *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \frac{4k+1}{(-27)^k}\gamma_k \equiv \left(\frac{-3}{p}\right) p \pmod{p^3}. \tag{3}$$

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results. We prove Theorem 1.1 in Section 3.

2 Preliminary results

The nth harmonic number is given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j},$$

with the convention that $H_0 = 0$. In order to prove Theorem 1.1, we require the following preliminary results.

Lemma 2.1. *For any non-negative integer n, we have*

$$\sum_{i=0}^{n} \frac{(-1)^i}{2i+1}\binom{n}{i}\binom{n+i}{i}\left(H_{2i+1} - H_i\right) = \frac{1}{(2n+1)^2} + \frac{2}{2n+1}\left(H_{2n} - H_n\right). \tag{4}$$

In fact, the identity (4) can be discovered and proved by the symbolic summation package `Sigma` developed by Schneider [16]. One can also refer to [9, 10] for the same approach to finding and proving identities of this type.

Lemma 2.2. *For any prime $p \geq 5$, we have*

$$\frac{(1/3)_{(p-1)/2}(2/3)_{(p-1)/2}}{(1)_{(p-1)/2}^2} \equiv \left(\frac{-3}{p}\right) p(3^p - 6p - 5) \pmod{p^3}, \tag{5}$$

and

$$\frac{1}{p}\binom{p-1}{(p-1)/2}^2 \binom{2p-2}{p-1}\binom{(5p-3)/2}{2p-2}$$
$$\equiv -8p\left(2^{4(p-1)} - 2^{p+1} + 4p + 4\right) \pmod{p^3}. \tag{6}$$

Proof. Note that

$$\frac{(1/3)_{(p-1)/2}(2/3)_{(p-1)/2}}{(1)_{(p-1)/2}^2} = \frac{\binom{p-1}{(p-1)/2}\binom{3(p-1)/2}{(p-1)/2}}{27^{(p-1)/2}}. \tag{7}$$

By [8, (49)], we have

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}. \tag{8}$$

Furthermore, we have

$$\begin{aligned}\binom{3(p-1)/2}{(p-1)/2} &= \frac{p \cdot (p+1) \cdots (p + (p-3)/2)}{1 \cdot 2 \cdots (p-1)/2} \\ &\equiv \frac{p}{(p-1)/2}\left(1 + pH_{(p-3)/2}\right) \\ &\equiv -2p - 2p^2\left(H_{(p-1)/2} + 3\right) \\ &\equiv 2p\left(2^p - 3p - 3\right) \pmod{p^3},\end{aligned} \tag{9}$$

where we have used the congruence [8, (45)]:

$$H_{(p-1)/2} \equiv \frac{2(1 - 2^{p-1})}{p} \pmod{p}. \tag{10}$$

Combining (7)–(9), we arrive at

$$\frac{(1/3)_{(p-1)/2}(2/3)_{(p-1)/2}}{(1)_{(p-1)/2}^2}$$
$$\equiv \frac{(-1)^{(p-1)/2} 2^{2p-1}\left(2^p - 3p - 3\right) p}{3^{3(p-1)/2}} \pmod{p^3}. \tag{11}$$

From the congruence $\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)^2\equiv 0 \pmod{p^2}$, we deduce that

$$(-3)^{(p-1)/2}\equiv\left(\frac{-3}{p}\right)\frac{3^{p-1}+1}{2}\pmod{p^2}. \tag{12}$$

Applying (12) and the Fermat's little theorem to the right-hand side of (11) gives

$$\frac{(1/3)_{(p-1)/2}(2/3)_{(p-1)/2}}{(1)^2_{(p-1)/2}}\equiv\left(\frac{-3}{p}\right)p(3^p-6p-5)\pmod{p^3},$$

which proves (5).

Note that $\binom{2p-2}{p-1}/p$ is always an integer. On the other hand, we have

$$\begin{aligned}\binom{(5p-3)/2}{2p-2}&=\frac{(2p-1)\cdot 2p\cdot(2p+1)\cdots(2p+(p-3)/2)}{1\cdot 2\cdots(p+1)/2}\\ &\equiv\frac{(2p-1)\cdot 2p}{(p-1)/2\cdot(p+1)/2}\left(1+2pH_{(p-3)/2}\right)\\ &\equiv 8p+16p^2\left(H_{(p-1)/2}+1\right)\\ &\equiv 8p\left(2p+5-2^{p+1}\right)\pmod{p^3}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{p}\binom{p-1}{(p-1)/2}^2\binom{2p-2}{p-1}\binom{(5p-3)/2}{2p-2}\\ &\equiv 8\left(2p+5-2^{p+1}\right)\binom{p-1}{(p-1)/2}^2\binom{2p-2}{p-1}\pmod{p^3}.\end{aligned} \tag{13}$$

Furthermore, by Wolstenholme's theorem [26], we have

$$\binom{2p-2}{p-1}=\frac{p}{2p-1}\binom{2p-1}{p-1}\equiv -p(2p+1)\pmod{p^3}. \tag{14}$$

Finally, combining (8),(13) and (14) gives

$$\frac{1}{p}\binom{p-1}{(p-1)/2}^2\binom{2p-2}{p-1}\binom{(5p-3)/2}{2p-2}\equiv -8p\left(2^{4(p-1)}-2^{p+1}+4p+4\right)\pmod{p^3},$$

where we have used the Fermat's little theorem. ∎

Lemma 2.3. *For any prime $p\geq 5$, we have*

$$p\sum_{k=0}^{p-1}\frac{(3k)!}{(2k+1)3^{3k}k!^3}\left(H_{3k}-H_k\right)\equiv 2(p-1)\left(\frac{-3}{p}\right)\pmod{p^2}. \tag{15}$$

Proof. Noting that

$$\frac{(3k)!}{3^{3k}k!^3} = \frac{(1/3)_k(2/3)_k}{(1)_k^2}, \tag{16}$$

we have

$$\sum_{k=0}^{p-1} \frac{(3k)!}{(2k+1)3^{3k}k!^3}\left(3H_{3k} - H_k\right) = \sum_{k=0}^{p-1} \frac{(1/3)_k(2/3)_k}{(2k+1)(1)_k^2} \sum_{j=0}^{k-1}\left(\frac{1}{1/3+j} + \frac{1}{2/3+j}\right). \tag{17}$$

Recall the following identity due to Tauraso [24, Theorem 1]:

$$\frac{(1/3)_k(2/3)_k}{(1)_k^2} \sum_{j=0}^{k-1}\left(\frac{1}{1/3+j} + \frac{1}{2/3+j}\right) = \sum_{j=0}^{k-1} \frac{(1/3)_j(2/3)_j}{(1)_j^2} \cdot \frac{1}{k-j}. \tag{18}$$

Substituting (18) into the right-hand side of (17) and exchanging the summation order gives

$$\begin{aligned}
&\sum_{k=0}^{p-1} \frac{(3k)!}{(2k+1)3^{3k}k!^3}\left(3H_{3k} - H_k\right)\\
&= \sum_{k=0}^{p-1}\sum_{j=0}^{k-1} \frac{(1/3)_j(2/3)_j}{(1)_j^2} \cdot \frac{1}{(2k+1)(k-j)}\\
&= \sum_{j=0}^{p-2} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2} \sum_{k=j+1}^{p-1}\left(\frac{1}{k-j} - \frac{2}{2k+1}\right)\\
&= \sum_{j=0}^{p-2} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{p-1} - 2H_{2p-1} + H_{p-1-j} + 2H_{2j+2} - H_{j+1}\right).
\end{aligned} \tag{19}$$

It follows from (16) and (19) that

$$\begin{aligned}
&p\sum_{k=0}^{p-1} \frac{(3k)!}{(2k+1)3^{3k}k!^3}\left(3H_{3k} - 3H_k\right)\\
&\equiv p\sum_{j=0}^{p-2} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{p-1} - 2H_{2p-1} + H_{p-1-j} + 2H_{2j+2} - H_{j+1} - 2H_j\right)\\
&\equiv 2p\sum_{j=0}^{p-2} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{2j+1} - H_j\right) - 2\sum_{j=0}^{p-1} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2} \pmod{p^2},
\end{aligned} \tag{20}$$

where we have used the facts that $H_{p-1} \equiv 0 \pmod{p^2}, H_{p-1-j} \equiv H_j \pmod{p}$ and $(1/3)_{p-1}(2/3)_{p-1} \equiv 0 \pmod{p^2}$.

Furthermore, we have

$$\begin{aligned}
&p\sum_{j=0}^{p-2}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{2j+1}-H_j\right)\\
&\equiv p\sum_{j=0}^{\lfloor p/3\rfloor}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{2j+1}-H_j\right)+\sum_{j=(p+1)/2}^{p-1}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\\
&+\frac{(1/3)_{(p-1)/2}(2/3)_{(p-1)/2}}{(1)_{(p-1)/2}^2}\left(\frac{1}{p}-H_{(p-1)/2}\right)\pmod{p^2},
\end{aligned}\tag{21}$$

where we have used the fact that $(1/3)_j(2/3)_j\equiv 0\pmod{p}$ for $j>\lfloor p/3\rfloor$.

By [2, Lemma 2.3], we have

$$\frac{(1/3)_j(2/3)_j}{(1)_j^2}\equiv(-1)^j\binom{\lfloor p/3\rfloor}{j}\binom{\lfloor p/3\rfloor+j}{j}\pmod{p},\tag{22}$$

for $0\le j\le\lfloor p/3\rfloor$. It follows from (4) and (22) that

$$\begin{aligned}
&p\sum_{j=0}^{\lfloor p/3\rfloor}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{2j+1}-H_j\right)\\
&\equiv p\sum_{j=0}^{\lfloor p/3\rfloor}\frac{(-1)^j}{2j+1}\binom{\lfloor p/3\rfloor}{j}\binom{\lfloor p/3\rfloor+j}{j}\left(H_{2j+1}-H_j\right)\\
&=p\left(\frac{1}{(2\lfloor p/3\rfloor+1)^2}+\frac{2}{2\lfloor p/3\rfloor+1}\left(H_{2\lfloor p/3\rfloor}-H_{\lfloor p/3\rfloor}\right)\right)\pmod{p^2}.
\end{aligned}\tag{23}$$

If $p\equiv 1\pmod{3}$, by $H_{p-1-j}\equiv H_j\pmod{p}$ we have

$$\text{LHS (23)}\equiv 9p\pmod{p^2}.$$

If $p\equiv 2\pmod{3}$, then

$$\text{LHS (23)}\equiv p\left(9-6\left(H_{\lfloor p/3\rfloor+1}-H_{\lfloor p/3\rfloor}\right)\right)\equiv -9p\pmod{p^2}.$$

It follows that

$$p\sum_{j=0}^{\lfloor p/3\rfloor}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\left(H_{2j+1}-H_j\right)\equiv 9p\left(\frac{-3}{p}\right)\pmod{p^2}.\tag{24}$$

By [19, Theorem 2.3] and [14, Remark 1.2], we have

$$\sum_{j=0}^{p-1}\frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},\tag{25}$$

and

$$\sum_{j=0}^{(p-1)/2} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2} \equiv \left(\frac{-3}{p}\right)\left(3^p+2-2^{p+1}\right) \pmod{p^2},$$

and so

$$\sum_{j=(p+1)/2}^{p-1} \frac{(1/3)_j(2/3)_j}{(2j+1)(1)_j^2} \equiv \left(\frac{-3}{p}\right)\left(2^{p+1}-3^p-1\right) \pmod{p^2}. \tag{26}$$

It follows from (6), (10), (20), (21), (24), (25) and (26) that

$$\begin{aligned}&\frac{3p}{2}\sum_{k=0}^{p-1} \frac{(3k)!}{(2k+1)3^{3k}k!^3}\left(H_{3k}-H_k\right)\\ &\equiv \left(\frac{-3}{p}\right)\left(9p+2^{p+1}-3^p-2+(3^p-6p-5)(2^p-1)\right)\\ &\equiv 3(p-1)\left(\frac{-3}{p}\right) \pmod{p^2},\end{aligned}$$

which is equivalent to (15). □

3 Proof of Theorem 1.1

We begin with the transformation formula due to Chan and Zudilin [5, Corollary 4.3]:

$$\gamma_n = \sum_{i=0}^{n} \binom{2i}{i}^2\binom{4i}{2i}\binom{n+3i}{4i}(-3)^{3(n-i)}. \tag{27}$$

Using (27) and exchanging the summation order, we obtain

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{4k+1}{(-27)^k}\gamma_k &= \sum_{k=0}^{p-1}\frac{4k+1}{(-27)^k}\sum_{i=0}^{k}\binom{2i}{i}^2\binom{4i}{2i}\binom{k+3i}{4i}(-3)^{3(k-i)}\\ &= \sum_{i=0}^{p-1}\frac{1}{(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\sum_{k=i}^{p-1}(4k+1)\binom{k+3i}{4i}.\end{aligned} \tag{28}$$

Note that

$$\sum_{k=i}^{n-1}(4k+1)\binom{k+3i}{4i} = \frac{(2n-1)(n-i)}{2i+1}\binom{n+3i}{4i}, \tag{29}$$

which can be easily proved by induction on n. Combining (28) and (29) gives

$$\sum_{k=0}^{p-1}\frac{4k+1}{(-27)^k}\gamma_k=(2p-1)\sum_{i=0}^{p-1}\frac{p-i}{(2i+1)(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\binom{p+3i}{4i}. \tag{30}$$

Furthermore, we have

$$\begin{aligned}&(-1)^i(p-i)\binom{2i}{i}^2\binom{4i}{2i}\binom{p+3i}{4i}\\&=\frac{(-1)^i p(p+3i)\cdots(p+1)(p-1)\cdots(p-i)}{i!^4}\\&\equiv\frac{p(3i)!}{i!^3}\left(1+p\left(H_{3i}-H_i\right)\right)\pmod{p^3}.\end{aligned}\tag{31}$$

Observe that none of the denominators on the right-hand side of (30) contain a multiple of p except for $i=(p-1)/2$. It follows from (30) and (31) that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{4k+1}{(-27)^k}\gamma_k&\equiv(2p-1)p\sum_{i=0}^{p-1}\frac{(3i)!}{(2i+1)3^{3i}i!^3}\left(1+p\left(H_{3i}-H_i\right)\right)\\&\quad-\frac{(2p-1)(3(p-1)/2)!}{3^{3(p-1)/2}((p-1)/2)!^3}\left(1+p\left(H_{3(p-1)/2}-H_{(p-1)/2}\right)\right)\\&\quad+\frac{(p+1)(2p-1)}{2p(-3)^{3(p-1)/2}}\binom{p-1}{(p-1)/2}^2\binom{2p-2}{p-1}\binom{(5p-3)/2}{2p-2}\pmod{p^3}.\end{aligned}\tag{32}$$

We can rewrite (5) and (25) as

$$\frac{(3(p-1)/2)!}{3^{3(p-1)/2}((p-1)/2)!^3}\equiv\left(\frac{-3}{p}\right)p(3^p-6p-5)\pmod{p^3}, \tag{33}$$

and

$$\sum_{i=0}^{p-1}\frac{(3i)!}{(2i+1)3^{3i}i!^3}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}. \tag{34}$$

Note that

$$p\left(H_{3(p-1)/2}-H_{(p-1)/2}\right)=1+2p\pmod{p^2}. \tag{35}$$

Substituting (6), (12), (15), (33), (34) and (35) into the right-hand side of (32) and using the Fermat's little theorem, we arrive at

$$\sum_{k=0}^{p-1}\frac{4k+1}{(-27)^k}\gamma_k\equiv\left(\frac{-3}{p}\right)p\pmod{p^3},\text{ as desired.}$$

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References

[1] G. Almkvist, W. Zudilin, Differential equations, mirror maps and zeta values, in *Mirror symmetry, Vol. V*, AMS/IP Studies in Advanced Mathematics **38**, American Mathematical Society, Providence, RI (2006), 481–515.

[2] T. Amdeberhan, R. Tauraso, Supercongruences for the Almkvist–Zudilin numbers, *Acta Arith.* **173** (2016), 255–268.

[3] H. H. Chan, S. Cooper, F. Sica, Congruences satisfied by Apéry-like numbers, *Int. J. Number Theory* **6** (2010), 89–97.

[4] H. H. Chan, H. Verrill, The Apéry numbers, the Almkvist–Zudilin numbers and new series for $1/\pi$, *Math. Res. Lett.* **16** (2009), 405–420.

[5] H. H. Chan, W. Zudilin, New representations for Apéry-like sequences, *Mathematika* **56** (2010), 107–117.

[6] J. Guillera, W. Zudilin, "Divergent" Ramanujan-type supercongruences, *Proc. Amer. Math. Soc.* **140** (2012), 765–777.

[7] V. J. W. Guo, q-Analogues of two "divergent" Ramanujan-type supercongruences, *Ramanujan J.* **52** (2020), 605–624.

[8] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. Math.* **39** (1938), 350–360.

[9] J.-C. Liu, Semi-automated proof of supercongruences on partial sums of hypergeometric series, *J. Symbolic Comput.* **93** (2019), 221–229.

[10] J.-C. Liu, Some supercongruences arising from symbolic summation, *J. Math. Anal. Appl.* **488** (2020), 124062.

[11] J.-C. Liu, Supercongruences for sums involving Domb numbers, *Bull. Sci. Math.* **169** (2021), 102992.

[12] J.-C. Liu, Ramanujan-type supercongruences involving Almkvist–Zudilin numbers, *Results Math.* **77** (2022), 67.

[13] L. Long, Hypergeometric evaluation identities and supercongruences, *Pacific J. Math.* **249** (2011), 405–418.

[14] G.-S. Mao, Z.-W. Sun, New congruences involving products of two binomial coefficients, *Ramanujan J.* **49** (2019), 237–256.

[15] S. Ramanujan, Modular equations and approximations to π, *Quart. J. Math. Oxford Ser.* **45** (1914), 350–372.

[16] C. Schneider, Symbolic summation assists combinatorics, *Sém. Lothar. Combin.* **56** (2007), B56b.

[17] N. J. A. Sloane, The on-line encyclopedia of integer sequences, `https://oeis.org`.

[18] Z.-H. Sun, Congruences for Domb and Almkvist–Zudilin numbers, *Integral Transforms Spec. Funct.* **26** (2015), 642–659.

[19] Z.-H. Sun, Super congruences involving Bernoulli polynomials, *Int. J. Number Theory* **12** (2016), 1259–1271.

[20] Z.-H. Sun, Super congruences concerning binomial coefficients and Apéry-like numbers, preprint, arXiv:2002.12072.

[21] Z.-H. Sun, New congruences involving Apéry-like numbers, preprint, arXiv:2004.07172.

[22] Z.-W. Sun, New series for powers of π and related congruences, *Electron. Res. Arch.* **28** (2020), 1273–1342.

[23] H. Swisher, On the supercongruence conjectures of Van Hamme, *Res. Math. Sci.* **2** (2015), 18.

[24] R. Tauraso, Supercongruences for a truncated hypergeometric series, *Integers* **12** (2012), A45.

[25] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, p-adic functional analysis, Nijmegen (1996), Lecture Notes in Pure and Appl. Math. **192**, Marcel Dekker, New York (1997), 223–236.

[26] J. Wolstenholme, On certain properties of prime numbers, *Quart. J. Pure Appl. Math.* **5** (1862), 35–39.

[27] D. Zagier, Integral solutions of Apéry-like recurrence equations, in *Groups and Symmetries*, CRM Proceedings and Lecture Notes **47**, American Mathematical Society, Providence, RI (2009), 349–366.

[28] W. Zudilin, Ramanujan-type supercongruences, *J. Number Theory* **129** (2009), 1848–1857.

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