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Graphs which are intervals

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Abstract

A graph G is called an interval if there exist $a, b \in V(G)$ such that G is the union of shortest paths connecting a and b. In this paper we show that

1. If G is an interval between a and b, then there exists a path H with diameter $d(H) = d(G)$ such that there is a homomorphism $f : G \to H$ and the distance $\rho(a, b) + 1 \leq |H| \leq |G| - 1$;
2. Every interval is a connected bipartite graph;
3. If G is an interval between a and b that is not a path, then G has a path P with internal vertices (if any) all of degree 2 such that deletion of the internal vertices of P from G gives rise to an interval (if $P = uv$ then $G - uv$ is an interval).

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1 Intervals

The surface of the sphere S in 3-dimensional Euclidean space has the property that if a, b are the end points of any diameter then S is the union of all geodesics connecting a and b. The metric is the intrinsic path metric of S; the distance between $x, y \in S$ is the length of a geodesic arc connecting x and y in S. A general question is: which metric spaces in which every two points are connected by a geodesic has this property. While we are not able to resolve the question in general, we are able to determine the graphs with this property.

Let G be a connected graph. If there exist $a, b \in V(G)$ such that the union of shortest paths connecting a and b is G, then G is called an *interval*. The vertices a, b are called *extremal*, or *diametrical* vertices of G. Our aim is to determine all intervals. The family of graphs we abbreviate as intervals in this note appear as important computational models in task scheduling [4] and parallel computing [4, 5, 6]. We cannot use the more appropriate term *geodesic graphs* as the term was already in use in a different context [2]. The term *interval* is not an abuse since each graph we call an interval is the union of shortest paths connecting a pair of vertices and hence it possesses a faithful embedding as an interval in a partially ordered set with a rank function.

Let G be a graph and let $A, B \subseteq V(G)$. Define $[A, B] = \{ab \in E(G) : a \in A, b \in B\}$. Note that we did not require that $A \cap B = \emptyset$. Note also that if a graph is not specified, then $[A, B]$ is the set of all edges with one end in A and the other end in B. Hence if $A \cap B = \emptyset$ and $|A| = m$, $|B| = n$, then $[A, B] = K_{m,n}$ the complete bipartite graph with parts of m and n vertices.

Let G be an interval. Then a replacement of a set of multiple edges between two vertices by a single edge gives rise to an interval. Hence we may assume that graphs in this note are simple. That is, the graphs have no loops or multiple edges.

We show some examples of intervals in the figure below.

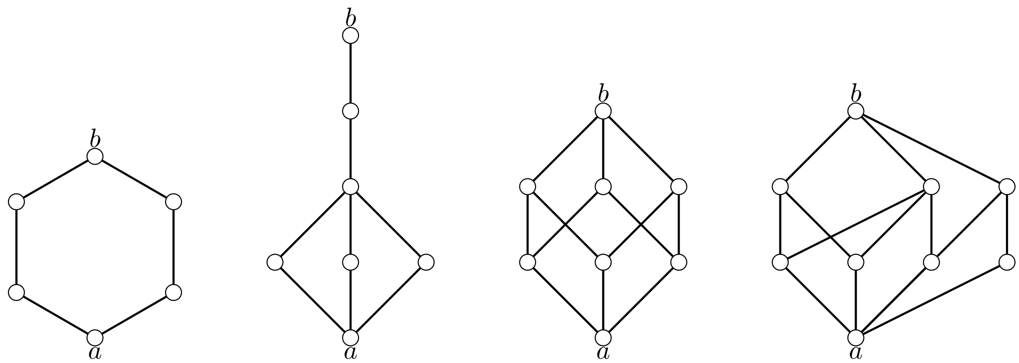

Figure 1. Examples of intervals.

Let G be an interval with extremal vertices a and b. For $u, v \in V(G)$, $\rho(u, v)$ denotes the length of a shortest path connecting u and v in G. For every integer $i \geq 0$, define the level set $V_i = \{x \in V(G) : \rho(a, x) = i\}$. Let H be an interval with level sets V_i, $i \geq 0$. Then the graph G, obtained from H by subdivision of every edge of $[V_i, V_{i+1}]$ for a fixed index i, by exactly r vertices each, is also an interval. A subdivision of this type is called *balanced*. For each $n \geq 1$, any balanced subdivision of the graph $K_{2,n}$ is an interval.

If H is an interval and $x, y \in V(H)$ with $k = \rho(x, y) > 0$, then the addition of a path of length k connecting x and y, internally disjoint from H, gives an interval G. More precisely, let k be a positive integer and H be an interval. Let $x \in V_i(H)$, $y \in V_{i+k}(H)$. Let $w_1, w_2, \ldots, w_{k-1} \notin V(H)$. Then $G = H \cup x w_1 w_2 \cdots w_{k-1} y$ is also an interval. Note that if $k = 1$ and $xy \notin E(G)$ then the path added is just the edge xy. The operation of addition of a path, as defined in this paragraph, produces an interval from a given interval. A main aim of this note is to prove that every interval may be obtained by a finite number of iterations of this operation, beginning with a path.

2 Homomorphisms

For a graph G, denote $|G| = |V(G)|$ and $\|G\| = |E(G)|$. Let G be an interval and $P \subseteq G$ be a path. If P is contained in a shortest path between a and b then P is called a *shortest path* path.

Let G, H be graphs and let $f : V(G) \to V(H)$ be a mapping. If $xy \in E(G)$ implies $f(x)f(y) \in E(H)$, then f is called a (graph) *homomorphism*. An injective homomorphism is called a *monomorphism* or an *embedding* (of G as a subgraph of H). A surjective homomorphism is called an *epimorphism*. It is well understood that if there exist a monomorphism $f : G \to H$ and a monomorphism $g : H \to G$, then G and H are isomorphic and this is denoted $G \simeq H$. Denote by $d(G)$ the diameter of a graph G.

Proposition 2.1. *Let G be an interval with extremal vertices a and b. Then $d(G) = \rho(a, b)$.*

Proof. The diameter of G bounds the distance between every pair of vertices in G from above. Since $a, b \in V(G)$, we have $\rho(a, b) \leq d(G)$.

It remains to show that for every $u, v \in V(G)$, $\rho(u, v) \leq \rho(a, b)$. Since every vertex x is on a shortest path connecting a and b, $\rho(a, b) = \rho(a, x) + \rho(x, b)$. It may be assumed that $\rho(a, u) \geq \rho(b, v)$. Then

$$\rho(u, v) \leq \rho(u, b) + \rho(b, v) = \rho(a, b) - \rho(a, u) + \rho(b, v) \leq \rho(a, b).$$

□

Theorem 2.2. *If G is an interval with extremal vertices a and b, then there exists a path H with $d(H) = d(G)$ and $\rho(a, b)+1 = |H| \leq |G|$ such that there is an epimorphism $f : G \to H$.*

Proof. Let G be an interval with extremal vertices a and b and suppose that $d = \rho(a, b)$. By the definition of an interval, G is the union of shortest paths connecting a and b. As defined above, let $V_i = \{x : \rho(a, x) = i\}$, $i = 0, 1, \ldots, d$. Then $\{V_0, V_1, \ldots, V_d\}$ is a partition of $V(G)$. Define H with $V(H) = \{V_0, V_1, \ldots, V_d\}$ and $E(H) = \{V_iV_{i+1} : 0 \leq i \leq d - 1\}$. Then H is path connecting $V_0 = \{a\}$ and $V_d = \{b\}$. Define $f : G \to H$ by $f(v) = V_i$ if $v \in V_i$. Then f is a mapping $V(G) \to V(H)$. For each $x, y \in V(G)$ with $xy \in E(G)$, by the definition of an interval, it may be assumed that $x \in V_i$ and $y \in V_{i+1}$ for some $i \geq 0$. By the definition of f, $f(xy) = f(x)f(y) = V_iV_{i+1} \in E(H)$. Hence f is an epimorphism from G onto H. We have $d(H) = \rho(a, b) = d(G)$ and $\rho(a, b) + 1 = |H| \leq |G| - 1$. The second inequality follows from the assumption that G is not a path and hence there exists i such that $|V_i| \geq 2$. □

The levels give a partial order on an interval G in which the function $\rho(a, x)$ acts as rank function. Let G be an interval and $u \in V(G)$. If $u \in V_i$ and the edges incident with u are all in $[V_{i-1}, u]$ then u is called a *local maximum*. Local minimum vertices are similarly defined. In intervals, global extrema are the only local extrema.

Lemma 2.3. *If G is an interval and if $u \in V(G) \setminus \{b\}$, then u is not a local maximum.*

Proof. Assume that $u \in V(G) \setminus \{b\}$ is a local maximum. By definition, every edge incident with u, the other end is in a lower level. No such edge is on a shortest path connecting a and b. This is a contradiction to the assumption that G is an interval. □

Similarly, an interval G with extremal vertices a and b has no local minimum other than a.

Theorem 2.4. *Let G be an interval. If G has a cycle then G has a cycle which is the union of two shortest paths.*

Proof. Let $a, b \in V(G)$ be the extremal vertices of G. For every $v \in V_i$, $0 \leq i < \rho(a, b)$, that is, $v \in V(G) \setminus \{b\}$, let $d^+(v) = |[v, V_{i+1}]|$. By Lemma 2.3, for every $v \in V(G) \setminus \{b\}$, $d^+(v) \geq 1$. If for every $v \in V(G) \setminus \{b\}$, $d^+(v) = 1$, then G is a path. Hence, if G contains a cycle, then there exists $v \in V(G) \setminus \{b\}$ such that $d^+(v) \geq 2$. Suppose that $v \in V_i$ and let

$x, y \in V_{i+1}$ with $vx, vy \in E(G)$. Start two walks W_1, W_2 up from v towards b; and suppose that for walks W_1, W_2, vx is the first edge of W_1 and vy is the first edge of W_2. By Lemma 2.3, if either of W_1 and W_2 reaches a vertex other than b, there is an edge connecting it to the next level higher. Complete W_1 and W_2 as shortest paths from v to b. Since both paths are shortest path, there exists $x \in V(G) \setminus \{v\}$ with a minimum level number such that $z \in W_1 \cap W_2$. Let P be the shortest path on W_1 from v to z, and Q be the shortest path on W_2 from v to z. Then $C = P \cup Q$ is a cycle in G. □

3 Leveled graphs

Let G be a graph. If there exist $a, b \in V(G)$ such that for each $x \in V(G)$ there exists a shortest path P connecting a and b such that $x \in V(P)$, then G is called a *leveled* graph.

Theorem 3.1. *If G is a leveled graph with extremal vertices a and b, then there exist an interval H and a set $S \subseteq \{uv \notin E(H) : \rho(a,u) = \rho(a,v)\}$, such that $G = H \cup S$ and $E(H) \cap S = \emptyset$.*

Proof is routine.

Every interval is a leveled graph. It is possible that a leveled graph is not bipartite, and hence not an interval. Let $e = ab \in E(K_4)$. Then the graph $G = K_4 - e$ is a leveled graph whose spanning interval is $K_{2,2}$ with extremal vertices a and b.

We now consider a generalization of intervals. Let G be a connected graph. If there exist $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$ and G is the union of shortest paths connecting A, B, then G is called a *semi-interval*. The sets A, B are called the extremal sets. The vertices of G may be partitioned into level sets from A to B.

Lemma 3.2. *Every connected bipartite graph is a semi-interval.*

Proof. Let G be a connected bipartite graph with bipartition $\{A, B\}$. Then G is a semi-interval with extremal sets A and B. □

Problem. *If G is a semi-interval with extremal sets A and B, then is it true that there exist $a \in V(G)$ and $C \subseteq V(G)$ such that G is a semi-interval with extremal sets $\{a\}$ and C?*

Let G be any connected graph. Let $x \in V(G)$ with the property that there exists a vertex $y \in V(G)$ such that $\rho(x, y) = d(G)$. A vertex x with this property may be called *extremal*, *peripheral* or *diametrical* vertex. Let $a \in V(G)$ an extremal vertex. For integers $i \geq 0$, let $V_i = \{x \in V(G) : \rho(a, x) = i\}$.

Then $V(G)$ may be partitioned into level sets $V_0, V_1, \ldots, V_d$. Suppose that for each $i \geq 0$, V_i is an independent set. Then there is a homomorphism $f : G \to H$ with $d(H) = d(G)$.

If $|V_d| = 1$ then G is a leveled graph. By Theorem 2.2, there exist a path H and a homomorphism $f : G \to H$ with $d(H) = d(G)$. If V_d is independent then there is a homomorphism $f : G \to H$ such that $d(H) = d(G)$. We do not state these statements explicitly as theorems.

4 Reduction

Let G be an interval. Since no shortest path uses an edge between two vertices of the same level, no interval has such an edge.

Proposition 4.1. *Every interval is a connected bipartite graph.*

Proof. Let G be an interval and suppose that a, b are two extremal vertices. Since for each vertex, there is a path connecting it to a, G is connected. With V_i as in the proof of Theorem 2.2, let

$$X = \bigcup_{i \text{ even.}} V_i,\ Y = \bigcup_{i \text{ odd.}} V_i.$$

Since each edge of G is between two vertices of consecutive levels, $\{X, Y\}$ is a bipartition of G such that $E(G|_X) = \emptyset = E(G|_Y)$. Hence G is bipartite. □

Note that not all connected bipartite graphs are intervals. The complete bipartite graph $K_{3,3}$ is not an interval. Let the two parts be denoted X and Y. For any $a, b \in V(K_{3,3})$, it may be assumed that $a \in X$. If $b \in X$, then the third vertex of X and each of the three edges incident with it cannot be on a shortest path connecting a and b. If $b \in Y$, then each vertex of X different from a and edges incident with it cannot be on a shortest path connecting a and b. Hence $K_{3,3}$ is not an interval.

Lemma 4.2. *Let G be an interval with extremal vertices a and b. If $u, v \in V(G)$ and $\rho(a, u) = \rho(a, v)$ then $uv \notin E(G)$.*

Proof. The assumption of the lemma implies that no shortest path connecting a and b in G contains uv. □

This theorem is closely related to Theorem 3.1.

Let G be an interval and P be a shortest path connecting u, v in G. If $|P| = 2$ then denote $I(P) = E(P) = uv$ and if $|P| \geq 3$ then denote $I(P) = P - \{u, v\}$. Let P, Q be shortest paths in G. If $I(P) \cap I(Q) = \emptyset$ then P, Q are said to be *internally disjoint* .

Theorem 4.3. *Let G be an interval with extremal vertices a and b. Suppose that G is not a path. Then there exists a shortest path P in G with each vertex of $I(P)$ (if any) of degree 2 in G, such that $G - I(P)$ is an interval with extremal vertices a and b.*

Proof. The proof is by induction on $|G| + \|G\|$. Since G is connected, $|G| + \|G\| \geq 2|G| - 1$. If $|G| + \|G\| < 8$, then we have $|G| \leq 4$. Since G is an interval but not a path, $G = K_{2,2}$ (the cycle of length 4) and the conclusion of the theorem in true. Assume therefore that $|G| + \|G\| \geq 8$ and that the statement of the theorem is true for every interval H with $|H| + \|H\| < |G| + \|G\|$. In this proof, for $v \in V(G)$, $d(v)$ denotes the degree of v in G.

Suppose that G is not a path. If $d(a) = 1$, then let the only neighbour of a be a'. Then $G - a$ is an interval with extremal vertices a', b, and $G - a$ is not a path. By the inductive hypothesis, there exists a shortest path P in $G - a$ with each vertex of $I(P)$ (if any) of degree 2 in $G - a$, such that $G - a - I(P)$ is an interval, with extremal vertices a', b. Since

$P \subseteq G - a \subseteq G$, P is a shortest path in G with each vertex if $I(P)$ (if any) of degree 2 in G. And $G - I(P)$ is an interval with extremal vertices a, b.

Assume, therefore, that $d(a), d(b) > 1$. Recall the definition of level sets V_i. As usual, for $v \in V(G)$, the set of neighbors of v is denoted by $N(v)$. For $v \in V_i$, $0 < i < \rho(a,b)$, define $d^-(v) = |N(v) \cap V_{i-1}|$ and for $0 \geq i < \rho(a,b)$, define $d^+(v) = |N(v) \cap V_{i+1}|$. In this proof only, we called $d^-(v)$ the *down-degree* of v, and $d^+(v)$ the *up-degree* of v. We may also agree that the up-degree of b and the down-degree of a are both 0. Since for every i and every $x, y \in V_i$, $xy \notin E(G)$, we have that for each $v \in V(G)$, $d(v)$ is the sum of the up- and the down-degrees of v.

Let P be a shortest path connecting vertices u, v with $\rho(a,u) < \rho(a,v)$. If
(1) either $u = a$ or $d(u) > 2$,
(2) either $v = b$ or $d(v) > 2$, and
(3) each vertex of $I(P)$ is of degree 2 in G;
then P is called an *admissible* path. There exists an admissible path in G. In fact, each $u \in V(G) \setminus \{b\}$ with either $u = a$ or $d(u) > 2$ is the lower end of an admissible path. This may be seen by beginning at u and traversing any edge incident with u and a vertex in the level immediately above u. Continue until b is reached or a vertex $v \in V(G) \setminus \{a, b\}$ with $d(v) > 2$ is reached. Similarly, each $v \in V(G) \setminus \{a\}$ such that either $v = b$ or $d(v) > 2$ is the upper end of an admissible path P in G.

Let P be an admissible path in G with ends u and v. Suppose that $\rho(a,u) < \rho(a,v)$.
Claim. If $d^+(u) \geq 2$ and $d^-(v) \geq 2$, then $G - I(P)$ is an interval with extremal vertices a, b.

As usual, if P is a path and $x, y \in V(P)$, then $P[x,y]$ denotes the sub-path from x to y in P. Let $H = G - I(P)$. A path in H is said to be *shortest path* in H, if it is shortest path in G. From this convention, it follows that any shortest path connecting a, b in H is a shortest path connecting a, b in H. To show that H is an interval with extremal vertices a and b, it suffices to show that each member of $V(H) \cup E(H)$ is on a shortest path path in H connecting a and b.

By our assumption, each vertex of $I(P)$ is of degree 2 in G and the two edges incident with each vertex of degree 2 of P are edges of P. Hence each path Q in G with $Q \nsubseteq P$ and $P \nsubseteq Q$ is internally disjoint from P. Hence such a path Q must be a path in H.

Let $c \in V(H) \cup E(H)$. Since $H \subseteq G$, there is a shortest path in G connecting a and b, that contains c. Since the paths P and Q are internally disjoint, $Q \subseteq H$. Hence $c \in Q$. That is, $c \in Q[a,u]$ or $c \in Q[v,b]$.

If $c \in Q[a,u]$, since $d^+(u) > 1$, there exists $e \in E(G)$ incident with u with its other end in the level above u. Since P, Q are internally disjoint, $e \notin E(P)$. Since e is contained in a shortest path connecting a and b in G, e is contained in a shortest path R' connecting u and b in G. Then $R = Q[a,u] \cup R'$ is a shortest path connecting a and b in G, internally disjoint from P. Hence $R \subseteq H$. Hence c is contained in a shortest path connecting a and b in H.

If $c \in Q[v,b]$ then the proof is similar. This completes the proof of claim.

It may, therefore, be assumed that

> (A) if P is an admissible path in G connecting u and v, with $\rho(a,u) < \rho(a,v)$, and if $d^+(u) \geq 2$ then $d^-(v) = 1$.

Since $d(a) = d^+(a) > 1$, an edge incident with a may be traversed into V_1. Continue the traversal up, starting at a, traversing vertices of degree 2. If b is reached, then the path traversed is an admissible path connecting a and b. Then $d(b) = d^-(b) > 1$ is a contradiction to the assumption (A).

Hence such a path terminates at a vertex w with $\rho(a, w) < \rho(a, b)$ and $d(w) > 2$. This path connecting a and w is admissible and $d^+(a) > 1$. By the assumption (A), $d^-(w) = 1$. Since $d(w) > 2$ and $d(w) = d^+(w) + d^-(w)$, we have $d^+(w) \geq 2$.

Begin a new admissible path Q, starting with w, by traversing up into the immediate level above w. If Q reaches b, since $d^+(w) \geq 2$ and $d^-(b) \geq 2$, this would contradict our the assumption (A) that if an admissible path in G connects u and v, with $\rho(a, u) < \rho(a, v)$, and if $d^+(u) \geq 2$ then $d^-(v) = 1$. If Q terminates at a vertex z with $d^-(z) = 1$ and $d^+(z) \geq 2$, then the process of traversal will be continued upwards from z.

Since the number of levels is finite, such a process will terminate at b. This is a contradiction to the assumption (A). □

Corollary 4.4. *Every interval that is not a path is obtained from a path with the same diameter by repeatedly adding paths of length $k \geq 1$ connecting vertices of level difference k, internally disjoint from the current interval.*

For the concepts of partially ordered sets, lattices and rank functions, the reader is referred to [1].

Corollary 4.5. *Let X be a finite partially ordered set with a rank function and with unique maximal and minimal elements. Then X is obtained from a chain by attachment of chains between two points of different rank with the length (the number of elements) of chain one bigger than the difference between ranks of the two points chosen.*

Note that an interval is not necessarily planar. In fact, not only can an interval contain a minor isomorphic to $K_{3,3}$, it can contain an induced subgraph $K_{3,3}$. Let $r \geq 2$ be an integer and the two parts in a bipartition of $K_{r,r}$ be denoted X and Y. Let $a, b \notin V(K_{r,r})$. Let $G = K_{r,r} \cup [a, X] \cup [b, Y]$. Then G is an interval and $K_{r,r} \subseteq G$ as an induced subgraph.

Since every interval is bipartite, no interval contains K_5 as a subgraph. An interval can contain a K_5 minor. The graph of Figure 2 has a K_5 minor.

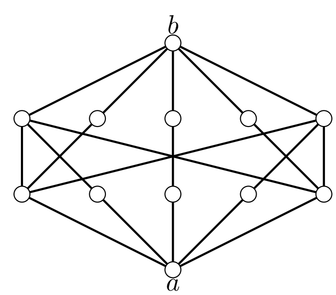

Figure 2. An interval with a K_5-minor.

5 Extremal intervals

Consider the graph G obtained by subdivision of each edge of $K_{2,n}$ by r vertices. Then $|G| = rn + n + 2$. Let a and b be the extremal vertices of G. Since each edge is an edge of a unique shortest path connecting a and b, for each $e \in E(G)$, $G - e$ is not an interval. Hence G is a minimal interval. We have not determined minimal intervals in this note.

Let H be any interval. Then the (edge) maximal interval containing H as a spanning subgraph may also be determined. Suppose that $\rho(a,b) = d$. Then

$$G = H \cup [a, V_1] \cup [V_1, V_2] \cup \cdots \cup [V_{d-1}, b]$$

Here the square brackets do not make reference to the graph H. That is, they are bipartite completions between consecutive levels. It is routine to verify that G is the (edge) maximum, with H as a spanning subgraph.

Comments and remarks

Theorem 4.3 determines the structure of intervals. By Theorem 4.3, the family of intervals has an almost partial ordering under subgraph inclusion, and hence under minor inclusion. If the operation of deletion of an end vertex of a path that is of degree 1 is introduced, then under the subgraph inclusion, every antichain is finite in the almost partial ordering. This makes the almost partial ordering a well quasi ordering. That is, the relation is reflexive, transitive and every antichain is finite. In addition to these conditions, the family also satisfies another important condition, namely, every strictly descending chain is finite (the well-known Dedekind descending chain condition). Corollary 4.5 provides a constructive definition of ranked lattices with unique maximum and minimum elements using the operation of attachment of chains. This corollary resembles the "ear decomposition" of bipartite graphs [3].

Note that the graph of the 3-cube and any even cycle have the property that every vertex is extremal. That is, every vertex is the end of a diameter. We conclude this note with a question.

Problem. *Determine intervals in which every vertex is extremal.*

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