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Results on some conjectures about binomial coefficients

by

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Abstract

Z. W. Sun, V. J. W. Guo and C. Krattenthaler proposed several conjectures about arithmetic properties of binomial coefficients. In this paper, we aim to give new results for these conjectures or their extensions. In particular, we partially solve a conjecture of Z. W. Sun.

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1 Introduction

Let n and k be nonnegative integers. The binomial coefficient $\binom{n}{k}$ is defined by $\frac{n!}{k!(n-k)!}$ if $k \leq n$, and is 0 otherwise. Binomial coefficients is an important class of integers in mathematics. Accordingly, it has many properties and appears in all kinds of mathematical fields.

For any prime p, we use $\nu_p(n)$ to denote the largest nonnegative integer e such that p^e divides n, that is, $p^e \mid n$ and $p^{e+1} \nmid n$. Here $\nu_p(n)$ is called the p-adic valuation of n. For the calculation of the p-adic valuation of the binomial coefficient, we have the following remarkable result of Kummer.

Theorem ([6], cf. [3]) *For any integers $0 \leq k \leq n$ and any prime p:*

$$\nu_p\left(\binom{n}{k}\right) = \sharp\{\textit{carries when adding } k \textit{ to } n-k \textit{ in base } p\}.$$

Arithmetic properties of binomial coefficients are studied extensively in literature and we refer the interested reader to consult articles [3], [2], [7]. Closely related to our object in this article, in 2014, Guo and Krattenthaler [4] proved that Conjecture 1.2 of Sun [8] is correct. Their result states that if a, b, n are positive integers and $(bn+1) \mid \binom{an+bn}{an}$ for all sufficiently large positive integers n, then each prime factor of a divides b. In other words, if a has a prime factor not dividing b, then there are infinitely many positive integers n such that $(bn+1) \nmid \binom{an+bn}{an}$.

Guo and Krattenthaler [4] proposed the following conjecture:

Conjecture 1.1. ([4, Conjecture 7.2]) *For any odd prime p, there are no positive integers $a > b$ such that*

$$\binom{an}{bn} \equiv 0 \pmod{pn-1}$$

for all $n \geq 1$.

Yaqubi and Mirzavaziri [10] proved Conjecture 1.1 under the additional hypothesis $ab \not\equiv 0 \pmod{p}$ and they also provided a partial proof if p divides a. In this article we prove the following general result.

Theorem 1.2. *Let p be a prime, and let $a, b, \gamma > 0, \beta \geq 0$ be integers. Suppose $a > b$ and $a\gamma \neq p^u - p$ for any positive integer $u \geq 2$. Then there exist infinitely many positive integers n for which*

$$\binom{an}{bn+\beta} \not\equiv 0 \pmod{pn-\gamma}.$$

When $\gamma = 1$, we obtain the following consequence that improves a theorem of Yaqubi and Mirzavaziri [10, Theorem 2.1].

Corollary 1.3. *Let p be a prime, and let $a, b, \beta \geq 0$ be integers. Suppose $a > b$ and $a \neq p^u - p$ for any positive integer $u \geq 2$. Then there exist infinitely many positive integers n for which*

$$\binom{an}{bn+\beta} \not\equiv 0 \pmod{pn-1}.$$

Using some properties of the p-adic valuation, Yaqubi and Mirzavaziri [10] confirmed another conjecture of Guo and Krattenthaler[4]: For any positive integer m, there are positive integers a and b such that $am > b$ and

$$\binom{amn}{bn} \equiv 0 \pmod{an-1}$$

for all $n \geq 1$ ([4, Conjecture 7.3]). Regarding this conjecture, we will prove in Section 4 the following result.

Theorem 1.4. *For any positive integer m, there exist positive integers a such that*

$$\binom{4amn}{an} \equiv 0 \pmod{(4an-1)(2an-1)}$$

for all $n \geq 1$. Furthermore, there exist positive integers a such that

$$\binom{12amn}{an} \equiv 0 \pmod{(12an-1)(6an-1)(4an-1)(3an-1)(2an-1)}$$

for all $n \geq 1$.

Guo and Krattenthaler [4] also proposed the following conjecture.

Conjecture 1.5. ([4, Conjecture 7.1]) *Let α, β, a, b be integers and let p be a prime number. Suppose $0 < b < a$ and p does not divide a. Then, for each $r = 0, 1, \ldots, p-1$, there are infinitely many positive integers n such that*

$$\binom{an+\alpha}{bn+\beta} \equiv r \pmod{p}.$$

For Conjecture 1.5, Vsemirnov [9] showed that $\binom{4n+\alpha}{2n+\beta} \equiv 0, 1, 4 \pmod 5$ if $(\alpha, \beta) \in \{(0,0), (1,0), (1,1)\}$ and $\binom{4n+\alpha}{2n+\beta} \equiv 0, 2, 3 \pmod 5$ when $(\alpha, \beta) \in \{(2,1), (3,1), (3,2)\}$, which can be used as counterexamples.

Much to our surprise, even knowing Conjecture 1.5 is false, proving the fact that there exists a positive integer n such that $\binom{an+\alpha}{bn+\beta} \not\equiv 0 \pmod p$ is not easy, and that is still open now. We will give a partial result below.

Theorem 1.6. *Let α, β, a, b be integers, and let p be a prime. Suppose $p > a > b > 0$. Then there exists a positive integer n such that*

$$\binom{an+\alpha}{bn+\beta} \not\equiv 0 \pmod p.$$

In 2012, Sun [8] proposed the following conjecture.

Conjecture 1.7. ([8, Conjecture 1.10]) *Let k and l be integers greater than one. If $\binom{kn}{n} \mid \binom{ln}{n}\binom{kln}{ln-1}$ for all $n \in \mathbb{N}$, then $k = l$ or $l = 2$ or $\{k, l\} = \{3, 5\}$. If $\binom{kn}{n} \mid \binom{ln}{n-1}\binom{kln}{ln}$ for all $n \in \mathbb{N}$, then $k = 2$ and $l + 1$ is a power of 2.*

In this paper, we confirm Conjecture 1.7 partially by proving the following two theorems.

Theorem 1.8. *Let k and l be integers greater than one. If $\binom{kn}{n} \mid \binom{ln}{n}\binom{kln}{ln-1}$ for all $n \in \mathbb{N}$, then $k = l$ or $l = 2$ or $\{k, l\} = \{3, 5\}$.*

Theorem 1.9. *Let k and l be integers greater than one with $k \neq l$. If $\binom{kn}{n} \mid \binom{ln}{n-1}\binom{kln}{ln}$ for all $n \in \mathbb{N}$, then $k = 2$ and $l + 1$ is a power of 2.*

Remark: To completely solve Conjecture 1.7, we need to show that for any positive integer $k \geq 3$, there exists a positive integer n such that

$$((k-1)n+1) \nmid \binom{k^2 n}{kn}.$$

However, we cannot do this now.

This paper is structured as follows. In Section 2, we state and prove several preliminary results. Following that, in Section 3, we focus on introducing a special set that consists of residue classes of binomial coefficients modulo a prime number p. In Section 4, we present the proofs of Theorems 1.2 and 1.4, along with additional results related to Theorem 1.2. Finally, in the last section, we provide the proofs for Theorems 1.8 and 1.9. Throughout this paper, for a real number x, we let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.

2 Preliminary works

In 1878, Lucas [5] established an important result about the congruence of binomial coefficients that we recall below.

Lemma 2.1. (Lucas [5]) *Let n, m be nonnegative integers and p be any prime. Then*

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\dots\binom{n_k}{m_k} \pmod{p},$$

where $n = n_0 + n_1 p + \cdots + n_k p^k$ and $m = m_0 + m_1 p + \cdots + m_k p^k$, $0 \le m_i, n_i < p$ for $i = 0, 1, \dots, k$ are the p-adic expansions of n and m, respectively.

As an immediately consequence of Lucas' result, we have the following lemma.

Lemma 2.2. *Let n and m be nonnegative integers, k and t_i be positive integers, where $1 \le i \le k$. Rewrite $n = n_0 + n_1 p^{t_1} + \cdots + n_k p^{t_1+\cdots+t_k}$ and $m = m_0 + m_1 p^{t_1} + \cdots + m_k p^{t_1+\cdots+t_k}$, where $0 \le m_k, n_k$ and $0 \le m_i, n_i < p^{t_{i+1}}$ for $0 \le i \le k-1$. Then we have*

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\dots\binom{n_k}{m_k} \pmod{p}.$$

For a rational number $x = m/n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we set $\nu_p(x) = \nu_p(m) - \nu_p(n)$ for any prime p. The following lemma follows from the definitions of $\nu_p(x)$ and $\lfloor x \rfloor$, and we will use it in this paper.

Lemma 2.3. (1) *A rational number x is an integer if and only if $\nu_p(x) \ge 0$ for all prime numbers p.*

(2) *Let n be a positive integer. Then, for any integer $m > 1$, we have*

$$\left\lfloor \frac{n+1}{m} \right\rfloor - \left\lfloor \frac{n}{m} \right\rfloor = \begin{cases} 1, & \text{if } m \mid (n+1), \\ 0, & \text{otherwise.} \end{cases}$$

We also need the following two results of Bober [1] and a result of Sun [8].

Lemma 2.4. (Bober [1, Lemma 3.2]) *Let $a_1, \dots, a_K$, $b_1, \dots, b_L$ be nonnegative integers and let*

$$u_n = \frac{(a_1 n)! \dots (a_K n)!}{(b_1 n)! \dots (b_L n)!}.$$

Then u_n is an integer for all n if and only if the function

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

is nonnegative for all x between 0 and 1.

Lemma 2.5. (Bober [1, Table 2]) *Let n be a positive integer. Then*

$$\frac{(15n)!(2n)!}{(10n)!(4n)!(3n)!} \in \mathbb{Z} \text{ and } \frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!} \in \mathbb{Z}.$$

Lemma 2.6. (Sun [8, Theorem 1.6]) *Let n be a positive integer. Then $(10n+1)\binom{3n}{n} \mid \binom{15n}{5n}\binom{5n-1}{n-1}$.*

The following result is a direct consequence of Bober's result in [1, Theorem 1.4].

Lemma 2.7. *Let k, l be positive integers greater than one. Then*

$$\frac{\binom{ln}{n}\binom{kln}{ln}}{\binom{kn}{n}} = \frac{(kln)!((k-1)n)!}{(kn)!((l-1)n)!((k-1)ln)!} \in \mathbb{N}$$

if and only if $k = l$ or $k = 2$ or $l = 2$ or $\{k, l\} = \{3, 5\}$.

To prove our main results, we also need the following technical lemmas on the floor function.

Lemma 2.8. *Let $n > 0$ and $m > 1$ be integers. Suppose $m \neq 7, 11$ and $m|(4n+1)$. Then*

$$\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + 1. \tag{1}$$

If $m = 7$ or 11, then $\left\lfloor \frac{15n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor = \left\lfloor \frac{10n}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor$.

Proof. For $m \neq 7, 11$, it is sufficient to prove that

$$\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1. \tag{2}$$

Since $m \mid (4n+1)$, there is a positive integer t such that $4n + 1 = mt$, where $t \equiv 1, 3 \pmod 4$. We have

$$\left\{ \frac{4n}{m} \right\} = \left\{ \frac{mt-1}{m} \right\} = \frac{m-1}{m}, \quad \left\{ \frac{2n}{m} \right\} = \left\{ \frac{4n}{2m} \right\} = \left\{ \frac{mt-1}{2m} \right\} = \frac{m-1}{2m},$$
$$\left\{ \frac{10n}{m} \right\} = \left\{ \frac{20n}{2m} \right\} = \left\{ \frac{5mt-5}{2m} \right\} = \frac{m-5}{2m} \ (\text{if}\, m > 3),$$
$$\left\{ \frac{3n}{m} \right\} = \left\{ \frac{12n+3-3}{4m} \right\} = \left\{ \frac{3mt-3}{4m} \right\} = \begin{cases} \frac{3m-3}{4m}, & \text{if } t \equiv 1 \pmod 4, \\ \frac{m-3}{4m}, & \text{if } t \equiv 3 \pmod 4, \end{cases}$$
$$\left\{ \frac{15n}{m} \right\} = \left\{ \frac{60n+15-15}{4m} \right\} = \left\{ \frac{15mt-15}{4m} \right\} = \begin{cases} \frac{3m-15}{4m}, & \text{if } t \equiv 1 \pmod 4 \text{ and } m > 3, \\ \frac{m-15}{4m}, & \text{if } t \equiv 3 \pmod 4,\ m > 11. \end{cases}$$

Therefore, we get

$$\left\{ \frac{15n}{m} \right\} + \left\{ \frac{2n}{m} \right\} = \left\{ \frac{10n}{m} \right\} + \left\{ \frac{4n}{m} \right\} + \left\{ \frac{3n}{m} \right\} - 1.$$

For $m = 3$, then $n \equiv 2 \pmod 3$ since $3|(4n+1)$. Hence $\left\{ \frac{15n}{3} \right\} + \left\{ \frac{2n}{3} \right\} = 0 + \frac{1}{3} = \frac{1}{3}$ and

$$\left\{ \frac{10n}{3} \right\} + \left\{ \frac{4n}{3} \right\} + \left\{ \frac{3n}{3} \right\} - 1 = \frac{2}{3} + \frac{2}{3} + 0 - 1 = \frac{1}{3}.$$

Hence the identity (2) holds for $m = 3$.

If $m = 7$, then $n \equiv 5 \pmod 7$ since $7|(4n+1)$. Hence $\left\{ \frac{15n}{7} \right\} + \left\{ \frac{2n}{7} \right\} = \frac{5}{7} + \frac{3}{7} = \frac{8}{7}$ and

$$\left\{ \frac{10n}{7} \right\} + \left\{ \frac{4n}{7} \right\} + \left\{ \frac{3n}{7} \right\} = \frac{1}{7} + \frac{6}{7} + \frac{1}{7} = \frac{8}{7}.$$

Consequently, we get $\left\{\frac{15n}{7}\right\}+\left\{\frac{2n}{7}\right\}=\left\{\frac{10n}{7}\right\}+\left\{\frac{4n}{7}\right\}+\left\{\frac{3n}{7}\right\}$.

If $m=11$, then $n\equiv 8 \pmod{11}$ since $11|(4n+1)$. Hence $\left\{\frac{15n}{11}\right\}+\left\{\frac{2n}{11}\right\}=\frac{10}{11}+\frac{5}{11}=\frac{15}{11}$ and

$$\left\{\frac{10n}{11}\right\}+\left\{\frac{4n}{11}\right\}+\left\{\frac{3n}{11}\right\}=\frac{3}{11}+\frac{10}{11}+\frac{2}{11}=\frac{15}{11}.$$

Therefore, we obtain $\left\{\frac{15n}{11}\right\}+\left\{\frac{2n}{11}\right\}=\left\{\frac{10n}{11}\right\}+\left\{\frac{4n}{11}\right\}+\left\{\frac{3n}{11}\right\}$. This proves the lemma. ∎

Proposition 2.9. *Let n be a positive integer. Then*

$$(4n+1)\mid 7\cdot 11\cdot\frac{(15n)!(2n)!}{(10n)!(4n)!(3n)!}. \tag{3}$$

Proof. Let

$$A_n:=\frac{7\cdot 11\cdot(15n)!(2n)!}{(4n+1)(10n)!(4n)!(3n)!}.$$

Then, for any prime p, we have

$$\nu_p(A_n)=\nu_p(7)+\nu_p(11)+\sum_{i=1}^{\infty}\left(\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n+1}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor\right).$$

By Lemmas 2.4 and 2.5, for any $x\in[0,1]$, we have

$$\lfloor 15x\rfloor+\lfloor 2x\rfloor-\lfloor 10x\rfloor-\lfloor 4x\rfloor-\lfloor 3x\rfloor\geq 0. \tag{4}$$

For a prime p and a positive integer i, if $p^i\not|(4n+1)$, then by Lemma 2.3 (2) and by applying (4) to $x=n/p^t$, we have

$$\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n+1}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor=\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor\geq 0;$$

If $p^i|(4n+1)$ and $p^i\neq 7,11$, then by Lemma 2.8, we have

$$\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n+1}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor=\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor-1=0;$$

If $p^i|(4n+1)$ and $p^i=7$ or 11, then by Lemma 2.8 again, we have

$$\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n+1}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor=\left\lfloor\frac{15n}{p^i}\right\rfloor+\left\lfloor\frac{2n}{p^i}\right\rfloor-\left\lfloor\frac{10n}{p^i}\right\rfloor-\left\lfloor\frac{4n}{p^i}\right\rfloor-\left\lfloor\frac{3n}{p^i}\right\rfloor-1=-1.$$

Therefore $\nu_p(A_n)\geq 0$ for any prime p. This completes the proof. ∎

Lemma 2.10. *Let $n>0$ and $m>1$ be integers. Suppose that $m\neq 7,9,11,13$ and $m\mid(2n+1)$. Then*

$$\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor=\left\lfloor\frac{12n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+1. \tag{5}$$

If $m=7$ or 9 or 11 or 13, then $\left\lfloor\frac{15n}{m}\right\rfloor+\left\lfloor\frac{4n}{m}\right\rfloor=\left\lfloor\frac{12n}{m}\right\rfloor+\left\lfloor\frac{5n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor$.

Proof. For $m \neq 7, 9, 11, 13$, it suffices to prove that

$$\left\{\frac{15n}{m}\right\} + \left\{\frac{4n}{m}\right\} = \left\{\frac{12n}{m}\right\} + \left\{\frac{5n}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1. \tag{6}$$

Since $m \mid (2n+1)$, $2n+1 = mt$, where $t \equiv 1 \pmod 2$. Suppose that $m = 3, 5, 7, 9, 11, 13$, since $m|(2n+1)$, we get $n \equiv 1 \pmod 3, 2 \pmod 5, 3 \pmod 7, 4 \pmod 9, 5 \pmod{11}, 6 \pmod{13}$, respectively. Hence

$$\left(\left\{\frac{15n}{m}\right\}, \left\{\frac{4n}{m}\right\}, \left\{\frac{12n}{m}\right\}, \left\{\frac{5n}{m}\right\}, \left\{\frac{2n}{m}\right\}\right) = \begin{cases} \left(0, \frac{1}{3}, 0, \frac{2}{3}, \frac{2}{3}\right), & \text{if } m = 3, \\ \left(0, \frac{3}{5}, \frac{4}{5}, 0, \frac{4}{5}\right), & \text{if } m = 5, \\ \left(\frac{3}{7}, \frac{5}{7}, \frac{1}{7}, \frac{1}{7}, \frac{6}{7}\right), & \text{if } m = 7, \\ \left(\frac{2}{3}, \frac{7}{9}, \frac{1}{3}, \frac{2}{9}, \frac{8}{9}\right), & \text{if } m = 9, \\ \left(\frac{9}{11}, \frac{9}{11}, \frac{5}{11}, \frac{3}{11}, \frac{10}{11}\right), & \text{if } m = 11, \\ \left(\frac{12}{13}, \frac{11}{13}, \frac{7}{13}, \frac{4}{13}, \frac{12}{13}\right), & \text{if } m = 13. \end{cases}$$

If $m \geq 15$, then we have

$$\begin{aligned} &\left\{\frac{4n}{m}\right\} = \left\{\frac{2mt-2}{m}\right\} = \frac{m-2}{m}, \quad \left\{\frac{2n}{m}\right\} = \left\{\frac{mt-1}{m}\right\} = \frac{m-1}{m}, \\ &\left\{\frac{12n}{m}\right\} = \left\{\frac{6mt-6}{m}\right\} = \frac{m-6}{m}, \quad \left\{\frac{15n}{m}\right\} = \left\{\frac{15mt-15}{2m}\right\} = \frac{m-15}{2m}, \\ &\left\{\frac{5n}{m}\right\} = \left\{\frac{5mt-5}{2m}\right\} = \frac{m-5}{2m}. \end{aligned}$$

Therefore, for $m \neq 7, 9, 11, 13$, we have

$$\left\lfloor\frac{15n}{m}\right\rfloor + \left\lfloor\frac{4n}{m}\right\rfloor = \left\lfloor\frac{12n}{m}\right\rfloor + \left\lfloor\frac{5n}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor + 1.$$

If $m = 7$ or 9 or 11 or 13, then $\left\lfloor\frac{15n}{m}\right\rfloor + \left\lfloor\frac{4n}{m}\right\rfloor = \left\lfloor\frac{12n}{m}\right\rfloor + \left\lfloor\frac{5n}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor$. This completes the proof. □

Lemma 2.11. *Let n and m be positive integers with $m > 1$ such that $m \mid (12n+1)$. Then*

$$\left\lfloor\frac{15n}{m}\right\rfloor + \left\lfloor\frac{4n}{m}\right\rfloor = \left\lfloor\frac{12n}{m}\right\rfloor + \left\lfloor\frac{5n}{m}\right\rfloor + \left\lfloor\frac{2n}{m}\right\rfloor + 1. \tag{7}$$

Proof. It is sufficient to prove that

$$\left\{\frac{15n}{m}\right\} + \left\{\frac{4n}{m}\right\} = \left\{\frac{12n}{m}\right\} + \left\{\frac{5n}{m}\right\} + \left\{\frac{2n}{m}\right\} - 1. \tag{8}$$

Now $m \mid (12n+1)$ implies that there exists a positive integer t such that $12n+1 = mt$, where $t \equiv 1, 5, 7, 11 \pmod{12}$. By calculation, we obtain

$$\left\{\frac{12n}{m}\right\} = \frac{m-1}{m},$$
$$\left\{\frac{15n}{m}\right\} = \left\{\frac{5mt-5}{4m}\right\} = \begin{cases} \frac{m-5}{4m}, & \text{if } t \equiv 1, 5 \pmod{12}, \\ \frac{3m-5}{4m}, & \text{if } t \equiv 7, 11 \pmod{12}, \end{cases}$$
$$\left\{\frac{4n}{m}\right\} = \left\{\frac{mt-1}{3m}\right\} = \begin{cases} \frac{m-1}{3m}, & \text{if } t \equiv 1, 7 \pmod{12}, \\ \frac{2m-1}{3m}, & \text{if } t \equiv 5, 11 \pmod{12}, \end{cases}$$
$$\left\{\frac{2n}{m}\right\} = \left\{\frac{mt-1}{6m}\right\} = \begin{cases} \frac{m-1}{6m}, & \text{if } t \equiv 1, 7 \pmod{12}, \\ \frac{5m-1}{6m}, & \text{if } t \equiv 5, 11 \pmod{12}. \end{cases}$$
$$\left\{\frac{5n}{m}\right\} = \left\{\frac{5mt-5}{12m}\right\} = \begin{cases} \frac{5m-5}{12m}, & \text{if } t \equiv 1 \pmod{12}, \\ \frac{m-5}{12m}, & \text{if } t \equiv 5 \pmod{12}, \\ \frac{11m-5}{12m}, & \text{if } t \equiv 7 \pmod{12}, \\ \frac{7m-5}{12m}, & \text{if } t \equiv 11 \pmod{12}. \end{cases}$$

Therefore, the equality (8) holds for all integers $n > 0$ and $m > 1$ with $m \mid (12n+1)$. Hence the lemma is proved. □

Proposition 2.12. *Let n be a positive integer. Then*

$$(2n+1) \mid 7 \cdot 9 \cdot 11 \cdot 13 \cdot \frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!}. \tag{9}$$

Proof. Let

$$B_n := \frac{7 \cdot 9 \cdot 11 \cdot 13 \cdot (15n)!(4n)!}{(2n+1)(12n)!(5n)!(2n)!}.$$

For any prime p, the p-adic valuation of B_n is given by

$$\begin{aligned} \nu_p(B_n) &= \nu_p(7) + \nu_p(9) + \nu_p(11) + \nu_p(13) \\ &\quad + \sum_{i=1}^{\infty} \left(\left\lfloor \frac{15n}{p^i} \right\rfloor + \left\lfloor \frac{4n}{p^i} \right\rfloor - \left\lfloor \frac{12n}{p^i} \right\rfloor - \left\lfloor \frac{5n}{p^i} \right\rfloor - \left\lfloor \frac{2n+1}{p^i} \right\rfloor \right). \end{aligned}$$

Using Lemma 2.10, by similar arguments to the proof of Proposition 2.9, we get $\nu_p(B_n) \geq 0$ for any odd prime p. This completes the proof. □

Similarly, by Lemma 2.11, we also have the following result.

Proposition 2.13. *Let n be a positive integer. Then*

$$(12n+1) \mid \frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!}. \tag{10}$$

3 A special subset of the residue class group $\mathbb{Z}_p$

In this section, we will consider a special subset of the residue class ring $\mathbb{Z}_p$. Throughout this section, we let $a > b$ be positive integers, α, β be any integers and p be prime satisfying $\gcd(p, a) = 1$. Let

$$A(a, b; \alpha, \beta; p) := \left\{ \binom{an+\alpha}{bn+\beta} \pmod p,\ n \in \mathbb{N} \right\},$$

and write $A(a, b; \alpha, \beta)$ for short when the prime p does not need to be mentioned, and $A(a, b)$ for short when $(\alpha, \beta) = (0, 0)$ in addition.

Let's begin with the following useful lemma.

Lemma 3.1. *(i) For any $x, y \in A(a, b)$, we have $xy \in A(a, b)$.*

(ii) For any $x \in A(a, b), y \in A(a, b; \alpha, \beta)$, we have $xy \in A(a, b; \alpha, \beta)$.

Proof. Since $x, y \in A(a, b)$, there are positive integers $n_1, n_2 \in \mathbb{N}$ such that

$$\binom{an_1}{bn_1} \equiv x \pmod p \ \text{ and } \ \binom{an_2}{bn_2} \equiv y \pmod p.$$

Let m be a positive integer with $p^m > an_2$. By Lemma 2.2, we have

$$\binom{a(n_1p^m + n_2)}{b(n_1p^m + n_2)} = \binom{an_1p^m + an_2}{bn_1p^m + bn_2} \equiv \binom{an_1}{bn_1}\binom{an_2}{bn_2} \equiv xy \pmod p.$$

This proves (i).

As for (ii), by the same argument as in (i), we choose m such that $p^m > \max\{an_2 + \alpha, bn_2 + \beta\}$. Then

$$\binom{a(n_1p^m + n_2) + \alpha}{b(n_1p^m + n_2) + \beta} = \binom{an_1p^m + (an_2 + \alpha)}{bn_1p^m + (bn_2 + \beta)} \equiv \binom{an_1}{bn_1}\binom{an_2 + \alpha}{bn_2 + \beta} \equiv xy \pmod p.$$

This completes the proof of (ii). □

Here is an example of using this useful lemma to prove that there exist infinitely many positive integers n such that

$$\binom{4n+21}{2n} \equiv r \pmod 5$$

for each $r = 0, 1, 2, 3, 4$. By the result of Vsemirnov [9], we have that $\binom{4n_1}{2n_1} \equiv 0, 1, -1 \pmod 5$ holds for any positive integer n_1. Note that when $n_2 = 2, n_3 = 68$, Lemma 2.1 gives

$$\binom{4n_2+21}{2n_2} \equiv \binom{4}{4}\binom{0}{0}\binom{1}{0} \equiv 1 \pmod 5 \text{ and } \binom{4n_3+21}{2n_3} \equiv \binom{3}{1}\binom{3}{2}\binom{1}{0}\binom{2}{1} \equiv 3 \pmod 5,$$

respectively. Hence, $\{0, 1, 2, 3, 4\} \subset A(4, 2; 21, 0; 5)$ by Lemma 3.1. That is, the conclusion is established.

Remark 3.2. *For any positive integers a, b, m and any prime p, by Lemma 2.2, we note that*
$$\binom{anp^m}{bnp^m} \equiv \binom{an}{bn} \pmod{p}.$$
This implies that, for any integer r, if there exists a positive integer n such that $\binom{an}{bn} \equiv r \pmod{p}$, then there are infinitely many positive integers n satisfying $\binom{an}{bn} \equiv r \pmod{p}$.

For a positive integer m, let $\varphi(m)$ denote the Euler function, that is, $\varphi(m)$ is the number of elements in the sequence $1, 2, \ldots, m$ which are relatively prime to m. We have the following proposition.

Proposition 3.3. *Let α, β, a, b be integers, and let p be a prime. Suppose $0 < b < a$ and p does not divide a. Then $0 \in A(a, b; \alpha, \beta)$.*

Proof. Since $\gcd(p, a) = 1$ and $\binom{an}{bn} = \binom{an}{(a-b)n}$, we may assume that $\gcd(ab, p) = 1$, then $b \mid p^{\varphi(b)} - 1$. Let $m \in \mathbb{N}$ and let n_m be the positive integer such that
$$bn_m = p^{m\varphi(b)} - 1.$$
As $bn_m = (p-1) + p(p-1) + \cdots + (p-1)p^{m\varphi(b)-1}$, we have $\binom{an_m}{bn_m} \not\equiv 0 \pmod{p}$ if and only if $an_m \equiv bn_m \equiv p^{m\varphi(b)} - 1 \pmod{p^{m\varphi(b)}}$. Hence $a \equiv b \pmod{p^{m\varphi(b)}}$, which is impossible for sufficiently large m. Therefore, there are infinitely many positive integers n such that $\binom{an}{bn} \equiv 0 \pmod{p}$. Hence the result follows from Lemma 3.1. □

The following corollary follows immediately from Lemma 3.1 and Theorem 3.3.

Corollary 3.4. *Let p be a prime and g be a primitive root modulo p. If there exists a positive integer n such that*
$$\binom{an}{bn} \equiv g \pmod{p},$$
then $r \in A(a, b)$ for each $r = 0, 1, \ldots, p-1$.

Moreover, Lemma 3.1 and Theorem 3.3 lead to the following two problems:

Problem 3.5. *Is it true that there exists an $n \in \mathbb{N}$ such that $\binom{an+\alpha}{bn+\beta} \not\equiv 0 \pmod{p}$?*

Problem 3.6. *What are the elements of the set $A(a, b)$?*

We present partial results about these two problems. Let p be a prime, by Wilson's Theorem one always has
$$\binom{p-1}{b} \equiv (-1)^b \pmod{p},$$
hence $(-1)^b \in A(p-1, b)$. Theorem 1.6 is a special solution to Problem 3.5 under the condition $p > a > b > 0$.

Proof of Theorem 1.6: Without loss of generality, we may assume that $\alpha \geq 0$. We can also suppose that $\beta \geq 0$ because $\binom{an+\alpha}{bn+\beta} = \binom{an+\alpha}{(a-b)n+\alpha-\beta}$. Since $\gcd(p, a) = 1$, we can find $0 \leq m_1 \leq p-1$ such that $am_1 + \alpha \equiv p-1 \pmod{p}$, so $am_1 + \alpha = \alpha_1 p + p - 1$

and $bm_1 + \beta = \beta_1 p + w_1$, where $\alpha_1, \beta_1 \geq 0$ and $0 \leq w_1 \leq p-1$. Assume that $\alpha = up + v$ and $\beta = sp + t$, where u, $v, s, t \geq 0$ and $0 \leq v, t \leq p-1$. Since $b < a < p$, we have

$$\alpha_1 = \left\lfloor \frac{am_1 + \alpha}{p} \right\rfloor = \left\lfloor \frac{am_1}{p} \right\rfloor + \left\lfloor \frac{\alpha}{p} \right\rfloor \leq u + p - 2$$

and

$$\beta_1 = \left\lfloor \frac{bm_1 + \beta}{p} \right\rfloor = \left\lfloor \frac{bm_1}{p} \right\rfloor + \left\lfloor \frac{\beta}{p} \right\rfloor + \varepsilon_1 \leq s + p - 2,$$

where $\varepsilon_1 \in \{0, 1\}$.

Let $n = n_1 p + m_1$. Then

$$\binom{an+\alpha}{bn+\beta} = \binom{an_1 p + am_1 + \alpha}{bn_1 p + bm_1 + \beta} \equiv \binom{an_1 p + \alpha_1 p + p - 1}{bn_1 p + \beta_1 p + w_1} \equiv \pm \binom{an_1 + \alpha_1}{bn_1 + \beta_1} \pmod{p}.$$

Repeating the above procedure, we finally have a nonnegative integer j such that $\alpha_j < p$, $\beta_j < p$, and it holds

$$\binom{an+\alpha}{bn+\beta} \equiv \pm \binom{an_j + \alpha_j}{bn_j + \beta_j} \pmod{p},$$

where

$$\alpha_j = \left\lfloor \frac{am_j + \alpha_{j-1}}{p} \right\rfloor \quad \text{and} \quad \beta_j = \left\lfloor \frac{bm_j + \beta_{j-1}}{p} \right\rfloor.$$

If $\alpha_j < \beta_j < p$, then

$$\alpha_{j+1} = \left\lfloor \frac{am_{j+1}}{p} \right\rfloor \quad \text{and} \quad \beta_{j+1} = \left\lfloor \frac{bm_{j+1}}{p} \right\rfloor + \varepsilon_{j+1},$$

where $\varepsilon_{j+1} \in \{0, 1\}$. Hence

$$p > \alpha_{j+1} + 1 \geq \beta_{j+1}. \tag{11}$$

If $\alpha_{j+1} < \beta_{j+1} < p$, then, by (11),

$$\alpha_{j+1} + 1 = \beta_{j+1} < p.$$

Since $b < a < p$, it holds

$$\alpha_{j+2} = \left\lfloor \frac{am_{j+2} + \alpha_{j+1}}{p} \right\rfloor = \left\lfloor \frac{am_{j+2} + \beta_{j+1} - 1}{p} \right\rfloor \geq \left\lfloor \frac{bm_{j+2}}{p} + \beta_{j+1} \right\rfloor = \beta_{j+2}.$$

Consequently, we can find a non-negative integer u such that $p > \alpha_u \geq \beta_u \geq 0$. Let $n_u = p^k$, where $k \geq 1$. Therefore, by Lemma 2.1,

$$\binom{an+\alpha}{bn+\beta} \equiv \pm \binom{an_u + \alpha_u}{bn_u + \beta_u} \equiv \pm \binom{a}{b} \binom{\alpha_u}{\beta_u} \not\equiv 0 \pmod{p}$$

as desired. This completes the proof of Theorem 1.6. □

Remark 3.7. *Theorem 1.6 gives a partial result for Problem 3.5. The general problem is difficult, we even do not know the answer of Problem 3.5 for* $p = 2$.

For Problem 3.6, we give in Theorem 3.8 the answer when $a \equiv b \pmod{p^{m+2}}$ and $b < p^m - 2p^{m-1}$.

Theorem 3.8. *Let p be an odd prime, $m \in \mathbb{N}$, and let a and b be positive integers such that $a > b$, $a \equiv b \pmod{p^{m+2}}$ and $b < p^m - 2p^{m-1}$. Then, for each $r = 0, 1, \ldots, p-1$, there exist infinitely many positive integers n for which*

$$\binom{an}{bn} \equiv r \pmod{p}.$$

Proof. Let u be the least nonnegative integer such that $p^m = bn_1 - u, n_1 \in \mathbb{N}$. Then we have $u < b < p^m - 2p^{m-1}$ and $bn_1 = p^m + u$. Since $b \equiv a \pmod{p^{m+2}}$, we have $an_1 = tp^{m+s} + p^m + u, p \nmid t, s \geq 2$. Let $n_2 = p^s + g, 0 \leq g \leq p-1$. Now we consider the residue class

$$\binom{(tp^{m+s} + p^m + u)n_2}{(p^m + u)n_2} \pmod{p}.$$

Then we have $(p^m + u)(p^s + g) = p^{m+s} + gp^m + u(p^s + g)$, $(tp^{m+s} + p^m + u)(p^s + g) = tp^{m+2s} + (tg+1)p^{m+s} + gp^m + u(p^s + g)$, and $gp^m + u(p^s + g) < p^{m+s}$ since $u < b < p^m - 2p^{m-1}$ and $s \geq 2$. Hence by Lemma 2.2

$$\binom{(tp^{m+s} + p^m + u)n_2}{(p^m + u)n_2} \equiv \binom{tg+1}{1}\binom{gp^m + u(p^s + g)}{gp^m + u(p^s + g)} \equiv tg + 1 \pmod{p}.$$

Since g ranges from 0 to $p-1$ and $\gcd(t, p) = 1$, we get the desired result by Remark 3.2. ∎

Next, we introduce several related examples.

Proposition 3.9. *Let p be a prime, α, t be positive integers with $p \nmid t$. Then, for each $r = 0, 1, \ldots, p-1$, there are infinitely many positive integers n for which*

$$\binom{(tp^\alpha + 1)n}{n} \equiv r \pmod{p}.$$

Proof. Let $n = p^\alpha + g$, where $0 \leq g \leq p-1$. Then

$$(p^\alpha + g)(tp^\alpha + 1) = tp^{2\alpha} + (tg+1)p^\alpha + g.$$

Now we choose the positive integer g such that $tg + 1$ is a primitive root modulo p (since we have $\varphi(p-1)$ ways to choose g), then we have

$$\binom{(p^\alpha + g)(tp^\alpha + 1)}{(p^\alpha + g)} \equiv \binom{g}{g}\binom{tg+1}{1} \equiv tg + 1 \pmod{p}.$$

In conclusion, Proposition 3.2 follows from Corollary 3.4 and Remark 3.2. ∎

Proposition 3.10. *Let p be a prime. Then, for each $r = 0, 1, \ldots, p-1$, we have $r \in A(2, 1)$.*

Proof. Let

$$M_n = \left\langle \binom{2}{1}, \binom{4}{2}, \ldots, \binom{2n}{n} \right\rangle$$

be the multiplicative subgroup of $\mathbb{Q}$ generated by $\binom{2}{1}, \binom{4}{2}, \ldots, \binom{2n}{n}$. Then we have $q \in M_n$ for any prime $q \leq 2n$. We prove this by induction on n.

We have $2 = \binom{2}{1} \in M_1$, $3 = \binom{4}{2}/\binom{2}{1} \in M_2$. Assume that the statement holds for n, that is, $q \in M_n$ for any prime $q \leq 2n$. If $2n+1$ is not a prime, then the statement is obviously true. If $2n+1$ is a prime, then

$$2n + 1 = \frac{(n+1)\binom{2n+2}{n+1}}{2\binom{2n}{n}} \in M_{n+1}.$$

Therefore, any prime $p \leq 2n$ belongs to M_n. Let g be a primitive root modulo p and t be a positive integer with $2t > p$. Then $g \in M_t$. From Corollary 3.4, the proof of the proposition is completed. □

4 Proofs of Theorems 1.2, 1.4, and some results related to Theorem 1.2

In this section, we will prove Theorems 1.2 and 1.4, and additionally obtain some results related to Theorem 1.2.

Proof of Theorem 1.1: Since $a\gamma \neq p^u - p$ for any positive integer $u \geq 2$, $p + a\gamma$ has a prime divisor $q \neq p$, and hence $q \nmid a\gamma$. Let

$$n_m = \frac{q^{m\varphi(a)} - 1}{a}, \quad m \in \mathbb{N}.$$

For a positive integer m with $(a-b)n_m \geq \beta$ and $bn_m + \beta \geq 0$, we have

$$\binom{an_m}{bn_m + \beta} = \binom{q^{m\varphi(a)} - 1}{bn_m + \beta} \not\equiv 0 \pmod{q}.$$

On the other hand, we have

$$pn_m - \gamma = p \cdot \frac{q^{m\varphi(a)} - 1}{a} - \gamma \equiv -\frac{a\gamma + p}{a} \equiv 0 \pmod{q}.$$

Therefore

$$\binom{an_m}{bn_m + \beta} \not\equiv 0 \pmod{pn_m - \gamma}$$

for large enough positive integers m. This completes the proof of Theorem 1.2. □

The following is a result similar to Theorem 1.2.

Theorem 4.1. *Let $a > b$ be positive integers, $\beta \geq 0$ and $\gamma > 0$ be integers and p be prime such that $a > p$ and $a\gamma - p \neq p^u$ for any nonnegative integer u. Then there exist infinitely many positive integers n for which*

$$\binom{an}{bn + \beta} \not\equiv 0 \pmod{pn + \gamma}.$$

Proof. Since $a\gamma - p \neq p^u$ for any nonnegative integer u, $a\gamma - p$ has a prime divisor $q \neq p$, and hence $q \nmid a\gamma$. Let

$$n_m = \frac{q^{m\varphi(a)} - 1}{a}, \quad m \in \mathbb{N}.$$

For the positive integer m with $(a-b)n_m \geq \beta$ and $bn_m + \beta \geq 0$, we have

$$\binom{an_m}{bn_m + \beta} = \binom{q^{m\varphi(a)} - 1}{bn_m + \beta} \not\equiv 0 \pmod{q}.$$

On the other hand, we have

$$pn_m + \gamma = p \cdot \frac{q^{m\varphi(a)} - 1}{a} + \gamma \equiv \frac{a\gamma - p}{a} \equiv 0 \pmod{q}.$$

Therefore

$$\binom{an_m}{bn_m + \beta} \not\equiv 0 \pmod{pn_m + \gamma}$$

for sufficient large positive integers m. This proves Theorem 4.1. ∎

Now we introduce the interesting example when $a = (p+1), b = 1, \beta = 0$, and $\gamma = 1$. Here we have

$$\binom{(p+1)n}{n} = \frac{pn+1}{n}\binom{(p+1)n}{n-1} \equiv 0 \pmod{pn+1}$$

since $\gcd(pn+1, n) = 1$. Hence

$$\binom{(p+1)n}{n} \equiv 0 \pmod{pn+1}$$

for all $n \geq 1$.

Proof of Theorem 1.4: Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$ be the sequence of prime numbers. Choose s such that $p_s \geq 8m - 1$ and put

$$a = \prod_{i=1}^{s} p_i.$$

By the choice of a, we have that $q > p_s \geq 8m-1$ for any prime divisor q of $(2an-1)(4an-1)$. We have, for a certain positive integer A,

$$\binom{4amn}{an} = \frac{8m^3 A(4an-1)(2an-1)}{(an-1)\cdots(an-m)\cdots(an-2m)}\binom{4amn-2m-1}{an-2m-1}.$$

For any positive integer t with $1 \leq t \leq 2m$, we have

$$\gcd(4an-1, an-t) = \gcd(4t-1, an-t) \leq 4t-1 \leq 8m-1 < q$$

and $\gcd(2an-1, an-t) \leq 2t-1 \leq 4m-1 < q$. Hence $\gcd((4an-1)(2an-1), \prod_{i=1}^{2m}(an-i)) = 1$, which implies that

$$\binom{4amn}{an} \equiv 0 \pmod{(4an-1)(2an-1)}$$

for all $n \geq 1$.

Similarly, choose s so that $p_s \geq 72m - 1$ and put

$$a = \prod_{i=1}^{s} p_i.$$

Now we have that $q > q_s \geq 72m - 1$ for any prime divisor q of $(12an-1)(6an-1)(4an-1)(3an-1)(2an-1)$. We have, for a certain positive integer B,

$$\binom{12amn}{an} = \frac{12m^6 B(12an-1)(6an-1)(4an-1)(3an-1)(2an-1)}{(an-1)\cdots(an-m)\cdots(an-2m)\cdots(an-6m)}\binom{12amn-6m-1}{an-6m-1}.$$

For any positive integer t with $1 \leq t \leq 6m$, we have

$$\gcd(12an-1, an-t) = \gcd(12t-1, an-t) \leq 12t-1 \leq 72m-1 < q,$$

and similarly,

$$\gcd(6an-1, an-t) < q,\ \gcd(4an-1, an-t) < q,\ \gcd(3an-1, an-t) < q,$$

and $\gcd(2an-1, an-t) < q$. Hence

$$\gcd((12an-1)(6an-1)(4an-1)(3an-1)(2an-1), \prod_{i=1}^{6m}(an-i)) = 1,$$

which implies

$$\binom{12amn}{an} \equiv 0 \pmod{(12an-1)(6an-1)(4an-1)(3an-1)(2an-1)}$$

for all $n \geq 1$. This proves Theorem 1.4. □

5 Proofs of Theorems 1.8 and 1.9

In this section, we will prove Theorems 1.8 and 1.9. Let $k, l > 1$ be fixed integers. Denote

$$U_n := \frac{\binom{ln}{n}\binom{kln}{ln}}{\binom{kn}{n}}. \tag{12}$$

Our strategy for proving these two theorems is as follows: We rewrite the formula in Theorems 1.8 and 1.9 in the form of

$$\frac{s}{t} \cdot U_n, \quad s, t \in \mathbb{N},\ \gcd(s, t) = 1.$$

If U_n is not an integer, we use a similar argument to the proof of Bober's Lemma [1, Lemma 3.3]. If U_n is an integer, then $k = l$ or $k = 2$ or $\{k, l\} = \{3, 5\}$ by Lemma 2.7. Finally, we prove the results case by case.

Proof of Theorem 1.8: Let $k, l > 1$ be fixed integers. Denote

$$C_n := \frac{\binom{ln}{n}\binom{kln}{ln-1}}{\binom{kn}{n}}.$$

Then

$$C_n = \frac{ln}{kln - ln + 1} U_n.$$

We first consider the case when $U_n \notin \mathbb{N}$. Our argument here is similar to the proof of Bober's Lemma [1, Lemma 3.3], and we state it as follows.

Let

$$f(x) := \lfloor klx \rfloor + \lfloor (k-1)x \rfloor - \lfloor kx \rfloor - \lfloor (l-1)x \rfloor - \lfloor (k-1)lx \rfloor.$$

Then, by Lemma 2.3, we know that there is at least one prime p, such that

$$\nu_p(U_n) = \sum_{\alpha=1}^{\infty} f\left(\frac{n}{p^\alpha}\right) < 0.$$

Since f is a step function, there is some interval, say $[\beta, \beta+\epsilon]$, such that $f(x) < 0$ for all $x \in [\beta, \beta+\epsilon]$. Besides, there exists $\delta > 0$, such that $f(x) = 0$ for all $x \in [0, \delta]$. Note that if we could find some m and p such that $\frac{m}{p} \in [\beta, \beta+\epsilon]$ and $\frac{m}{p^2} \in [0, \delta]$, then we would have $f\left(\frac{m}{p}\right) < 0$ and $f\left(\frac{m}{p^\alpha}\right) = 0$ for all $\alpha \geq 2$. That is, if m and p simultaneously satisfy $p\beta \leq m \leq p(\beta+\epsilon)$ and $0 \leq m \leq p^2\delta$ for fixed ϵ and δ, then

$$\nu_p(U_m) = \sum_{\alpha=1}^{\infty} f\left(\frac{m}{p^\alpha}\right) < 0.$$

For prime p large enough, say $p > P_1$, we get $p^2\delta > p(\beta+\epsilon)$ and for p large enough, say $p > P_2$, we have $p\epsilon > 2$. Therefore, there are at least two consecutive integers in the interval $(p\beta,\, p\beta + p\varepsilon)$. We choose the integer in the interval which is relatively prime to p, say m_0. Therefore, for any $p > P = \max\{P_1, P_2, l\}$, there exists an integer m_0 such that $\gcd(m_0, p) = 1$ and $\nu_p(U_{m_0}) < 0$.

Let $U_{m_0} = \frac{A}{B}$, where $A, B \in \mathbb{N}$ and $\gcd(A, B) = 1$. Then $p | B$ and $\gcd(lm_0, p) = 1$. Since

$$C_{m_0} = \frac{lm_0}{klm_0 - lm_0 + 1}\frac{A}{B},$$

it follows that C_{m_0} is not an integer. In other words, we can find m_0 such that $\binom{km_0}{m_0} \nmid \binom{lm_0}{m_0}\binom{klm_0}{lm_0-1}$ in this case.

Next we deal with the case when $U_n \in \mathbb{N}$. By Lemma 2.7, we have $k = l$ or $k = 2$ or $l = 2$ or $\{k, l\} = \{3, 5\}$. We divide the remaining proof into five cases.

Case 1: $k = l$. Then, by the definitions of U_n and C_n, we find that

$$C_n = \frac{kn}{k^2n - kn + 1} \cdot U_n = \frac{kn}{k^2n - kn + 1} \cdot \binom{k^2n}{kn} = \binom{k^2n}{kn-1}.$$

Hence

$$C_n = \frac{kn}{k^2n - kn + 1} U_n \in \mathbb{N}.$$

Case 2: $k = 3, l = 5$. Then

$$U_n = \frac{\binom{5n}{n}\binom{15n}{5n}}{\binom{3n}{n}} = \frac{(15n)!(2n)!}{(10n)!(4n)!(3n)!}, \quad kln - ln + 1 = 10n + 1.$$

Lemma 2.6 implies that

$$(10n+1) \mid \frac{\binom{5n}{n}\binom{15n}{5n}}{5\binom{3n}{n}}$$

for all $n \in \mathbb{N}$, so

$$C_n = \frac{5n}{10n+1} U_n \in \mathbb{N}.$$

Case 3: $k = 5, l = 3$. Then

$$U_n = \frac{\binom{3n}{n}\binom{15n}{3n}}{\binom{5n}{n}} = \frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!}, \quad kln - ln + 1 = 12n + 1.$$

Note that Proposition 2.13 implies that $(12n+1) | U_n$. It follows that

$$C_n = \frac{3n}{12n+1} U_n \in \mathbb{N}.$$

Case 4: $k = 2$. Then

$$U_n = \frac{\binom{ln}{n}\binom{2ln}{ln}}{\binom{2n}{n}}, \quad kln - ln + 1 = ln + 1.$$

In view of the case $k = l$, we may suppose that $l \geq 3$. We take a prime number p that satisfies $p = lr - 1$, where $l, r \in \mathbb{N}$ and $r > 4$. Now consider the situation when $n = 2p - r$. Thus

$$\left\{\frac{ln}{p}\right\} = \frac{p-1}{p}, \left\{\frac{2ln}{p}\right\} = \frac{p-2}{p}, \left\{\frac{n}{p}\right\} = \frac{p-r}{p}, \left\{\frac{2n}{p}\right\} = \frac{p-2r}{p}, \left\{\frac{(l-1)n}{p}\right\} = \frac{r-1}{p}.$$

At this point, notice that

$$\left\{\frac{2ln}{p}\right\} + \left\{\frac{n}{p}\right\} - \left\{\frac{ln}{p}\right\} - \left\{\frac{2n}{p}\right\} - \left\{\frac{(l-1)n}{p}\right\} = 0,$$

which means that

$$\left\lfloor\frac{2ln}{p}\right\rfloor + \left\lfloor\frac{n}{p}\right\rfloor = \left\lfloor\frac{ln}{p}\right\rfloor + \left\lfloor\frac{2n}{p}\right\rfloor + \left\lfloor\frac{(l-1)n}{p}\right\rfloor.$$

Take notice of

$$ln + 1 = l(2p - r) + 1 = 2lp - p = (2l - 1)p$$

and

$$2ln + 2 = (4l - 2)p = (4l - 2)(lr - 1) < l^2r^2 - 2lr + 1 = p^2,$$

which means that $2ln < p^2$, so $\nu_p(U_n) = 0$ and $\nu_p\left(\frac{U_n}{ln+1}\right) < 0$. That is to say, $ln + 1 \nmid U_n$, which proves that

$$C_n = \frac{ln}{ln+1} U_n \notin \mathbb{N}.$$

Case 5: $l=2$. Then

$$U_n=\frac{\binom{2n}{n}\binom{2kn}{2n}}{\binom{kn}{n}},\quad kln-ln+1=2kn-2n+1.$$

By Example 1.9 in Sun[8], we can get that $C_n\in\mathbb{N}$ in this situation.

This completes the proof of Theorem 1.8. $\square$

Proof of Theorem 1.9: Let

$$D_n:=\frac{\binom{ln}{n-1}\binom{kln}{ln}}{\binom{kn}{n}}.$$

Then

$$D_n=\frac{n}{ln-n+1}U_n.$$

The proof of the case of $U_n\notin\mathbb{N}$ is completely analogue to that of the proof in Theorem 1.8. Consequently, we only point out the essential differences between the two proofs.

Suppose that $U_n=\frac{A}{B}\notin\mathbb{N}$, where $A,B\in\mathbb{N}$ and $\gcd(A,B)=1$. Then every prime $p>P=\max\{P_1,P_2\}$ occurs as a factor in B. In addition,

$$T_{m_0}=\frac{m_0}{lm_0-m_0+1}\frac{A}{B}.$$

Therefore, if prime $p>\max\{P_1,P_2\}$ with $(p,m_0)=1$, then T_{m_0} is not an integer. That is to say, we can find m_0 such that $\binom{km_0}{m_0}\nmid\binom{lm_0}{m_0-1}\binom{klm_0}{lm_0}$ in this case.

Assume now $U_n\in\mathbb{N}$. Similarly, by Lemma 2.7, we have $k=l$ or $k=2$ or $l=2$ or $\{k,l\}=\{3,5\}$. Hence we divide the remaining proof into four cases by the assumption that $k\neq l$.

Case 1: $k=3$, $l=5$. Then

$$U_n=\frac{\binom{5n}{n}\binom{15n}{5n}}{\binom{3n}{n}}=\frac{(15n)!(2n)!}{(10n)!(4n)!(3n)!},\quad ln-n+1=4n+1.$$

Proposition 2.9 implies $D_n\notin\mathbb{N}$ in this case.

Case 2: $k=5$, $l=3$. Then

$$U_n=\frac{\binom{3n}{n}\binom{15n}{3n}}{\binom{5n}{n}}=\frac{(15n)!(4n)!}{(12n)!(5n)!(2n)!},\quad ln-n+1=2n+1.$$

Proposition 2.12 shows $D_n\notin\mathbb{N}$ in this case.

Case 3: $l=2$. Then

$$U_n=\frac{\binom{2n}{n}\binom{2kn}{2n}}{\binom{kn}{n}},\quad ln-n+1=n+1.$$

We can conclude from Sun's result [8, Theorem 1.3(i)] that there are integers $n\geq 0$ such that

$$(n+1)\mid(2k+1)\frac{\binom{2n}{n}\binom{2kn}{2n}}{\binom{kn}{n}},$$

while

$$(n+1) \nmid \frac{\binom{2n}{n}\binom{2kn}{2n}}{\binom{kn}{n}}.$$

Hence

$$D_n = \frac{n}{n+1}\frac{\binom{2n}{n}\binom{2kn}{2n}}{\binom{kn}{n}} \notin \mathbb{N}.$$

Case 4: $k = 2$. Then

$$U_n = \frac{\binom{ln}{n}\binom{2ln}{ln}}{\binom{2n}{n}}, \quad ln - n + 1 = ln - n + 1.$$

We can conclude from Sun's result [8, Theorem 1.3(ii)] that there are integers $n \geq 0$ such that

$$(ln - n + 1) \mid (l+1)' \frac{\binom{2ln}{ln}\binom{ln}{n}}{\binom{2n}{n}},$$

while

$$(ln - n + 1) \nmid \frac{\binom{2ln}{ln}\binom{ln}{n}}{\binom{2n}{n}},$$

where $(l+1)'$ is the odd part of $(l+1)$. Note if $(l+1)$ is a power of 2, then $(l+1)' = 1$, thereby

$$D_n = \frac{n}{ln - n + 1}\frac{\binom{ln}{n}\binom{2ln}{ln}}{\binom{2n}{n}} \in \mathbb{N}.$$

In conclusion, the proof of Theorem 1.9 is completed. □

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