Bull. Math. Soc. Sci. Math. Roumanie
Tome 67 (115), No. 4, 2024, 449–466

On a family of numerical semigroups

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Abstract

In this paper we introduce a family of numerical semigroups whose elements, given nonnegative integers a, r, q, j_1 satisfying certain conditions, are denoted by $\mathrm{L}(a, r, q, j_1)$. For such a class of numerical semigroups we compute its main invariants and verify Wilf's conjecture. If L is a semigroup of the family, $\mathcal{A}$ is a numerical semigroup transform introduced in a previous paper by the first author, we study the numerical semigroup $\mathcal{A}(L)$ and provide a subfamily for which $\mathcal{A}$ produces a decreasing of the embedding dimension.

Key Words: Numerical semigroup, transforms, Wilf's conjecture, embedding dimension.
2020 Mathematics Subject Classification: Primary 20M14; Secondary 05C25, 11D07.

1 Introduction

A numerical semigroup is a submonoid of $\mathbb{N}$ having finite complement in it. This topic is widely studied by several authors from different points of view and perspective. The book [18] is a very good reference for an overview of basic definitions, properties and insights. It is well known that every numerical semigroup $S \subseteq \mathbb{N}$ can be generated by a finite set of its elements, in particular there exists a unique finite subset A of the semigroup, with minimal cardinality, such that every element of the semigroup can be expressed as a linear combination of elements in A with coefficients in $\mathbb{N}$. In such a case, A is called the set of the minimal generators of S and its cardinality, denoted by $\mathrm{e}(S)$, is called the *embedding dimension* of S. If $\mathcal{S}$ is the set of all numerical semigroups and $\mathcal{S}'$ a subset of it, a *numerical semigroup transform* is any function $\mathcal{F} : \mathcal{S}' \to \mathcal{S}$. Different transforms of semigroups are introduced by many authors, for instance in [1, 2, 3], or also in [6, 13] in a more general setting. Such transforms are associated to arrangements of the set of numerical semigroups in a family of graphs, that are in general rooted trees. Such arrangements allow to study some features of numerical semigroups and to produce a desired family of numerical semigroups. In this paper, inspired by [5], we introduce a class of numerical semigroups and study the behaviour of the embedding dimension on the set of numerical semigroups with respect to some particular numerical semigroup transforms. In [5] a transform, denoted by $\mathcal{A}$, is introduced and its behaviour with respect to the embedding dimension is studied, with some consequences on a well known conjecture on numerical semigroups, namely *Wilf's conjecture* (see [8, 22] or also [7, 15] for the introduction to such a conjecture in more general settings). In particular, it is shown that if S is a numerical semigroup then the inequality

$e(S) > e(\mathcal{A}(S))$ can occur only in some particular conditions that we describe. The main purpose of this work is to provide and study an infinite family of numerical semigroups that allows to find infinite numerical semigroups S satisfying the condition $e(S) > e(\mathcal{A}(S))$. The paper is organized as follows. In Section 2 we recall some basic definitions on numerical semigroups and the main results from [5] that will be used. In Section 3 we introduce the main object of this paper, that is the family of numerical semigroups whose elements, given nonnegative integers a, r, q, j_1 satisfying certain conditions, are denoted by $L(a, r, q, j_1) = L$. We study such a class computing its main invariants and proving that all semigroups of such a class satisfy Wilf's conjecture. In Section 4 we study the transformed semigroup $\mathcal{A}(L)$, where $\mathcal{A}$ is a numerical semigroup transform introduced in [5]. For $\mathcal{A}(L)$ we compute the main invariants and we prove that satisfies Wilf's conjecture. Moreover we provide the conditions on a, r, q, j_1 such that $e(L) > e(\mathcal{A}(L))$, and show that the number of the numerical semigroups L satisfying such an inequality is infinite. Section 5 contains some considerations linked to the obtained results.

2 Preliminaries and known results

Recall that a numerical semigroup S is a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is a finite set. It is well known that every numerical semigroup admits a unique finite minimal system of generators, that is, there exists a finite subset $G(S)$ of S such that every element of S is obtained as a linear combination of elements in $G(S)$ with coefficients in $\mathbb{N}$ and it is minimal in the sense that no proper subset of $G(S)$ has the same property. The elements in $G(S)$ are often called *minimal generators*. Obviously an element $s \in S$ is not a minimal generator if and only if $s = s_1 + s_2$ with $s_1, s_2 \in S \setminus \{0\}$. If a set A generates a numerical semigroup S we usually write $S = \langle A \rangle$. Moreover every (minimal or not) system of generators of a numerical semigroup is characterized by the fact that the greatest common divisor of all its elements is 1. For these and other interesting properties related to numerical semigroups a very good reference is [18]. Some invariants are related to a numerical semigroup S. We provide here a list of the most important of them that are useful for this paper:

- $H(S) = \mathbb{N} \setminus S$ is called the set of *gaps* of S.
- $g(S) = |H(S)|$ is called the *genus* of S.
- $F(S) = \max(H(S))$ if $S \neq \mathbb{N}$, conventionally $F(\mathbb{N}) = -1$. It is called the *Frobenius number* of S.
- $m(S) = \min(S \setminus \{0\})$ is called the *multiplicity* of S.
- $n(S) = |\{s \in S \mid s < F(S)\}|$, often referred as the number of *left elements* of S, if $S \neq \mathbb{N}$. Conventionally $n(\mathbb{N}) = 0$.
- $e(S) = |G(S)|$, the number of minimal generators, is called the *embedding dimension* of S.

Observe that if for some $s \in S$ we have $\{s, s+1, \ldots, s+m(S)-1\} \subset S$ then $s+n \in S$ for all $n \in \mathbb{N}$. We provide now some known results that we need for the forthcoming sections. The first one is well known.

Proposition 2.1 ([18], Exercise 2.1). *Let S be a numerical semigroup and $x \in S$. $S \setminus \{x\}$ is a numerical semigroup if and only if x is a minimal generator of S.*

A numerical semigroup is called *irreducible* if it cannot be expressed as an intersection of two numerical semigroups properly containing it. An irreducible numerical semigroup S is called *symmetric* if $\mathrm{F}(S)$ is odd, *pseudo-symmetric* if $\mathrm{F}(S)$ is even. There are several characterizations for irreducible numerical semigroups. A useful result is the following:

Proposition 2.2 ([18], Corollary 4.5). *Let S be a numerical semigroup. Then*

1. *S is symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$*
2. *S is pseudo-symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$*

Some useful subsets of $\mathrm{H}(S)$ are:

- $\mathrm{PF}(S) = \{h \in \mathrm{H}(S) \mid h + s \in S \text{ for all } s \in S\}$ that is the set of *pseudo-Frobenius* elements of S;
- $\mathrm{SG}(S) = \{h \in \mathrm{H}(S) \mid 2h \in S,\ h + s \in S \text{ for all } s \in S\} \subseteq \mathrm{PF}(S)$ that is the set of *special gaps* of S.

The number $\mathrm{t}(S) = |\,\mathrm{PF}(S)|$ is called the *type* of S. The following nice results on special gaps can be found in [18, 19].

Proposition 2.3. *Let S be a numerical semigroup and $x \in \mathrm{H}(S)$. Then*

1. *$S \cup \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{SG}(S)$.*
2. *S is irreducible if and only if $\mathrm{SG}(S) = \{\mathrm{F}(S)\}$.*

Some invariants of numerical semigroups are involved in a famous conjecture, widely studied by several authors:

Conjecture 2.4 (Wilf's conjecture [22]). *Let S be a numerical semigroup. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq \mathrm{F}(S) + 1$$

or equivalently

$$(\mathrm{e}(S) - 1)\,\mathrm{n}(S) \geq \mathrm{g}(S)$$

Wilf's conjecture is satisfied by several classes of numerical semigroups, but it has not been proved to be true for all numerical semigroups. Some results are contained for instance in [10, 11, 12, 17, 20, 21]. For a more complete and exhaustive survey about the study of Wilf's conjecture see [8]. We consider in particular the following:

Theorem 2.5 ([10]). *Let S be a numerical semigroup. S satisfies Wilf's conjecture if one of the following conditions holds:*

i) S is irreducible.

ii) $\mathrm{n}(S) \geq \frac{\mathrm{F}(S)+1}{4}$ *or equivalently* $3\,\mathrm{n}(S) \geq \mathrm{g}(S)$.

iii) $\mathrm{e}(S) \geq \mathrm{t}(S) + 1$

The family of numerical semigroups that we introduce in the next section is inspired by some arguments concerning the paper [5]. We summarize such arguments in the remainder of this section.

Recall that a numerical semigroup S is called *ordinary* if there exists $c \in \mathbb{N}$ such that $S = \{s \in \mathbb{N} \mid s \geq c\} \cup \{0\}$ and it is denoted by $S = \{0, c, \rightarrow\}$. A numerical semigroup $S \subseteq \mathbb{N}$ having only one gap greater than its multiplicity is called *almost-ordinary*. In such a case $S = \{0, g, g+1, \ldots, g+n-2, g+n, \rightarrow\}$ with $g \in \mathbb{N}$, $g > 2$ and $n \in [2, g]$. In [5] a numerical semigroup is called *special* if it is irreducible, ordinary or almost-ordinary and the following transform is introduced and studied.

Definition 2.6. Let $\mathcal{H}$ be the set of all non special numerical semigroups and $\mathcal{S}$ be the set of all numerical semigroups. We denote by $\mathcal{A}$ the transform $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{S}$ defined by $\mathcal{A}(S) = (S \cup \{h\}) \setminus \{\mathrm{m}(S)\}$, with $h = \max(\mathrm{SG}(S) \setminus \{\mathrm{F}(S)\})$.

Proposition 2.7 ([5], Proposition 4.7)**.** *Let S be a non special numerical semigroup, $h = \max(\mathrm{SG}(S) \setminus \{\mathrm{F}(S)\})$ and $T = \mathcal{A}(S)$. Then $2\,\mathrm{m}(S)$ and h are minimal generators of T.*

Lemma 2.8 ([5], Lemma 4.8)**.** *Let S be a non special numerical semigroup, $h = \max(\mathrm{SG}(S) \setminus \{\mathrm{F}(S)\})$ and $T = \mathcal{A}(S)$. Let $\{\mathrm{m}(S) = n_1 < n_2 < \ldots < n_t\}$ be the set of minimal generators of S. If $\mathrm{m}(S) + h \geq n_r$ for some $r \in \{1, \ldots, t\}$, then $n_2, n_3, \ldots, n_r$ are minimal generators of T. In particular, if $\mathrm{m}(S) + h \geq n_{t-1}$, then $\mathrm{e}(T) \geq \mathrm{e}(S)$.*

Theorem 2.9 ([5], Theorem 4.9)**.** *Let S be a non special numerical semigroup and $T = \mathcal{A}(S)$. If there exists $x \in [\mathrm{F}(S) - \mathrm{m}(S) + 1, \mathrm{F}(S)[\cap \mathrm{H}(S)$, then $\mathrm{e}(T) \geq \mathrm{e}(S)$.*

By the previous result, in order to study the connection between $\mathrm{e}(S)$ and $\mathrm{e}(\mathcal{A}(S)$, it suffices to consider the semigroups S such that $[\mathrm{F}(S) - \mathrm{m}(S) + 1, \mathrm{F}(S)[\subset S$. In such a case it may occur $\mathrm{e}(\mathcal{A}(S)) < \mathrm{e}(S)$, as the following example shows.

Example 2.10. Let S be the numerical semigroup generated by the set $[761, 768] \cup [11546, 12305]$. Computations on such a semigroup are obtained by the package `numericalsgps` [9] in the computer algebra system `GAP` [14].

```
gap> G:=Concatenation([761..768],[11546..12305]);;
gap> s:=NumericalSemigroup(G);
gap> Length(SpecialGaps(s));
648
gap> h:=SpecialGaps(s)[647];
11537
gap> t:=AddSpecialGapOfNumericalSemigroup(11537,s);
<Numerical semigroup>
gap> t:=RemoveMinimalGeneratorFromNumericalSemigroup(761,t);
<Numerical semigroup with 762 generators>
gap> EmbeddingDimension(s);
```

```
655
gap> EmbeddingDimension(t);
652
```

So, in such a case, $\max(\mathrm{SG}(S) \setminus \{\mathrm{F}(S)\}) = 11537$. If $T = \mathcal{A}(S)$, then $652 = \mathrm{e}(T) < \mathrm{e}(S) = 655$. We observe also that $\mathrm{m}(S) + h = n_{t-7}$, where $\{n_1 < n_2 < \ldots < n_t\}$ is the set of minimal generators of S.

Let S be a non ordinary numerical semigroup. The Frobenius number of the semigroup $S \cup \{\mathrm{F}(S)\}$ is called the *sub-Frobenius number of* S, and we denote it by $\mathrm{u}(S)$. In particular $\mathrm{u}(S) = \max(\mathrm{H}(S) \setminus \{\mathrm{F}(S)\})$. In [3] and [4] the following transform is defined and studied:

Definition 2.11. Let $\mathcal{Q}$ be the set of all non ordinary and non almost-ordinary numerical semigroups. $\mathcal{B} : \mathcal{Q} \to \mathcal{S}$ is the transform defined by $\mathcal{B}(S) = (S \cup \{\mathrm{u}(S)\}) \setminus \{\mathrm{m}(S)\}$.

Observe that, if S is a non ordinary and non almost-ordinary numerical semigroup, then $\mathcal{A}(S) \neq \mathcal{B}(S)$ if and only if $\mathrm{u}(S) = \mathrm{F}(S) - \mathrm{m}(S)$. Moreover $\mathrm{e}(\mathcal{B}(S)) \geq \mathrm{e}(S)$ ([5, Proposition 6.7]).

From now on, if $a, b \in \mathbb{N}$ we denote $[a, b[\; = \{x \in \mathbb{N} \mid a \leq x < b\}$ and $[a, b] = \{x \in \mathbb{N} \mid a \leq x \leq b\}$.

As described in [3, 4, 5], it is possible to associate a family of graphs to the two transforms $\mathcal{A}$ and $\mathcal{B}$ mentioned above. Let $g \in \mathbb{N}$ with $g > 3$ and $n \in [2, g]$. We denote by $S_{g,n}$ the almost-ordinary numerical semigroup having genus g and $\mathrm{n}(S_{g,n}) = n$. Furthermore, let $\mathcal{H}_{g,n}$ be the set whose elements are $S_{g,n}$ and all non special numerical semigroups S such that $\mathrm{g}(S) = g$ and $\mathrm{n}(S) = n$. If $g > 3$ and $n \in [2, g-2]$ we define the oriented graph $\mathcal{T}_{g,n} = (\mathcal{H}_{g,n}, \mathcal{E})$, where $\mathcal{E}$ is the set of all pairs $(S, \mathcal{A}(S))$. If $(S, T) \in \mathcal{E}$ we say that S is a *child* of T. A numerical semigroup without children is called a *leaf*. It is proved in [5, Theorem 5.6] that the graph $\mathcal{T}_{g,n}$ is a rooted tree, where the root is the almost-ordinary semigroup $S_{g,n}$. Similarly, considering the set $\mathcal{N}_{g,n}$ of all numerical semigroups having genus g and number of left elements n, if $g > 3$ and $n \in [2, g]$, we define the oriented graph $\mathcal{T}'_{g,n} = (\mathcal{N}_{g,n}, \mathcal{E}')$, where $\mathcal{E}'$ is the set of all pairs $(S, \mathcal{B}(S))$. It is proved in [5, Theorem 6.5] that the graph $\mathcal{T}'_{g,n}$ is a rooted tree whose root is the almost-ordinary semigroup $S_{g,n}$. Such rooted trees are related to Wilf's conjecture by the following results.

Proposition 2.12 ([5], Theorem 5.9)**.** *Let $\mathcal{F}_{g,n}$ be the set of all leaves of the tree $\mathcal{T}_{g,n}$. Suppose that:*

1. *All semigroups $S \in \mathcal{F}_{g,n}$ satisfy Wilf's conjecture.*
2. *$\mathcal{A}(S)$ satisfies Wilf's conjecture for all $S \in \mathcal{F}_{g,n}$ such that $[\mathrm{F}(S) - \mathrm{m}(S) + 1, \mathrm{F}(S)[\; \subseteq S$*

Then all semigroups in $\mathcal{H}_{g,n}$ satisfy Wilf's conjecture.

Proposition 2.13 ([5], Corollary 6.8)**.** *Let $\mathcal{F}'_{g,n}$ be the set of all leaves of the tree $\mathcal{T}'_{g,n}$ and suppose that all $S \in \mathcal{F}'_{g,n}$ satisfy Wilf's conjecture. Then all semigroups in $\mathcal{N}_{g,n}$ satisfy Wilf's conjecture.*

The previous results are a consequence of the fact that the inequality $\mathrm{e}(\mathcal{A}(S)) < \mathrm{e}(S)$ can occur only if S is a leaf of $\mathcal{T}_{g,n}$. For any $S \in \mathcal{Q}$ we have $\mathrm{e}(\mathcal{B}(S) \geq \mathrm{e}(S)$. The numerical semigroups we introduce in the next section are related to this argument, since some leaves of the trees $\mathcal{T}_{g,n}$ and $\mathcal{T}'_{g,n}$ consist of such semigroups. We will recognize this fact by the following results.

Corollary 2.14 ([5], Corollary 5.7). *A numerical semigroup $T \in \mathcal{H}_{g,n}$ is a leaf in $\mathcal{T}_{g,n}$ if and only if for all $h \in \mathrm{SG}(T)$ it is verified $h > \mathrm{m}(T)$ or for every minimal generator y of $T \cup \{h\}$, with $y \neq h$ and $y < \mathrm{F}(T)$, one of the following holds:*

- *If $T \cup \{h\}$ is irreducible, then $y < \frac{\mathrm{F}(T)}{2}$.*
- *If $T \cup \{h\}$ is not irreducible, then $y < \max(\mathrm{SG}(T \cup \{h\}) \setminus \{\mathrm{F}(T)\})$.*

Corollary 2.15 ([5], Corollary 6.6). *A numerical semigroup $T \in \mathcal{N}_{g,n}$ is a leaf in $\mathcal{T}'_{g,n}$ if and only if T does not have minimal generators in the interval $[\mathrm{u}(S), \mathrm{F}(S)]$ or, for all minimal generators $y \in [\mathrm{u}(S), \mathrm{F}(S)]$ and for all $h \in \mathrm{SG}(T \setminus \{y\})$, we have $h > \mathrm{m}(T)$.*

3 The family $\mathcal{L}$ of numerical semigroups $\mathrm{L}(a, r, q, j_1)$

In this section we introduce a family of numerical semigroups that allows to obtain an infinite number of numerical semigroups L such that $\mathrm{e}(L) > \mathrm{e}(\mathcal{A}(L))$, giving a pattern, inspired by Example 2.10, to build such a numerical semigroup L. Moreover we study Wilf's conjecture for such a family of numerical semigroups. We start recalling the following results about numerical semigroups generated by intervals:

Proposition 3.1 ([16]). *Let $a, r \in \mathbb{N} \setminus \{0\}$, with $r < a$, and let $\mathrm{S}(a, r)$ be the numerical semigroup generated by $\{a, a+1, a+2, \ldots, a+r\}$. Then*

$$\mathrm{S}(a, r) = \left(\bigcup_{k=0}^{p-1} [ka, k(a+r)] \right) \cup [pa, +\infty[$$

with $p = \lceil \frac{a-1}{r} \rceil$. In particular $\mathrm{F}(\mathrm{S}(a, r)) = pa - 1$.

Definition 3.2. Let a, r, q, j_1 be non negative integers satisfying the following conditions:

- $a > 2$, $r > 0$ and $q > 2$.
- $qr < a - 2$.
- $j_1 \leq a - 2 - qr$

Let $j_2 \in \mathbb{N}$ be such that $j_1 + j_2 = a - 2 - qr$. We define $\mathrm{L}(a, r, q, j_1)$ as the semigroup generated by the set:

$$\{a, a+1, \ldots, a+r\} \;\cup$$
$$\{qa - j_1, qa - j_1 + 1, \ldots, qa, qa+1, \ldots, q(a+r), q(a+r)+1, \ldots, q(a+r)+j_2\}$$

and let $\mathcal{L}$ be the set of the numerical semigroups $\mathrm{L}(a, r, q, j_1)$.

For instance, the semigroup of Example 2.10 is $\mathrm{L}(761,7,16,630)$.

Remark 3.3. Observe that $\mathrm{S}(a,r) \subseteq \mathrm{L}(a,r,q,j_1)$, since $\{a,a+1,\ldots,a+r\} \subseteq \mathrm{L}(a,r,q,j_1)$. Moreover, under the above hypotheses, $q < \frac{a-2}{r} < \lceil \frac{a-1}{r} \rceil$, in particular all elements in $]q(a+r),(q+1)a[$ do not belong to $\mathrm{S}(a,r)$ and, since $(q+1)a > q(a+r)+j_2+1$, $q(a+r)+j_2+1 \notin \mathrm{S}(a,r)$.

Proposition 3.4. *Under the assumptions on a,r,q,j_1 of the previous definition, we have:*

$$\mathrm{L}(a,r,q,j_1) = \left(\bigcup_{k=0}^{q-1}[ka,k(a+r)]\right) \cup [qa-j_1, q(a+r)+j_2] \cup [q(a+r)+j_2+2,+\infty[$$

Proof. Let $L = \mathrm{L}(a,r,q,j_1)$. By definition, and the fact that $\mathrm{S}(a,r) \subseteq L$, it is clear that $[qa-j_1, q(a+r)+j_2] \subseteq L$ and $\left(\bigcup_{k=0}^{q-1}[ka,k(a+r)]\right) \subseteq L$. We prove that $[q(a+r)+j_2+2,+\infty[\subseteq L$. First, observe that $(q(a+r)+j_2)-(qa-j_1) = a-2$, and $[q(a+r)+j_2+2, q(a+r)+j_2+a] = [qa-j_1, q(a+r)+j_2]+a \subseteq L$. Moreover also $q(a+r)+j_2+a+1 \in L$. It means that $[q(a+r)+j_2+2, q(a+r)+j_2+a+1] \subseteq L$ and such an interval has size $a-1 = \mathrm{m}(L)-1$, so every element greater than $q(a+r)+j_2+2$ belongs to L. In order to prove that L is contained in the union of those sets it suffices to consider the sums $x+y$ where $x \in \{a,a+1,\ldots,a+r\}$ and $y \in [qa-j_1, q(a+r)+j_2]$. These sums are in $[q(a+r)+j_2+2,+\infty[$. Moreover $q(a+r)+j_2+1 \notin L$, in fact $q(a+r)+j_2+1 \notin \mathrm{S}(a,r)$ and it cannot be obtained by the generators of L. □

If $L = \mathrm{L}(a,r,q,j_1)$, from the previuos proposition it follows that $\mathrm{F}(L) = q(a+r)+j_2+1$ and $[\mathrm{F}(L)-\mathrm{m}(L)+1, \mathrm{F}(L)[\subset L$. In particular, it is easy to see that all elements greater than $\mathrm{F}(L)$ are not minimal generators and $\mathrm{F}(L)+1 > 3\,\mathrm{m}(L)$.

Proposition 3.5. *Let $L = \mathrm{L}(a,r,q,j_1)$ and let $\mathrm{G}(L)$ be the minimal set of generators of L. Then:*

1. *If $j_1 = 0$ then $\mathrm{G}(L) = \{a,a+1,\ldots,a+r, q(a+r)+1, q(a+r)+2,\ldots,q(a+r)+j_2\}$.*
2. *If $j_1 = a-2-qr$ then $\mathrm{G}(L) = \{a,a+1,\ldots,a+r, qa-j_1, qa-j_1+1,\ldots,qa-1\}$.*
3. *If $0 < j_1 < a-2-qr$ then $\mathrm{G}(L) = \{a,a+1,\ldots,a+r, qa-j_1, qa-j_1+1,\ldots,qa-1, q(a+r)+1, q(a+r)+2,\ldots,q(a+r)+j_2\}$.*

Moreover L has the following invariants:

i) $\mathrm{e}(L) = a-(q-1)r-1$.

ii) $\mathrm{F}(L) = (q+1)a-j_1-1 = q(a+r)+j_2+1$.

iii) $\mathrm{n}(L) = a + \dfrac{(q-1)(2+rq)}{2}$.

iv) $\mathrm{g}(L) = \dfrac{(q-1)(2a-2-rq)}{2} + a - j_1 = \dfrac{(q-1)(2a-2-rq)}{2} + rq + j_2 + 2$.

Proof. It is trivial that $[a, a+r] \subset \mathrm{G}(L)$.
Suppose $j_1 \neq 0$ and let $i \in \{1, \ldots, j_1\}$. We show that $qa - i$ is a minimal generator of L, proving that $qa - i \notin S(a, r) = \langle a, \ldots, a+r \rangle$. By the hypotheses we have $q < \frac{a-2}{r} < \lceil \frac{a-1}{r} \rceil$ and $(q-1)(a+r) < qa - i < qa$ (since $i \leq j_1 \leq a - 2 - qr < a + r - qr$). Hence $qa - i \notin S(a, r)$ for all $i \in \{1, \ldots, j_1\}$ and $[qa - j_1, qa - 1] \subset \mathrm{G}(L)$ in (2) and (3). In a similar way, if $j_1 \neq a - 2 - qr$ and $k \in \{1, \ldots, j_2\}$, we prove that $q(a+r) + k$ is a minimal generator of L by showing that $q(a+r)+k \notin S(a, r)$. In fact $q(a+r) < q(a+r)+k < (q+1)a$ (since $k \leq j_2 \leq a - 2 - qr < a - qr$). So $q(a+r) + k \notin S(a, r)$ for all $k \in \{1, \ldots, j_2\}$ and $[q(a+r)+1, q(a+r)+j_2] \subset \mathrm{G}(L)$ in (1) and (3). Obviously the elements in $[qa, q(a+r)] \subset \mathrm{S}(a, r)$ are not minimal generators. In particular $\mathrm{e}(L) = r + 1 + j_1 + j_2 = a - (q-1)r - 1$, that is i) is true.
From Proposition 3.4 it follows that $\mathrm{F}(L) = (q+1)a - j_1 - 1 = q(a+r) + j_2 + 1$. In particular,

$$\begin{aligned} \mathrm{n}(L) &= 1 + \textstyle\sum_{i=1}^{q-1}(ir+1) + a - 1 \\ &= a + (q-1) + r\textstyle\sum_{i=1}^{q-1} i \\ &= a + (q-1) + r\frac{q(q-1)}{2} \end{aligned}$$

Finally, $\mathrm{g}(L)$ can be computed as $\mathrm{F}(L) + 1 - \mathrm{n}(L)$. □

We want to verify Wilf's conjecture for the family $\mathcal{L}$, proving that the last condition of Theorem 2.5 is true. We need some technical lemmas to compute the type of such numerical semigroups.

Notations:
Let $L = \mathrm{L}(a, r, q, j_1)$ and $R_k = [ka, k(a+r)]$, with $0 \leq k \leq q-1$.
Let $x \in \mathbb{N}$.

- $x > R_k$ means $x > k(a+r) = \max R_k$.
- $x < R_k$ means $x < ka = \min R_k$.

So if $x \in \mathrm{H}(L)$ we have only one of the following possibilities:

1. $R_{k-1} < x < R_k$ for some k with $1 \leq k \leq q-1$
2. $R_{q-1} < x < qa - j_1$
3. $x = \mathrm{F}(L)$

In case (2), observe that it is not hard to check that the inequality $qa - j_1 > (q-1)(a+r)$ is true, since $j_1 \leq a - 2 - qr$.

Lemma 3.6. *Let $L = \mathrm{L}(a, r, q, j_1)$. Then*

$$\{x \in \mathrm{H}(L) \mid x < R_{q-2}\} \cap \mathrm{PF}(L) = \emptyset$$

.

Proof. Let $x \in \mathrm{H}(L)$ and suppose $R_{k-1} < x < R_k$ for some $1 \leq k \leq q-2$. It is easy to see that $x + a < R_{k+1}$. Now we can consider the following two cases:
1) If $x + a > k(a + r)$, then $R_k < x + a < R_{k+1}$, that is $x + a \in \mathrm{H}(L)$, so $x \notin \mathrm{PF}(L)$.
2) If $x+a \leq k(a+r)$, then $x+(a+r) < ka+rk+2r \leq ka+r(q-2)+2r \leq ka+rq < (k+1)a$, in particular $R_k < x + (a + r) < R_{k+1}$, that is $x + (a + r) \in \mathrm{H}(L)$, so $x \notin \mathrm{PF}(L)$. □

Remark 3.7. Let $L = \mathrm{L}(a, r, q, j_1)$ and let $g = \mathrm{g}(L)$ and $n = \mathrm{n}(L)$. L is a leaf of $\mathcal{T}_{g,n}$ by Lemma 3.6 and Corollary 2.14. Moreover L is a leaf also in $\mathcal{T}'_{g,n}$, by Corollary 2.15. In fact $\mathrm{u}(L) = qa - j_1 - 1$, so if y is a minimal generator of L greater than $\mathrm{u}(L)$ then $y > R_{q-1}$, in particular all $n \in \mathbb{N}$ with $R_1 < n < R_2$ are gaps of $L \setminus \{y\}$. Such a semigroup has not special gaps smaller than $\mathrm{m}(L)$ since, by the proof of Lemma 3.6, if $x < \mathrm{m}(L)$ there exists $s \in R_1$ such that $R_1 < x + s < R_2$.

Lemma 3.8. *Let $L = \mathrm{L}(a, r, q, j_1)$. Then*

$$|\{x \in \mathrm{H}(L) \mid R_{q-2} < x < R_{q-1}\} \cap \mathrm{PF}(L)| = j_1$$

.

Proof. If $j_1 = 0$ we observe that if $y \in \mathbb{N}$ and $R_{q-1} < y < qa$, then $y \in \mathrm{H}(L)$. By the same argument of the proof of Lemma 3.6, we can show that for every $x \in \mathrm{H}(L)$, with $R_{q-2} < x < R_{q-1}$, then $x \notin \mathrm{PF}(L)$.
If $j_1 > 0$ consider the element $f = (qa-j_1)-a$. It is easy to check that $f < R_{q-1}$. Moreover we have $f > R_{q-2}$. In fact, if $f \leq R_{q-2}$ then $j_1 \geq a - qr + 2r > a - 2 - qr$, contradicting the definition of j_1. If $x \in \mathrm{H}(L)$ and $R_{q-2} < x < f$, we prove that $x \notin \mathrm{PF}(L)$ in both cases:
a) If $x + a > (q - 1)(a + r)$, then $R_{q-1} < x + a < qa - j_1$, in particular $x + a \in \mathrm{H}(L)$.
b) If $x + a \leq (q - 1)(a + r)$, let $i = \min\{n \in \mathbb{N} \mid x + (a + n) > (q - 1)(a + r)\}$. Observe that $i \in \{1, \dots, r\}$, since $x + (a + r) > (q - 1)(a + r)$. Furthermore $i < f - x$, since $x + (a + f - x) = f + a = qa - j_1 > (q - 1)(a + r)$. In particular, $x + (a + i) < qa - j_1$, so we have $R_{q-1} < x + (a + i) < qa - j_1$, that is $x + (a + i) \in \mathrm{H}(L)$ with $a + i \in L$.
If $x \in \mathrm{H}(L)$ and $f \leq x < R_{q-1}$, we prove that $x \in \mathrm{PF}(L)$. In fact, for every $s \in L$ we have $x + s > qa - j_1$, so

$$x \notin \mathrm{PF}(L) \text{ if and only if } \mathrm{F}(L) - x \in L.$$

But $\mathrm{F}(L) - x \leq \mathrm{F}(L) - f < 2a - 1$ and $\mathrm{F}(L) - x > \mathrm{F}(L) - (q - 1)a = 2a - j_1 - 1 > a + 1 + qr > a + r$, that is $a + r < \mathrm{F}(L) - x < 2a - 1$ and this means $\mathrm{F}(L) - x \notin L$. So $\{x \in \mathrm{PF}(L) \mid R_{q-2} < x < R_{q-1}\} = [f, (q - 1)a[$, in particular $|\{x \in \mathrm{H}(L) \mid R_{q-2} < x < R_{q-1}\} \cap \mathrm{PF(L)}| = j_1 = |[f, (q - 1)a[|$. □

Lemma 3.9. *Let $L = \mathrm{L}(a, r, q, j_1)$. Then*

$$|\{x \in \mathrm{H}(L) \mid R_{q-1} < x < qa - j_1\} \cap \mathrm{PF}(L)| = a - 2 - qr - j_1$$

Proof. Let $x \in \mathrm{H}(L)$ with $R_{q-1} < x < qa - j_1$. Observe that $x \in \mathrm{PF}(L)$ if and only if $x + s \in L$ for every minimal generator s of L. If s is a minimal generator of L with $s \notin \{a, a+1, \dots, a+r\}$, by Proposition 3.5 it is not hard to check that $x + s > (q - 1)(a+$

$r) + qa - j_1 > \mathrm{F}(L)$. So, if $x \notin \mathrm{PF}(L)$ then $x + i = \mathrm{F}(L)$ for some $i \in \{a, a+1, \dots, a+r\}$. Moreover for every $i \in \{a, a+1, \dots, a+r\}$ we have that $R_{q-1} < \mathrm{F}(L) - i < qa - j_1$. This means that all $x \in \mathrm{H}(L)$ with $R_{q-1} < x < qa - j_1$ are in $\mathrm{PF}(L)$ except for $r+1$ elements, in particular $|\{x \in \mathrm{H}(L) \mid R_{q-1} < x < qa-j_1\} \cap \mathrm{PF}(L)| = qa-j_1-((q-1)(a+r)+1)-(r+1) = a - 2 - qr - j_1$. ∎

Theorem 3.10. *Let $L = \mathrm{L}(a, r, q, j_1)$. Then $\mathrm{t}(L) = a - qr - 1$.*

Proof. By previous lemmas we have that $\mathrm{PF}(L) = (\{x \in \mathrm{H}(L) \mid R_{q-2} < x < R_{q-1}\} \cup \{x \in \mathrm{H}(L) \mid R_{q-1} < x < qa-j_1\}) \cap \mathrm{PF}(L) \cup \{\mathrm{F}(L)\}$. So $\mathrm{t}(L) = |\,\mathrm{PF}(L)| = j_1+a-2-qr-j_1+1 = a - qr - 1$ ∎

Corollary 3.11. $\mathrm{L}(a, r, q, j_1)$ *satisfies Wilf's conjecture.*

Proof. Let $L = \mathrm{L}(a, r, q, j_1)$. By Theorem 3.10 and Proposition 3.5 we have $\mathrm{e}(L) \geq \mathrm{t}(L)+1$, so L satisfies Wilf's conjecture by Theorem 2.5 (iii). ∎

4 The numerical semigroups $\mathcal{A}(L)$ with $L = \mathrm{L}(a, r, q, j_1)$

Now we want to study $\mathcal{A}(L)$, where $L = \mathrm{L}(a, r, q, j_1) \in \mathcal{L}$. We start by computing the element $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\})$. Then we obtain the embedding dimension of $\mathcal{A}(L)$ and we study Wilf's conjecture by the same strategy used before.

Proposition 4.1. *Let $L = \mathrm{L}(a, r, q, j_1)$ and $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\})$. Then*

$$h = \begin{cases} qa - j_1 - r - 2 & \text{if } \; j_1 < a - qr - 2 \\ (q-1)a - 1 & \text{if } \; j_1 = a - qr - 2, \; q > 3 \end{cases}$$

If $j_1 = a - qr - 2$ and $q = 3$, then L is pseudo-symmetric.

Proof. Suppose $j_1 < a - qr - 2$. Let $h = qa - j_1 - r - 2$, in this case h is a gap of L since $(q-1)(a+r) < h < qa - j_1$. If x is a gap greater than h, then $x = qa - j_1 - i$ with $i \in \{1, \dots, r+1\}$, but $x + (a+i-1) = \mathrm{F}(L)$ so $x \notin \mathrm{PF}(L)$. We have that $h + a + i \in [qa - j_1, q(a+r) + j_2]$ for $i \in \{0, 1, \dots, r\}$ so $h \in \mathrm{PF}(L)$. Moreover $2h \notin L$ if and only if $2h = \mathrm{F}(L)$ and this is equivalent to $j_1 = (q-1)a - 2r - 3 > a - qr - 3$, a contradiction. So $qa - j_1 - r - 2 = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\})$.
Suppose $j_1 = a - qr - 2$, then $qa - j_1 - r - 2 = (q-1)(a+r)$. By the same arguments as before we conclude that all the gaps greater than $qa - j_1 - r - 2$ are not in $\mathrm{PF}(L)$. The integer $h = (q-1)a - 1$ is the maximum of the set $\{x \in \mathrm{H}(L) \mid x < qa - j_1 - r - 2\}$. Observe that $h + a + i \in [qa - j_1, q(a+r) + j_2]$ for all $i \in \{0, 1, \dots, r\}$ and $h + 2a > \mathrm{F}(L)$, so $h \in \mathrm{PF}(L)$. Then $h \notin \mathrm{SG}(L)$ if and only if $2h = \mathrm{F}(L)$ and this is equivalent to $qr = (q-2)a - 3$. In particular, if $q > 3$ we obtain $qr \geq a - 2$, that is a contradiction, so in this case $(q-1)a - 1 = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\})$. If $q = 3$ we have that $3r - a + 3 = 0$ and it is not hard to see that $2\,\mathrm{g}(L) = \mathrm{F}(L) + 2$, that is L is pseudo-symmetric. ∎

Observe that, if $L = \mathrm{L}(a, r, q, j_1)$ then, by the previous result, $\mathcal{A}(L) \neq \mathcal{B}(L)$, since $\mathrm{u}(L) = qa - j_1 - 1$. So we can ask under which conditions $\mathrm{e}(\mathcal{A}(L)) < \mathrm{e}(L)$.

Proposition 4.2. *Let $L = \mathrm{L}(a, r, q, j_1)$ with $j_1 = a - qr - 2$ and $q > 3$. Then $\mathrm{e}(\mathcal{A}(L)) > \mathrm{e}(L)$, in particular $\mathcal{A}(L)$ satisfies Wilf's conjecture.*

Proof. Let $t = \mathrm{e}(L)$, let $\{n_1 < n_2 < \ldots < n_t\}$ be the minimal set of generators of L and $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$. By Proposition 4.1 and Proposition 3.5, $\mathrm{m}(L) + h = qa - 1 = n_t$. By Proposition 2.7 and Lemma 2.8 we have $\mathrm{e}(\mathcal{A}(L)) \geq |\{n_2, n_3, \ldots, n_t, h, 2n_1\}| > \mathrm{e}(L)$. □

Lemma 4.3. *Let $L = \mathrm{L}(a, r, q, j_1)$. Then $2a + 1$ and $3a$ are minimal generators of $\mathcal{A}(L)$.*

Proof. Let $T = \mathcal{A}(L)$. If $2a + 1 = s + t$ with $s, t \in T \setminus \{0\}$ then $s \geq a + 1 = \mathrm{m}(T)$ and $t \geq a + 1$, that is $s + t \geq 2a + 2 > 2a + 1$, a contradiction. Now suppose that $3a = s + t$, $s, t \in T \setminus \{0\}$. In this case $s < 3a$, so the only possibilities are $s \in [a+1, a+r]$ or $s \geq 2a$, and the same occurs for t. If $s, t \in [a + 1, a + r]$, then $s + t \leq 2(a + r) < 3a$. If $s \in [a + 1, a + r]$ and $t \geq 2a$, then $s + t > 3a$ and the same occurs if both s and t are greater than $2a$. So also $3a$ is a minimal generator of T. □

Lemma 4.4. *Let $L = \mathrm{L}(a, r, q, j_1)$ and let $x \in \mathbb{N}$ such that $ka \leq x \leq k(a + r)$, for some $k \in \{2, \ldots, q\}$. If $x \notin \{2a, 2a + 1, 3a\}$, then x is not a minimal generator of $\mathcal{A}(L)$.*

Proof. Let $T = \mathcal{A}(L)$. We show that every $x \in T$, such that $ka \leq x \leq k(a + r)$, $k \in \{2, \ldots, q\}$ and $x \notin \{2a, 2a+1, 3a\}$, is a sum of nonzero elements in T. Suppose $x \in T$, with $2a < x \leq 2(a+r)$ (that is $k = 2$), then $x = 2a + i$ with $i \in \{1, 2, \ldots, 2r\}$. If $i \leq r + 1$, then $x = (a + 1) + (a + i - 1)$, if $i \geq r + 2$, then $x = (a + r) + (a + i - r)$, so we conclude for $k = 2$. Now, suppose $x \in T$ with $ka \leq x \leq k(a + r)$ and $k \geq 3$. In such a case we can write $x = ka + jr + i$, with $j \in \{0, 1, \ldots, k - 1\}$ and $i \in \{0, \ldots, r\}$. So we consider the following cases:

1. If $i = 0$:
   - if $j = k - 1$, then $x = (k - 2)(a + r) + (2a + r)$.
   - if $0 < j \leq k - 2$, then $x = (k - j)a + j(a + r)$.
   - if $j = 0$, then $x = ka$ and, if $k \neq 3$, it is easy to see that x is a linear combination of $2a$ and $3a$.
2. If $i \neq 0$:
   - if $j = k - 1$, then $x = (k - 1)(a + r) + (a + i)$.
   - if $j = k - 2$, then $x = (k - 2)(a + r) + (2a + i)$.
   - if $j \leq k - 3$, then $x = (k - j - 1)a + j(a + r) + (a + i)$

This concludes the proof. □

Lemma 4.5. *Let $L = \mathrm{L}(a, r, q, j_1)$, with $j_1 < a - 2 - qr$, and let $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$. If $x \in \mathcal{A}(L)$ such that $h + a < x < \mathrm{F}(\mathcal{A}(L))$, then x is not a minimal generator of $\mathcal{A}(L)$*

Proof. By Proposition 4.1, $h = qa - j_1 - r - 2$. Let $T = \mathcal{A}(L)$. We have $\mathrm{F}(T) = q(a+r)+j_2+1$ and $h+a = (q+1)a - j_1 - (r+2) = q(a+r)+j_2-r$. So we can suppose $x = q(a+r)+j_2-r+i$ with $i \in \{1, 2, \ldots, r\}$, in particular $x = h + (a + i)$, that is a sum of elements in T. □

Lemma 4.6. *Let $L = \mathrm{L}(a, r, q, j_1)$. Every $x \in \mathcal{A}(L)$, with $x > \mathrm{F}(\mathcal{A}(L)) + 1$, is not a minimal generator of $\mathcal{A}(L)$.*

Proof. Let $T = \mathcal{A}(L)$ and $x \in T$. We know that $\mathrm{F}(T) = \mathrm{F}(L) = q(a + r) + j_2 + 1$. Observe that $\mathrm{m}(T) = a + 1$ and, if $x > \mathrm{F}(T) + \mathrm{m}(T)$, it is easy to see that x is not a minimal generator of T. If $x = \mathrm{F}(T) + \mathrm{m}(T)$, then $x = q(a + r) + j_2 + (a + 2) \in T$. If $\mathrm{F}(L) + 1 < x < \mathrm{F}(T) + \mathrm{m}(T)$, then $x = q(a + r) + j_2 + 2 + i$ with $i \in \{1, 2, \ldots, a - 1\}$. In such a case, $x = (qa - j_1 + i - 1) + (a + 1)$, that is a sum of elements in T. □

We can now describe the set of minimal generators of $\mathcal{A}(L)$ in terms of the minimal generators of L.

Theorem 4.7. *Let $L = \mathrm{L}(a, r, q, j_1)$ with $0 < j_1 < a-2-qr$. Let $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$ and let $\mathrm{G}(L)$ and $\mathrm{G}(\mathcal{A}(L))$ be the sets of minimal generators of L and $\mathcal{A}(L)$, respectively. If $\mathrm{G}(L) = \{n_1 < n_2 < \cdots < n_t\}$ then:*

1. *If $r < j_2$, $\mathrm{G}(\mathcal{A}(L)) = \{n_2, n_3, \ldots, n_{t-r}, 2a, 3a, 2a + 1, h, (q + 1)a - j_1\}$.*
2. *If $r \geq j_2$, $\mathrm{G}(\mathcal{A}(L)) = \{n_2, n_3, \ldots, n_{t-j_2}, 2a, 3a, 2a + 1, h, (q + 1)a - j_1\}$.*

Proof. Let $T = \mathcal{A}(L)$. The elements $2a, h$, by Lemma 2.7, and $2a+1, 3a$ by Lemma 4.3, are minimal generators of T. By Proposition 4.1(3), we have $h + a = (q + 1)a - j_1 - (r + 2) = q(a + r) + j_2 - r$. We examine two cases.
1) If $r < j_2$ then, by Proposition 3.5, $h + a = n_{t-r}$. So $n_2, n_3, \ldots, n_{t-r}$ are minimal generators of T, by Lemma 2.8. Moreover $qa - j_1, qa - j_1 + 1, \ldots, qa - 1, q(a + r) + 1, \ldots, q(a + r) + j_2 - r$ belong to $\{n_{r+1}, \ldots, n_{t-r}\}$ (see Proposition 3.5).
2) If $r \geq j_2$, then $qa < h + a \leq q(a + r)$. In particular $n_{t-j_2} < h + a < n_{t-j_2+1}$ with $n_{t-j_2} = qa - 1$ and $n_{t-j_2+1} = q(a+r)+1$. So, by Lemma 2.8, $n_2, n_3, \ldots, n_{t-j_2}$ are minimal generators of T.
To conclude, in both cases, thanks to Lemmas 4.4, 4.5 and 4.6, we have to examine only the element $x = \mathrm{F}(T) + 1 = (q + 1)a - j_1$, that we prove to be a minimal generator of T. We have to check that it is not possible $x = s + t$, with $s, t \in T \setminus \{0\}$, that is equivalent to $x - s \notin T$, for every $s \in T \setminus \{0\}$ with $s < x$. We can consider the following cases:

- If $s \in [qa - j_1, \mathrm{F}(T)[$ then $x - s \leq a$, so $x - s \notin T$.
- If $s = h$, then $x - h = a + r + 2 \notin T$.
- If $s = a + i$ with $i \in \{1, \ldots, r\}$, then $x - s = qa - j_1 - i$. In particular, $qa - j_1 - i > qa - a + qr + 2 - r = (q-1)(a+r)+2 > (q-1)(a+r)$. But $(q-1)(a+r) < x - s < qa - j_1$, that is $x - s \notin T$ (in fact if $x - s = h$ then $x - h \in T$, that is a contradiction from the previous step).

- If $s \in [ka, k(a+r)]$ with $k \in \{2, \ldots, q-1\}$, then we can write $s = ka + jr + i$ with $j \in \{0, \ldots, k-1\}$ and $i \in \{0, \ldots, r\}$. So $x - s = (q-k+1)a - j_1 - jr - i$ and we have $(q-k+1)a - j_1 - jr - i = (q-k)a + a - j_1 - jr - i \geq (q-k)a + a - j_1 - (k-1)r - r = (q-k)a + a - j_1 - kr > (q-k)a + qr + 2 - kr = (q-k)a + (q-k)r + 2 > (q-k)(a+r)$. In particular, $(q-k)(a+r) < x - s < (q-k+1)a$, that is $x - s \notin T$.

□

Theorem 4.8. *Let* $L = \mathrm{L}(a, r, q, j_1)$ *with* $j_1 = 0$. *Let* $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$, *let* $\mathrm{G}(L)$ *and* $\mathrm{G}(\mathcal{A}(L))$ *be the sets of minimal generators of* L *and* $\mathcal{A}(L)$, *respectively. If* $\mathrm{G}(L) = \{n_1 < n_2 < \cdots < n_t\}$, *then:*

1. *If* $r < j_2$, $\mathrm{G}(\mathcal{A}(L)) = \{n_2, n_3, \ldots, n_{t-r}, 2a, 3a, 2a+1, h\}$.
2. *If* $r \geq j_2$, $\mathrm{G}(\mathcal{A}(L)) = \{n_2, n_3, \ldots, n_{t-j_2}, 2a, 3a, 2a+1, h\}$.

Proof. The thesis follows as in the proof of Theorem 4.7, but in this case $\mathrm{F}(\mathcal{A}(L)) + 1 = (q+1)a$, that trivially is not a minimal generator. □

Easy consequences of the previous theorems are the following.

Corollary 4.9. *Let* $L = \mathrm{L}(a, r, q, j_1)$ *with* $0 < j_1 < a - 2 - qr$. *Then:*

1. *If* $r < j_2$, $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L) - r + 4$.
2. *If* $r \geq j_2$, $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L) - j_2 + 4$.

Corollary 4.10. *Let* $L = \mathrm{L}(a, r, q, j_1)$ *with* $j_1 = 0$. *Then:*

1. *If* $r < j_2$, $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L) - r + 3$.
2. *If* $r \geq j_2$, $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L) - j_2 + 3$.

Remark 4.11. From the previous results it is possible to describe a numerical semigroup L such that $\mathrm{e}(L) > \mathrm{e}(\mathcal{A}(L))$, with the difference $\mathrm{e}(L) - \mathrm{e}(\mathcal{A}(L))$ as large as wanted. It suffices to fix r and to consider a convenient numerical semigroup $\mathrm{L}(a, r, q, j_1)$, for instance with $j_1 = 0$ and the parameters a and q in order to have $r < j_2 = a - qr - 2$, satisfying Corollary 4.10(1).

Let $L = \mathrm{L}(a, r, q, j_1)$ with $j_1 < a - qr - 2$, $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$ and $T = \mathcal{A}(L)$. Under the same notations used before, we can see that if $x \in \mathrm{H}(T)$, then we have only one of the following possibilities:

1. $x < a + 1$
2. $R_{k-1} < x < R_k$ for some k with $2 \leq k \leq q - 1$.
3. $R_{q-1} < x < qa - j_1$ with $x \neq h$.
4. $x = \mathrm{F}(T)$.

Lemma 4.12. *Let $L = \mathrm{L}(a, r, q, j_1)$, with $j_1 < a - qr - 2$ and $T = \mathcal{A}(L)$. Then:*

1. *If $j_1 \neq 0$, then $\{x \in \mathrm{H}(T) \mid x < R_{q-2}\} \cap \mathrm{PF}(T) = \{a, (q-2)a - 1\}$.*
2. *If $j_1 = 0$, then $\{x \in \mathrm{H}(T) \mid x < R_{q-2}\} \cap \mathrm{PF}(T) = \{a\}$.*

Proof. Obviously, $a \in \mathrm{PF}(T)$ since $T \cup \{a\}$ is a numerical semigroup. Let $x \in \mathrm{H}(T)$ be such that $x < R_{q-2}$ and $x \neq a$, then $(k-1)(a+r) < x < ka$ for some $k \in \{1, 2, \ldots, q-2\}$. We consider the following cases:

- Suppose that $x < ka - 1$, then $x + (a+1) < (k+1)a$. If $x + (a+1) > k(a+r)$, we conclude that $x + (a+1) \notin T$, that is $x \notin \mathrm{PF}(T)$. If $x + (a+1) \leq k(a+r)$, then $x + (a+r) = x + (a+1) + r - 1 \leq k(a+r) + r - 1 < k(a+r) + 2r - 1 \leq ka + (q-2)r + 2r - 1 = ka + qr - 1 < ka + a - 2 - 1 < (k+1)a$. In particular, $k(a+r) < x + (a+r) < (k+1)a$, that is $x + (a+r) \notin T$ and $x \notin \mathrm{PF}(T)$.
- If $x = ka - 1$ with $k \neq q-2$, then $x + 2a = (k+2)a - 1$. In particular, $(k+1)(a+r) < x + 2a < (k+2)a$, that is, $x + 2a \notin T$. So $x \notin \mathrm{PF}(T)$.
- If $x = (q-2)a - 1$, then $x + a + i = (q-1)a + i - 1 \in T$, for $i \in \{1, \ldots, r\}$. Moreover:
  - a) If $j_1 \neq 0$ then $x + 2a = qa - 1 \in T$. Notice that $j_2 < a - qr - 2$. In particular, $j_2 \leq a - qr - 3$. From this fact it is easy to obtain that if s is a minimal generator of T such that $s \geq 3a$, then $x + s \geq x + 3a = (q+1)a - 1 \geq \mathrm{F}(T) + 1$. So $x \in \mathrm{PF}(T)$.
  - b) If $j_1 = 0$, then $x + 2a = qa - 1 \notin T$, that is $x \notin \mathrm{PF}(T)$

□

Lemma 4.13. *Let $L = \mathrm{L}(a, r, q, j_1)$ with $j_1 < a - qr - 2$, $h = \max(\mathrm{SG}(L) \setminus \{\mathrm{F}(L)\}$ and $T = \mathcal{A}(L)$. Then:*

$$\{x \in \mathrm{H}(T) \mid x > R_{q-2}\} \cap \mathrm{PF}(T) = \mathrm{PF}(L) \setminus \{h\} \cup \{\mathrm{F}(L) - a\}$$

.

Proof. Let $x = \mathrm{F}(L) - a = qa - j_1 - 1$. Then $x \in \mathrm{H}(T)$ and $x > R_{q-2}$. If $s \in T$, then $s > a$. In particular, $x + s > \mathrm{F}(L) = \mathrm{F}(T)$, that is, $x + s \in T$. So $\mathrm{F}(L) - a \in \mathrm{PF}(T)$.
Let $x \in \mathrm{PF}(L) \setminus \{h\}$, then $x \notin T$ and $x > R_{q-2}$ from Lemma 3.6. We prove that $x \in \mathrm{PF}(T)$, so let $s \in T$ and consider the element $x + s$.
If $s \neq h$, then $s \in L$ and $x + s \in L$, since $x \in \mathrm{PF}(L)$. Moreover $x + s > a$, so $x + s \in T$.
If $s = h$ then, by Proposition 4.1, $x + h > (2q-3)(a+r) + j_2$. So, if $q > 3$, then $x + h > \mathrm{F}(T)$, that is, $x + h \in T$. If $q = 3$, then $x + h \geq \mathrm{F}(T)$. If $x = \mathrm{F}(T) - h = a + r + 1$ we obtain a contradiction, since $(a+r+1) + (a+r) = 2(a+r) + 1 \notin L$, that is $x \notin \mathrm{PF}(L)$. Necessarily $x + h > \mathrm{F}(T)$, that is, $x + h \in T$. Therefore we have $x + s \in T$ for any s in T and this means $x \in \mathrm{PF}(T)$.
In order to conclude the proof, it suffices to consider $x \in \{y \in \mathrm{H}(T) \mid y > R_{q-2}\} \cap \mathrm{PF}(T)$ with $x \neq \mathrm{F}(L) - a$ and prove that $x \in \mathrm{PF}(L) \setminus \{h\}$. It is obviously $x \neq h$. Suppose that $x \notin \mathrm{PF}(L)$. So there exists $s \in L \setminus \{0\}$ such that $x + s \notin L$ and we have the following possibilities:

1) $R_{q-2} < x+s < R_{q-1}$.

2) $R_{q-1} < x+s < qa-j_1$.

3) $x+s = \mathrm{F}(L) = \mathrm{F}(T)$.

1) If $s \neq a$ then $s \in T$, in particular $x+s \notin T$, that is a contradiction, since $x \in \mathrm{PF}(T)$. If $s = a$ observe that $x+a < (q-1)a-1$, otherwise we have $x+a = (q-1)a-1$, that is, $x = (q-2)a-1 < (q-2)(a+r)$. So $R_{q-2} < x+(a+1) < R_{q-1}$. In particular $x+(a+1) \notin T$, that is a contradiction, since $x \in \mathrm{PF}(T)$.
2) Suppose $s \neq a$. Since $x \in \mathrm{PF}(T)$ the only possibility is $x+s = h$. If $s \geq 2a$, then $x = h-s < (q-2)(a+r)$, that is a contradiction. So $s < 2a$, that is, $s = a+i$ with $i \in \{1,\dots,r\}$. If $i < r$, then $x+(a+i+1) = h+1 \notin T$, that is a contradiction. If $i = r$, consider that $x+(a+r-1) = h-1$ and, since $j_1 < a-qr-2$, then $h-1 \notin T$, that is a contradiction, since $x \in \mathrm{PF}(T)$. Suppose $s = a$, then $x+a+1 = h+1 \notin T$, that is a contradiction.
3) It is not possible, since $x \in \mathrm{PF}(T)$ and $x \neq \mathrm{F}(L)-a$.
We have obtained a contradiction in all possible cases, so $x \in \mathrm{PF}(L)$. □

Theorem 4.14. *Let $L = \mathrm{L}(a,r,q,j_1)$ with $j_1 < a-qr-2$. Then:*

1. *If $j_1 \neq 0$, then $\mathrm{t}(\mathcal{A}(L)) = \mathrm{t}(L)+2$*
2. *If $j_1 = 0$, then $\mathrm{t}(\mathcal{A}(L)) = \mathrm{t}(L)+1$*

Proof. It easily follows from Lemma 4.12 and Lemma 4.13. □

Theorem 4.15. *Let $L = \mathrm{L}(a,r,q,j_1)$, with $j_1 < a-qr-2$. Then $\mathrm{e}(\mathcal{A}(L)) \geq \mathrm{t}(\mathcal{A}(L))+1$, and $\mathcal{A}(L)$ satisfies Wilf's conjecture.*

Proof. If $j_1 \neq 0$, then $\mathrm{t}(\mathcal{A}(L)) = \mathrm{t}(L)+2 = a-qr+1$ and, by Corollary 4.9,

- if $r < j_2$, then $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L)-r+4 = a-qr+3$. In particular $\mathrm{e}(\mathcal{A}(L)) > \mathrm{t}(\mathcal{A}(L))+1$.
- if $r \geq j_2$, then $\mathrm{e}(\mathcal{A}(L)) = \mathrm{e}(L)-j_2+4 = a-(q-1)r-j_2+3$. In particular the inequality $\mathrm{e}(\mathcal{A}(L)) \geq \mathrm{t}(\mathcal{A}(L))+1$ is equivalent to $r \geq j_2-1$, that is true in this case.

The same argument holds for $j_1 = 0$, by Theorem 4.14 and Corollary 4.10. □

5 Final considerations

We noticed in Remark 3.7 that $L = \mathrm{L}(a,r,q,j_1)$ is a leaf in $\mathcal{T}_{g,n}$ and in $\mathcal{T}'_{g,n}$, with $g = \mathrm{g}(L)$ and $n = \mathrm{n}(L)$ and we proved that L satisfies Wilf's conjecture. Consider the semigroups $S = \mathrm{S}(a,r)$, $0 < r \leq a-1$, generated by intervals. The cases $r = a-1$ and $r = a-2$ correspond, respectively, to ordinary and almost-ordinary semigroups. It is known that S satisfies Wilf's conjecture by [20, Proposition 20]. We want to prove that $S = \mathrm{S}(a,r)$, $0 < r < a-2$, is a leaf in $\mathcal{T}_{g,n}$ and in $\mathcal{T}'_{g,n}$ and that $\mathcal{A}(S) = \mathcal{B}(S)$, if S is not irreducible.

Proposition 5.1. *Let $a, r \in \mathbb{N} \setminus \{0\}$, with $r < a-2$ and let $\mathrm{S}(a,r)$ be the numerical semigroup generated by $\{a, a+1, a+2, \ldots, a+r\}$. Let $g = \mathrm{g}(\mathrm{S}(a,r))$ and $n = \mathrm{n}(\mathrm{S}(a,r))$. Then $\mathrm{S}(a,r)$ is a leaf in $\mathcal{T}_{g,n}$ and in $\mathcal{T}'_{g,n}$.*

Proof. Let $S = \mathrm{S}(a,r)$ and consider the notation $R_k = [ka, k(a+r)]$, $k \in \{0, \ldots, q-1\}$. By the same argument in the proof of Lemma 3.6, it can be shown that $\{x \in \mathrm{H}(S) \mid x < R_{q-1}\} \cap \mathrm{PF}(S) = \emptyset$, so S is a leaf in $\mathcal{T}_{g,n}$. S is also a leaf in $\mathcal{T}'_{g,n}$ by Corollary 2.15, since the greatest minimal generator is $a+r$ and $\mathrm{u}(S) > a+r$. □

Proposition 5.2. *Let $a, r \in \mathbb{N} \setminus \{0\}$, with $1 < r < a-2$, and let $\mathrm{S}(a,r)$ be a not irreducible numerical semigroup generated by $\{a, a+1, a+2, \ldots, a+r\}$. Then $\mathcal{A}(\mathrm{S}(a,r)) = \mathcal{B}(\mathrm{S}(a,r))$.*

Proof. Let $S = \mathrm{S}(a,r)$. Since $\mathrm{F}(S) = pa - 1$, $p = \lceil \frac{a-1}{r} \rceil$, it suffices to prove that $\mathrm{F}(S) - 1 \notin S$. By [16, Corollary 2], it suffices to prove that $a - 2 > (p-1)r$, that is equivalent to $\frac{a-2}{r} > \lceil \frac{a-1}{r} \rceil - 1$. Since S is not irreducible, by [16, Theorem 6] $\frac{a-2}{r}$ is not an integer. It is easy to see that $\frac{a-1}{r} - \frac{a-2}{r} < 1$, moreover it is not possible that $\frac{a-2}{r} < n < \frac{a-1}{r}$ for some positive integer n. So we can conclude that $\frac{a-2}{r} > \lceil \frac{a-1}{r} \rceil - 1$. □

The family $\mathcal{L}$ of numerical semigroups $\mathrm{L}(a, r, q, j_1)$ permits to provide examples of numerical semigroups S such that $\mathrm{e}(\mathcal{A}(S)) < \mathrm{e}(S)$. It could be nice to find other families of numerical semigroups having this property. Furthermore it could be interesting to find necessary conditions such that $\mathrm{e}(\mathcal{A}(S)) \geq \mathrm{e}(S)$, since at the moment we know only sufficient conditions (see, for instance, Lemma 2.8 and Theorem 2.9).

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Received: 12.09.2022
Revised: 30.01.2023
Accepted: 01.02.2023

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