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On finite groups with operators
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Abstract

Let G be a finite group with $|G| \geq 3$ and let A be a non–trivial subgroup of $Aut(G)$. The number $s(g) = |\{(x, \alpha) \in G \times A \mid g = [x, \alpha]\}|$ is determined for an arbitrary $g \in G$. The key to do this is an unpublished observation made by I. M. Isaacs in a particular case.

If C is the set of all commutators $[x, \alpha]$ with $x \in G$ and $\alpha \in A$ and F is the subgroup of the fixed points of A in G, we obtain a general and sharp lower bound for $|F \cap C|$. This is used to define a particular type of action of A on G, called here *uniform* action.

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1 Introduction

Throughout this paper G is a finite group with $|G| \geq 3$ and with the group operation written multiplicatively. Since $Aut(G)$ is non–trivial, we let A denote an arbitrary non–trivial subgroup of $Aut(G)$. For $(x, \alpha) \in G \times A$ we write $x^\alpha := \alpha(x)$ and $[x, \alpha] := x^{-1}x^\alpha$, calling the latter element the *commutator* of the pair (x, α).

The set of commutators is denoted here by C; if $g \in G$ we consider the *source* of g, which is the set $S(g) := \{(x, \alpha) \in G \times A \mid g = [x, \alpha]\}$ and we let $s(g) = |S(g)|$. By definition, $s(g) = 0$ precisely when $g \in G \setminus C$.

Since A is non–trivial, $|C| > 1$ and if $g \in C$ we solve the old combinatorial problem of expressing $s(g)$ in purely group–theoretical terms depending on g, G and A.

A particular case of this problem was considered by G. Frobenius at the end of the 19th century. His results show that $Frob(g) := \frac{1}{|G|}|\{(x, y) \in G \times G \mid g = [x, y]\}|$ can be expressed in terms of characters for every $g \in G$; if $Irr(G)$ is the set of the irreducible complex characters of G,

$$Frob(g) = \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)}.$$

A direct consequence of Theorem 3.1 expresses $Frob(g)$ in group–theoretical terms in two different ways.

If $F := C_G(A)$ is the subgroup of the fixed points of A in G it is important to evaluate $|F \cap C|$. A general and sharp lower bound for $|F \cap C|$ is given in Theorem 3.3 and this is used to define a special type of action of A on G called here $uniform$ action. Uniform action generalizes (in the precise sense of Theorem 3.3) both fixed–point–free action and co–prime action.

The unexplained notation is mostly standard and it follows that in [6]. The methods are dictated by the general context; they are elementary, for only basic counting principles and basic properties of automorphisms and commutators are used.

2 Preliminaries

This is a long, boring and, at times, even pedantic section. A lot of notation is needed because we deal with a handful of actions of various groups on sets. We also fix notation related to our abstract group G, to commutators and fixed points and we list the results needed from the literature.

If $1 \in X \subseteq G$, we write $X^\star := X \setminus \{1\}$. We abuse notation a bit and write $X = 1$ instead of $X = \{1\}$ and $X \neq 1$ instead of $X \neq \{1\}$. As usual, $|X|$ is the number of elements in X.

If $x, y \in G$, $[x, y] := x^{-1}y^{-1}xy$, $\mathcal{C} := \{[x, y] \mid x, y \in G\}$ and $G' = \langle \mathcal{C} \rangle$. We write $Z := Z(G)$ and if H is a subgroup of G we write $H \leqslant G$. If $H \leqslant G$, then $Z(H), C_G(H), N_G(H)$ stand for the center of H, the centralizer of H in G and the normalizer of H in G respectively.

We write $I := Inn(G)$ to denote the group of inner automorphisms of G, so $I \cong G/Z$ and if $g \in G$ we let i_g denote the inner automorphism of G induced by g via conjugation. Then $[x, y] = [x, i_y]$ and if $g \in G$ we write $Cl_G(g) := \{g^x \mid x \in G\}$. The number of conjugacy classes in G is $k(G)$.

Throughout, p is a prime number, $\pi(G)$ is the set of the prime divisors of $|G|$ and $Syl_p(G)$ is the set of all Sylow p–subgroups of G.

We need some notation related to the action of a group D of permutations on a set X. If $(x, d) \in X \times D$, $C_D(x) := \{e \in D \mid x^e = x\}$ is a subgroup of D and $C_X(d) := \{y \in X \mid y^d = y\} \subseteq X$. The D–orbit of x in X is the set $O_D(x) := \{x^e \mid e \in D\}$ and $|O_D(x)| = |D : C_D(x)|$. We write $T_D(X)$ for a set of representatives for the D–orbits in X and $t_D(X) := |T_D(X)|$.

A D–map is a map f defined on X, which takes complex values and it is constant on the D–orbits in X. If f is a D–map, then

$$\sum_{x \in X} f(x) = \sum_{x \in T_D(X)} |D : C_D(x)| f(x).$$

Now our A acts naturally on G with $\alpha \in A$ sending $x \in G$ to $x^\alpha := \alpha(x)$. The A–orbit of $x \in G$ is $O_A(x) = \{x^\alpha \mid \alpha \in A\}$ and $x^{-1}O_A(x) = \{[x, \alpha] \mid \alpha \in A\} \subseteq C$.

We let $T := T_A(G)$ and $t(G) := t_A(G) = |T|$. If $\alpha \in A$, $C_G(\alpha)$ is a subgroup of G and so the intersection of all subgroups $C_G(\alpha)$ for $\alpha \in A$ is the group of all fixed points of A in G, denoted here by F.

Note that A also acts naturally on the set $G/F := \{xF \mid x \in G\}$ of the left cosets of F in G, with $\alpha \in A$ sending $xF \in G/F$ to $(xF)^\alpha := x^\alpha F$. We write $t(G/F) := t_A(G/F)$. But A also acts on A via conjugation and for $\alpha, \beta \in A$ we let $\alpha^\beta := \beta^{-1}\alpha\beta$.

Another action is that of A on $G \times A$, where $(x, \alpha)^\beta := (x^\beta, \alpha^\beta)$. The action of A on $G \times A$ induces an action of A on C, with $\beta \in A$ sending $[x, \alpha] \in C$ to $[x^\beta, \alpha^\beta]$. Recall the definition of $S(g)$ and observe that this action implies $s(g) = s(g^\beta)$ for every $g \in G$ and $\beta \in A$. The map $g \to s(g)$ is thus an A–map.

Observe that the map $g \to s(g)$ is also a *measure* on G and as such, if $X \subseteq G$, $s(X) := \sum_{x \in X} s(x)$. It is clear that $s(G) = |G||A|$ and that $s(X) = s(X \cap C)$. As a consequence of the Cauchy–Frobenius lemma, $s(1) = |A|t(G)$.

If $g \in C$, the set $S(g)$ of all pairs (x, α) with $g = [x, \alpha]$ is a relation in $G \times A$. We let $L(g)$ denote the projection of $S(g)$ in G and $R(g)$ denote the projection of $S(G)$ in A. Simply said, $L(g)$ is the set of all x's and $R(g)$ is the set of all α's appearing in the equality $g = [x, \alpha]$. It is clear that $L(1) = G$, $R(1) = A$ and that if $g \in C^\star$, then $L(g) \subseteq G \setminus F$ and $R(g) \subseteq A^\star$.

Moreover, if $g = [x, \alpha]$, then $|\{\beta \in A \mid g = [x, \alpha] = [x, \beta]\}| = |C_A(x)|$ and $|\{y \in G \mid g = [x, \alpha] = [y, \alpha]\}| = |C_G(\alpha)|$.

If $g = [x, \alpha] \in F \cap C$ and if $f \in F$, then $g^f = [xf, \alpha] \in F \cap C$ and so, if $F \cap C$ is a group, then it is a normal subgroup of F. Also, if $g = [x, \alpha] \in F \cap C$, an induction argument gives $g^k = [x, \alpha^k]$ for every integer k. In particular, $|g|$ divides $|\alpha|$ and this means that $F \cap C = 1$ whenever $(|F|, |A|) = 1$.

If $g \in C$, then $C_A(g)$ plays a crucial rôle in the determination of $s(g)$ and we let $B := C_A(g)$.

The key remark needed to determine $s(g)$ is that if $g = [x, \alpha]$ and if $\beta \in B$ we have $[x, \alpha] = g = g^\beta = [x^\beta, \alpha^\beta]$. This shows that B leaves both $L(g)$ and $R(g)$ invariant and so both $L(g)$ and $R(g)$ are unions of B-orbits. We let $U(g) := T_B(L(g))$ denote a set of representatives for the B–orbits in $L(g)$ and $V(g) := T_B(R(g))$ denote a set of representatives for the B–orbits in $R(g)$.

In the unpublished note [8], I.M. Isaacs made the B-remark indicated in the previous paragraph in the particular context of G acting on G via conjugation. He proved that $|C_G(g)|$ divides $|G|Frob(g)$ in an elementary way, avoiding characters and using the fact that $C_G(g)$ acts on what is denoted here by $L(g)$. His argument is adapted here for the general case and for both $L(g)$ and $R(g)$ in the short proof of Theorem 3.1.

We now list some needed results. They are all related to the pair (G, A). The first result appears in [3].

Lemma 2.1. *If $\alpha \in A$, then*

$$|C_G(\alpha)| = |C_I(\alpha)||Z : Z_\alpha|,$$

where $Z_\alpha := Z \cap \{[g, \alpha] \mid g \in G\}$ is a subgroup of Z.

Let Ω denote the set of all A–orbits in G. It was observed in [2] that F acts on Ω, with $f \in F$ sending $O_A(x) \in \Omega$ to $O_A(xf) = O_A(x)f$. Let $C_F(O_A(x))$ denote the stabilizer in F of $O_A(x)$.

Lemma 2.2. $C_F(O_A(x)) = F \cap x^{-1}O_A(x) \cong C_A(xF)/C_A(x)$.

From [5] we collect something needed in the proof of Theorem 3.3.

Lemma 2.3. $t_F(\Omega) = t(G/F)$.

Using 2.2, M. Isaacs proved in [7] that if $K := \langle C \rangle$, then:

Lemma 2.4. *$t(G)$ is a multiple of $|F/(F \cap K)|$.*

Finally, the next result was obtained in [4]:

Lemma 2.5. *$s(F) = s(F \cap C) = |A||F|t(G/F)$.*

As a consequence, since clearly $s(F) \geq s(1)$, we derive that $|F|t(G/F) \geq t(G)$ and the equality holds if and only if $F \cap C = 1$.

3 Results

The first main result expresses $s(g)$ in terms of centralizers.

Theorem 3.1. *Let $g \in C$ and $B = C_A(g)$. Then:*

$$\sum_{x \in U(g)} \frac{|C_A(x)|}{|C_B(x)|} = \frac{s(g)}{|B|} = \sum_{\alpha \in V(g)} \frac{|C_G(\alpha)|}{|C_B(\alpha)|}$$

and so

$$s(g) = \sum_{x \in U(g)} |C_A(g)C_A(x)|.$$

Proof. The proof depends on some of the previous remarks; we compute the value of $s(g)$ in two ways, after observing that $x \to |C_A(x)|$ is a B–map on $L(g)$ and $\alpha \to |C_G(\alpha)|$ is a B–map on $R(g)$.

On one hand,

$$s(g) = \sum_{x \in L(g)} |C_A(x)| = \sum_{x \in U(g)} |B : C_B(x)||C_A(x)| = |B| \sum_{x \in U(g)} |C_A(x) : C_B(x)|.$$

It follows that

$$\frac{s(g)}{|B|} = \sum_{x \in U(g)} |C_A(x) : C_B(x)|$$

is a positive integer and so $|B| = |C_A(g)|$ divides $s(g)$. On the other hand,

$$s(g) = \sum_{\alpha \in R(g)} |C_G(\alpha)| = \sum_{\alpha \in V(g)} |B : C_B(\alpha)||C_G(\alpha)| = |B| \sum_{\alpha \in V(g)} \frac{|C_G(\alpha)|}{|C_B(\alpha)|},$$

whence

$$\frac{s(g)}{|B|} = \sum_{\alpha \in V(g)} \frac{|C_G(\alpha)|}{|C_B(\alpha)|}.$$

The last equality in the statement is just a rewriting of the first and the proof is complete. □

More can be said in particular cases; some of the consequences of Theorem 3.1 are presented here as a string of remarks.

a) If $g \in C$ and if $B = C_A(g) = 1$, Theorem 3.1 gives

$$s(g) = \sum_{x \in U(g)} |C_A(x)| = \sum_{\alpha \in V(g)} |C_G(\alpha)|.$$

In this situation, $|O_A(g)| = |A|$, $g \in C^\star$ and so $|C| > |A|$.

b) If $g \in F \cap C$, then $B = C_A(g) = A$ and the B-orbits in $L(g)$ are in fact A–orbits. By Theorem 3.1, $s(g) = |A||U(g)|$. If $g = 1$, this gives the well-known result $s(1) = |A|t(G)$ mentioned already in Section 2. When $1 \neq g \in F \cap C$, then $s(g) = |A||U(g)| \leq |A|(t(G) - |F|)$.

c) If A is abelian, or, more generally, if $B = C_A(g) \leqslant Z(A)$, then $C_B(\alpha) = B$ for every $\alpha \in V(g)$ and Theorem 3.1 implies that $|F|$ divides $s(g) = \sum_{\alpha \in V(g)} |C_G(\alpha)|$.

d) Theorem 3.1 shows that $|C_A(g)|$ divides $s(g)$ for every $g \in G$ – recall that $s(g) = 0$ for $g \notin C$. Therefore, the fraction $m(g) := \frac{|A| - s(g)}{|C_A(g)|}$ is an integer and our previous remarks show that $g \to m(g)$ is an A–map on G.

e) We are now in a position to give a group–theoretical interpretation for the rational number $Frob(g)$ when $g \in \mathcal{C}$.

To do this we need first to break the symmetry involved in the definition of $\mathcal{C}$. This is done by considering $A := I$ and observing that $\mathcal{C} = C = C(G, I) = \{[x, \alpha] \mid (x, \alpha) \in G \times I\}$. Then, $s(g) = s_I(g) = |\{(x, \alpha) \in G \times I \mid g = [x, \alpha]\}| = \frac{1}{|Z|}|G|Frob(g) = |I|Frob(g)$ and so $Frob(g) = \frac{1}{|I|}s(g)$. Combining Lemma 2.1 with Theorem 3.1 one obtains

$$Frob(g) = \frac{1}{|I|} \sum_{x \in U(g)} |C_I(g) C_I(x)| = \frac{1}{|I|} \sum_{\alpha \in V(g)} |C_I(g) C_I(\alpha)||Z : Z_\alpha|.$$

If $1 \neq g \in Z \cap \mathcal{C}$ we can say much more. With $A := I$ we have $F := Z$ and $C := \mathcal{C}$. Remark b) gives $Frob(g) = |U(g)| \leq k(G) - |Z|$ since in this particular case $t(G) = k(G)$.

Lemma 2.1 combined with Theorem 3.1 shows that $Frob(g) = \sum_{\alpha \in V(g)} |Z : Z_\alpha|$. Since $Z_\alpha \subseteq Z \cap \mathcal{C}$, one obtains that $Frob(g) \geq |V(g)| \frac{|Z|}{|Z \cap \mathcal{C}|}$.

Summarizing, if $1 \neq g \in Z \cap \mathcal{C}$, $Frob(g)$ is an integer and

$$|V(g)| \frac{|Z|}{|Z \cap \mathcal{C}|} \leq Frob(g) \leq k(G) - |Z|.$$

As we will see later, this string of inequalities is sharp, for there are examples when these inequalities become equalities for *every* $g \in (Z \cap \mathcal{C})^\star$.

f) In simple cases when G is abelian we can compute the value of $s(g)$ exactly. For example, let $G = (Z_n, +)$ be the additive group of residues classes modulo n and let $A := Aut(G)$. Then $|A| = \varphi(n)$ where φ is Euler's totient function and T can be taken as the set $\{[d] \mid d|n\}$ where $[d]$ is the residue class corresponding to d. It is then sufficient to compute $s([d])$. As simple as it looks, this seems to be a hard exercise in combinatorial modular number theory.

The second main result is surprising. It holds for every non–trivial finite group G and for every subgroup A of $Aut(G)$, that is, including the trivial case when $A = 1$.

It shows that for an arbitrary $A \leqslant Aut(G)$ the integer $m(g)$ defined in remark d) has an unexpected property: the average value of $m(g)$ as g runs over $T^\star$ is equal to 1. As such, it is a far reaching extension of the main result in [1].

Theorem 3.2. *Let G be a non-trivial group and let $A \leqslant Aut(G)$. Then,*

$$\sum_{g \in T^\star} m(g) = |T^\star|.$$

Proof. Recalling the definition of $m(g)$, it is clear that the statement is true if $A = 1$, so there is no loss in assuming that $A \neq 1$.

It is clear that

$$|G| = 1 + \sum_{g \in T^\star} |A : C_A(g)|.$$

Since $g \to m(g)$ is an A–map,

$$|G||A| = s(G) = \sum_{g \in G} s(g) = \sum_{g \in T} |A : C_A(g)| s(g) = |A| \sum_{g \in T} \frac{s(g)}{|C_A(g)|}$$

and one obtains

$$|G| = \sum_{g \in T} \frac{s(g)}{|C_A(g)|} = \frac{s(1)}{|A|} + \sum_{g \in T^\star} \frac{s(g)}{|C_A(g)|} = t(G) + \sum_{g \in T^\star} \frac{s(g)}{|C_A(g)|}.$$

Comparing the two expressions for $|G|$ gives the stated equality since $|T^\star| = t(G) - 1$. □

In the important particular case when $|A| = p$ and $F = 1$ it is easy to check that $m(g) = 1$ for every $g \in T^\star$.

Returning to our general pair (G, A) and, more specifically, to the natural action of A on G, two very special kinds of actions were considered in the literature over the past 100 years.

The first kind of action is the fixed–point–free action in which $F = 1$. Then, there is the co-prime action, in which $|A|$ and $|G|$ are co-prime. The interested reader may consult [6] for details. Both these types of actions, as we have seen in Section 2, imply that $F \cap C = 1$. However, depending on the choice of A, one is rather frequently forced to deal with the situation in which $F \cap C \neq 1$; group co-homology is just one important example that comes to mind.

The variety of situations covered here makes it impossible to determine exactly $|F \cap C|$. As such, even a lower general bound for $|F \cap C|$, depending only on G and A, would be of interest. The third main result gives such a sharp lower bound for $|F \cap C|$. As usual in abstract algebra, general inequalities are always saying something interesting about structure in the equality case. This is in striking contrast with the inequalities concerning real numbers, where the equality usually holds only in trivial situations.

Theorem 3.3.

$$|F \cap C| \geq \frac{|F|(t(G/F) - 1)}{t(G) - |F|}$$

and the equality holds if and only if $F \cap C = 1$ or one of the following equivalent conditions is satisfied:

i) $s(g) = |A|(t(G) - |F|)$ for every $g \in (F \cap C)^\star$.

ii) $F \cap C = x^{-1} O_A(x)$ for every $x \in G \setminus F$.

Proof. The remarks made in Section 2 show that the inequality we have to prove becomes an equality when $F \cap C = 1$.

From now on, we assume that $F \cap C \neq 1$ and write $Y := (F \cap C)^\star$. Everything we need here was presented in Section 2. The remarks in Section 2 imply that $s(Y) = |A|(|F|t(G/F) - t(G))$. If $g \in Y$, remark b) implies that $s(g) \leq |A|(t(G) - |F|)$. Thus $s(Y) \leq |Y||A|(t(G) - |F|)$ and a short calculation gives the stated inequality and also shows that equality holds if and only if i) is satisfied.

To prove ii), we need the information from Lemma 2.2 and lemma 2.3. Observe first that

$$t(G) = |\Omega| = \sum_{O_A(x) \in T_F(\Omega)} |F : C_F(O_A(x))|.$$

By lemma 2.2, $C_F(O_A(x)) = F \cap x^{-1}O_A(x) \subseteq F \cap C$, so $|F : C_F(O_A(x))| \geq \frac{|F|}{|F \cap C|}$.

Now, recalling these definitions, $C_F(O_A(1)) = 1$, so $|O_F(O_A(1))| = |F|$. Also, by Lemma 2.3, $|T_F(\Omega)^\star| = t(G/F) - 1$. Putting all these observations together, a short calculation shows that

$$t(G) - |F| \geq (t(G/F) - 1)\frac{|F|}{|F \cap C|}.$$

This proves the general inequality in the case $F \cap C \neq 1$ and also shows that equality holds if and only if ii) is satisfied. ◻

Some remarks are needed to better understand the meaning of Theorem 3.3.

g) The bound given in Theorem 3.3 is clearly a sharp one, for we get equality in Theorem 3.3 whenever $F \cap C = 1$. A very simple example shows that the bound can be attained even when $F \cap C \neq 1$. Just take $G := S_3$ denote the smallest non–abelian group and let A be the subgroup of order 3 of $Aut(G) = Inn(G)$. In this case, $F = C \cong A$ is the normal subgroup of G of order 3, $t(G) = 4$ and $t(G/F) = 2$.

If G is a p-group of order p^2 or p^3, it is also easy to verify that if $|A| = p$, then A acts uniformly on G.

h) There are many instances when $I := Inn(G)$ satisfies the extreme case of equality in Theorem 3.3, so it seems that equality holds in Theorem 3.3 rather frequently. As explained before stating Theorem 3.3, we are interested here only in the case $Z \cap \mathcal{C} \neq 1$.

This suggests the following definition: we say that A acts $uniformly$ on G if $F \cap C \neq 1$ and also $|F \cap C|(t(G) - |F|) = |F|(t(G/F) - 1)$.

Theorem 3.3 and one of our preliminary remarks in Section 2 imply that if A acts uniformly on G, then $F \cap C$ is a normal subgroup of F. Moreover, if also $F \neq N_G(F)$, then $F \cap C = F \cap x^{-1}O_A(x)$ for some $x \in N_G(F) \setminus F$. It is easy to verify that $x^{-1}O_A(x) \subseteq C_G(F)$ and in this case $F \cap C$ is a subgroup of $Z(F)$.

We will consider two familiar cases in which we have control over the action of A: they are $|A| = p$ and $A = I$.

When $|A| = p$ the situation is almost under complete control.

Corollary 3.4. *If $|A| = p$, then A acts uniformly on G if and only if $p = |C| = |F \cap C| \geq |G/F|$. Ipso facto, C is a normal subgroup of G.*

Proof. Since $|A| = p$, it follows that $t(G) = |F| + \frac{|G|-|F|}{p}$ and that if $x \in G \setminus F$, then $|x^{-1}O_A(x)| = |A| = p$.

If A acts uniformly on G and $g \in (F \cap C)^\star$, Theorem 3.3 gives $s(g) = |A|(t(G) - |F|) = |G| - |F|$. Also, remark c) implies $s(g) = |F||V(g)|$ since $F = C_G(\alpha)$ for every $\alpha \in V(g)$.

Thus $|V(g)| = |G/F| - 1 \leq |A| - 1 = p - 1$ and $|G/F| \leq p$. This forces A to act trivially on G/F. Consequently, $C \subseteq F$ and, since the action is uniform, $|C|(t(G) - |F|) = |F|(t(G/F) - 1) = |F|(|G/F| - 1) = |G| - |F| = p(t(G) - |F|)$, so $|C| = p$.

Conversely, suppose that $|C| = p$ and $|G/F| \leq p$. Then A acts trivially on G/F, so $C \subseteq F$. If $x \in G \setminus F$, then $F \cap x^{-1}O_A(x) = x^{-1}O_A(x)$ is a non–trivial subgroup of F by Lemma 2.2 and it is obviously contained in C. This forces $C = F \cap C = F \cap x^{-1}O_A(x)$ for every $x \in G \setminus F$ and Theorem 3.3 ensures that A acts uniformly on G.

Finally, if $g = [x, \alpha] \in C^\star$, observe that $L(g) = G \setminus F$. Indeed, $s(g) = |A||U(g)| = |L(g)| = p(t(G) - |F|) = |G| - |F|$.

Also, $[x^2, \alpha] = [x, \alpha]^x[x, \alpha]$, so $[x, \alpha]^x = g^x = [x^2, \alpha][x, \alpha]^{-1} \in C$. Thus $g^x \in C$ for every $x \in L(G) = G \setminus F$. Since $G = \langle G \setminus F \rangle$ we obtain that C is a normal subgroup of G, as asserted. ∎

Corollary 3.5. *If $G' \neq 1$ and I acts uniformly on G, then there exists $p \in \pi(G)$ and $P \in Syl_p(G)$ such that $G = ZP$. In this case $Z \cap \mathcal{C}$ is an elementary abelian p–group.*

Proof. In this case $F = Z$, $C = \mathcal{C}$, $t(G) = k(G)$ and $t(G/F) = k(G/Z)$. Also, $F \cap x^{-1}O_A(x) = Z \cap x^{-1}Cl_G(x) = Z \cap \{[x, g] \mid g \in G\} = Z \cap \{[g, x] \mid g \in G\} = Z_{i_x}$ is a subgroup of Z.

Pick $a \in G \backslash Z$ such that $a^p \in Z$ where $p \in \pi(G/Z)$ and let $\alpha := i_a$. Since I acts uniformly on G, Theorem 3.3 ensures that $Z \cap \mathcal{C} = Z_\alpha$. Since $|\alpha| = p$, $Z \cap \mathcal{C}$ is an elementary abelian p–subgroup of Z and p is the unique prime dividing $|G/Z|$. Thus G/Z is a p–group and G is nilpotent. Moreover, if $P \in Syl_p(G)$, then $G = ZP = O_{p'}(G) \times P$ and $O_{p'}(G)$ is contained in Z. ∎

We complete the long chain of remarks by adding a few more.

i) When $|A| = p$ and A acts uniformly on G, Corollary 3.4 ensures that $|G/F| \leq p$ and we have seen in remark g) that it is possible to have $|G/F| < p$. When G is a p–group, then $|G/F| = p$ and C is contained in $Z(G)$.

It is not clear if F is a normal subgroup of G when G is not a p–group and an interesting question is to see if the hypothesis of Corollary 3.4 ensures the solvability of G.

j) Corollary 3.5 reduces the situation when I acts uniformly on G to the case when G is a p–group. When G is a p–group of nilpotency class two and I acts uniformly on G, then $\mathcal{C} \subseteq Z$ and Theorem 3.3 implies that $|\mathcal{C}| = |Cl_G(x)|$ for every $x \in G \setminus Z$. It follows that

$$|\mathcal{C}| = \frac{|G| - |Z|}{k(G) - |Z|}.$$

k) As mentioned in Lemma 2.4, $t(G)$ is always a multiple of $|F/(F \cap K)|$ where $K = \langle C \rangle$. When A acts uniformly on G , the definition given in remark h) shows that $t(G)$ is a multiple of $\frac{|F|}{|F \cap C|}$.

l) When $|A| = p$ and $F \cap C \neq 1$, all non-trivial elements in $F \cap C$ have order p and so $F \cap C$ is a union of n cyclic groups of order p, where $n = \frac{|F \cap C|-1}{p-1}$. If X/F is the set of all fixed points of A in G/F and if $g \in (F \cap C)^\star$, observe that $\langle g \rangle = \{[x, \alpha] \mid \alpha \in A\}$ for at least one element x of $X \setminus F$ and thus $n \leq |X/F| - 1$.

If $|A| = 2$ and $F \cap C \neq 1$, then $|V(g)| = |A^\star| = 1$ for every $g \in (F \cap C)^\star$ and one obtains $|F \cap C| = |X/F| \leq |G/F|$.

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