Bull. Math. Soc. Sci. Math. Roumanie
Tome 67 (115), No. 3, 2024, 355–373

Mean value of Ramanujan sum and Cochrane sums

by
Xiao Wang(1), Jiankang Wang(2), Zhefeng Xu(3)

Abstract

The main purpose of this paper is to study the mean value of Ramanujan sum $R_q(a)$ and generalized Cochrane sum $C(h,q,m,n)$ over incomplete intervals. Let $\sum_a'$ denote the summation over all a such that $(a,q)=1$, k,l,s be positive integers and λ_1,λ_2 be real numbers with $0<\lambda_1,\lambda_2\le 1$. Define

$$W_q(a,h,k,m,n)=\sum_{a=1}^{q}{}' C(ah,q,m,n)R_q^k(a+1).$$

Some interesting mean value formulas for

$$\sum_{b\le\lambda_1 q}{}' \sum_{d\le\lambda_2 q}{}' b^l d^s W_q(a,bd,k,m,n)$$

will be given by using mean value of Dirichlet L-function.

Key Words: Dirichlet L-function, Ramanujan sum, Cochrane sums, mean value.
2020 Mathematics Subject Classification: Primary 11M20; Secondary 11L05, 11F20.

1 Introduction

For positive integers q and c, the Ramanujan sum $R_q(c)$ is defined as

$$R_q(c)=\sum_{a=1}^{q}{}' e^{\frac{2\pi iac}{q}}=\sum_{d|(c,q)} d\mu\left(\frac{q}{d}\right),$$

where $\mu(n)$ is Möbius function and $\sum_a'$ denotes the summation over all a such that $(a,q)=1$.

The Ramanujan sum is an interesting and important object in number theory, which can be used to solve the problem of the number of solutions of congruence equations(see [1,8]). At the same time, it has close relationship with Dedekind sum and Hardy sums.

In 2005, H. Liu and W. Zhang [5] obtained the mean value of Ramanujan sum and Dedekind sum:

$$s(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where q is a positive integer, h is an arbitrary integer and

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The result is as follow:

Proposition 1.1 *Let $q \geq 3$ be a square-full number (i.e., for any prime p, $p \mid q$, if and only if $p^2 \mid q$), we have*

$$\sum_{h=1}^{q}{}' s(h,q) R_q(h-1) = \frac{\phi^2(q)}{12} \prod_{p \mid q} \left(1 + \frac{1}{p}\right),$$

where $\phi(q)$ is Euler function.

In 2014, D. Han and W. Zhang [4] proved the following identity:

Proposition 1.2 *For any odd prime p and any integer $\alpha \geq 2$, we have*

$$\begin{aligned} &\sum_{h=1}^{p^\alpha}{}' s(h,p^\alpha) R_{p^\alpha}^2(h+1) \\ &= -\frac{1}{12} p^{2\alpha} \phi(p^\alpha) \left[\left(1 - \frac{1}{p^\alpha}\right)\left(1 - \frac{2}{p^\alpha}\right) - \frac{1}{p}\left(1 - \frac{1}{p^2}\right)\right]. \end{aligned}$$

For the results on Hardy sums see [2, 3, 10, 13].

In 2000, Cochrane introduced the following sum analogous to the Dedekind sum,

$$C(h,q) = \sum_{a=1}^{q}{}' \left(\left(\frac{\bar{a}}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where $\bar{a}$ is defined by the equation $a\bar{a} \equiv 1 \pmod q$. Furthermore, the generalized Cochrane sum is

$$C(h,q,m,n) = \sum_{a=1}^{q}{}' \bar{B}_m\left(\frac{\bar{a}}{q}\right) \bar{B}_n\left(\frac{ah}{q}\right),$$

where

$$\bar{B}_n(x) = \begin{cases} B_n(x - [x]), & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

called the n-th periodic Bernoulli function. Clearly, $C(h,q;1,1) = C(h,q)$ is the classical Cochrane sum.

Naturally, we might consider whether the mean value of Ramanujan sum and Cochrane sums would yield similar results? In fact, in the case of one variable, it is difficult to get a asymptotic formula for the mean value as

$$\sum_{h=1}^{q}{}' C(h,q,m,n) R_q^k(h+1).$$

Let k, l, s be positive integers and λ_1, λ_2 be real numbers with $0 < \lambda_1, \lambda_2 \leq 1$. In this paper, we will study the mean value

$$\sum_{b\leq\lambda_1 q}{}' \sum_{d\leq\lambda_2 q}{}' b^l d^s W_q(a, bd, k, m, n),$$

where

$$W_q(a, h, k, m, n) = \sum_{a=1}^{q}{}' C(ah, q, m, n) R_q^k(a+1),$$

by using mean value theorems of Dirichlet L-function. Noting that if $(h, q) = 1$,

$$\begin{aligned} C(h, q, n, m) &= \sum_{a=1}^{q}{}' \bar{B}_n\left(\frac{\bar{a}}{q}\right) \bar{B}_m\left(\frac{ah}{q}\right) = \sum_{a=1}^{q}{}' \bar{B}_n\left(\frac{a}{q}\right) \bar{B}_m\left(\frac{\bar{a}h}{q}\right) \\ &= \sum_{a=1}^{q}{}' \bar{B}_n\left(\frac{ah}{q}\right) \bar{B}_m\left(\frac{\bar{a}}{q}\right) = C(h, q, m, n), \end{aligned}$$

without losing generality we assume $m \geq n$. Denote

$$\xi(m, n, q) = \frac{\zeta^2(n+1)\zeta^2(m+1)}{\zeta(n+m+2)} \prod_{p|q} \frac{(p^{m+1}-1)^2(p^{n+1}-1)^2}{p^{m+n+2}(p^{m+n+2}-1)},$$

where $\zeta(s)$ is the Riemann-zeta function. We draw the following conclusions:

Theorem 1.1 *For any positive odd integers m, n with $m \geq n$,*

(i) if $q \geq 3$ is a square-full integer, then we have

$$\sum_{b=1}^{q}{}' \sum_{d=1}^{q}{}' bd W_q(a, bd, 1, m, n) = \frac{2m!n!q^4\phi(q)}{(2\pi i)^{m+n}\pi^2}\xi(m, n, q) + O\big(q^{4+\varepsilon}\big);$$

(ii) if $q \geq 3$ is an odd square-full integer, then we have

$$\sum_{b\leq\frac{q}{4}}{}' \sum_{d\leq\frac{q}{4}}{}' W_q(a, bd, 1, m, n) = \frac{m!n!q^2\phi(q)}{(2\pi i)^{m+n}\pi^2} H(m, n)\xi(m, n, q) + O\big(q^{2+\varepsilon}\big);$$

(iii) if $q > 3$ is a square-full integer and $3 \nmid q$, then we have

$$\sum_{b\leq\frac{q}{3}}{}' \sum_{d\leq\frac{q}{3}}{}' W_q(a, bd, 1, m, n) = \frac{9m!n!q^2\phi(q)}{2(2\pi i)^{m+n}\pi^2}\xi(m, n, 3q) + O\big(q^{2+\varepsilon}\big);$$

(iv) if $q > 3$ is an odd square-full integer and $3 \nmid q$, then we have

$$\sum_{b\leq\frac{q}{4}}{}' \sum_{d\leq\frac{q}{3}}{}' W_q(a, bd, 1, m, n) = \frac{3m!n!q^2\phi(q)}{(2\pi i)^{m+n}\pi^2} Q(m, n)\xi(m, n, q) + O\big(q^{2+\varepsilon}\big);$$

where

$$H(m,n) = \frac{2^{-m-3n-1} + 2^{-3m-n-1}(2^m+1)^2 + 2^{-2m-2n-1}(2^{m+1}+1)}{2^{m+n+2}-1} \times \frac{(2^{m+1}-1)^2(2^{n+1}-1)^2}{2^{m+n+2}-1},$$

$$Q(m,n) = \frac{(2^{m+1}-1)(2^{n+1}-1)(1+2^{-m}+2^{-n}-2^{-m-n-1})}{2^{m+n+2}-1} \times \frac{(3^{m+1}-1)(3^{n+1}-1)}{3^{m+n+2}-1}.$$

Theorem 1.2 *Let α be a positive integer and $H(m,n)$ be the same as before, for any positive odd integers m,n with $m \geq n$, we have*

(i) if $p \geq 3$ is a prime,

$$\sideset{}{'}\sum_{b=1}^{p^\alpha} \sideset{}{'}\sum_{d=1}^{p^\alpha} bdW_{p^\alpha}(a,bd,2,m,n) = \frac{2m!n!p^{5\alpha}\phi(p^\alpha)}{(2\pi i)^{m+n}\pi^2} \cdot \frac{2p^{m+n-1}-p^{m+n}-1}{p-p^{m+n}} \xi(m,n,p^\alpha) + O\big(p^{5\alpha+\varepsilon}\big);$$

(ii) if $p \geq 3$ is an odd prime,

$$\sideset{}{'}\sum_{b\leq \frac{q}{4}} \sideset{}{'}\sum_{d\leq \frac{q}{4}} W_{p^\alpha}(a,bd,2,m,n) = \frac{m!n!p^{3\alpha}\phi(p^\alpha)}{(2\pi i)^{m+n}\pi^2} \cdot \frac{2p^{m+n-1}-p^{m+n}-1}{p-p^{m+n}} H(m,n)\xi(m,n,p^\alpha) + O\big(p^{3\alpha+\varepsilon}\big).$$

From Theorems 1.1 and 1.2, we immediately deduce the following corollary:

Corollary 1.3 *Let $p > 3$ be a prime, then we have*

$$\sum_{a\leq p-1} \sideset{}{'}\sum_{b\leq p/3} \sideset{}{'}\sum_{d\leq p/3} C(abd,p)R_p(a+1) = -\frac{4}{81}p^3 + O\big(p^{2+\varepsilon}\big),$$

$$\sum_{a\leq p-1} \sideset{}{'}\sum_{b\leq p/4} \sideset{}{'}\sum_{d\leq p/3} C(abd,p)R_p(a+1) = -\frac{3}{64}p^3 + O\big(p^{2+\varepsilon}\big),$$

$$\sum_{a\leq p-1} \sideset{}{'}\sum_{b\leq p/4} \sideset{}{'}\sum_{d\leq p/4} C(abd,p)R_p^2(a+1) = -\frac{45}{1024}p^4 + O\big(p^{3+\varepsilon}\big),$$

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} bdC(abd,p)R_p^2(a+1) = -\frac{5}{144}p^6 + O\big(p^{5+\varepsilon}\big).$$

Let $s \leq 2$ and $l \leq 2$, for any positive integer k, the mean value

$$\sideset{}{'}\sum_{b=1}^{p^\alpha} \sideset{}{'}\sum_{d=1}^{p^\alpha} b^l d^s W_{p^\alpha}(a,bd,k,m,n)$$

can be determined in the same way of Theorem 1.2. But the calculation will be very complicated if $k \geq 4$.

2 Some Lemmas

Before proving the main theorems, we give some useful lemmas in this section.

Lemma 2.1 *Let integer $q \geq 3$ and $(h, q) = 1$. Then for any odd numbers m, n, we have*

$$C(h, q; m, n) = \frac{4m!n!}{(2\pi i)^{m+n}\phi(q)} \sum_{\substack{\chi \ mod \ q \\ \chi(-1)=-1}} \bar{\chi}(h) \left(\sum_{r=1}^{\infty} \frac{G(\chi, r)}{r^m} \right) \left(\sum_{s=1}^{\infty} \frac{G(\chi, s)}{s^n} \right),$$

where χ denotes an odd Dirichlet character modulo q and $G(\chi, n) = \sum_{b=1}^{q} \chi(b) e(\frac{bn}{q})$ denotes the Gauss sum corresponding to χ.

Proof See Theorem 2.1 of [6]. □

Lemma 2.2 *Let $p \geq 3$ be a prime and α, i be positive integers with $\alpha \geq 2$ and $i < \alpha$. Then we have the identity*

$$\sum_{a=1}^{p^\alpha}{}' \bar{\chi}(a) R_{p^\alpha}^2(a+1) = \begin{cases} \chi(-1) p^{2\alpha} \left(1 - \frac{2}{p}\right), & \text{if } \chi \text{ is a primitive character mod } p^\alpha, \\ \chi(-1) p^{2\alpha} \left(1 - \frac{1}{p}\right), & \text{if } \chi \text{ is a primitive character mod } p^i. \end{cases}$$

Note: if $\alpha = 1$, then we have

$$\sum_{a=1}^{p}{}' \bar{\chi}(a) R_p^2(a+1) = \chi(-1) p(p-2).$$

Proof See Lemma 1 of [4]. □

Lemma 2.3 *Let q be a positive integer, χ be a primitive character modulo q with $\chi(-1) = -1$. Then we have the identity*

$$\sum_{b=1}^{q} b\bar{\chi}(b) = \frac{qi}{\pi} \tau(\bar{\chi}) L(1, \chi),$$

where $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof See [9]. □

Lemma 2.4 *Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \ mod \ q}{}^{*} \chi(r) = \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d),$$

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right),$$

where $\sum_{\chi \bmod q}^{*}$ *denotes the summation over all primitive characters mod q, and J(q) denotes the number of all primitive characters mod q.*

Proof See Lemma 3 of [15]. □

Lemma 2.5 *Suppose χ is an odd character modulo q, generated by the primitive character χ_m modulo m. Then we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{q}{m}\left(\prod_{\substack{p|q \\ p\nmid m}}(1-\chi_m(p))\right)\left(\sum_{a=1}^{m} a\chi_m(a)\right).$$

Proof See Lemma 6 of [12]. □

Lemma 2.6 *For any positive integer k, let χ be a non-primitive character modulo k, and k^* denote the conductor of χ with $\chi = \chi_1\chi_{k^*}^*$. If $(n,k) > 1$, then we have*

$$G(\chi,n) = \begin{cases} \bar{\chi}^*\left(\frac{n}{(n,k)}\right)\chi^*(\frac{k}{k^*(n,k)})\mu(\frac{k}{k^*(n,k)})\phi(k)\phi^{-1}(\frac{k}{(n,k)})\tau(\chi^*), & \text{if } k^* = \frac{k_1}{(n,k_1)}, \\ 0, & \text{otherwise,} \end{cases}$$

where χ_1 denotes the principal character modulo k, k_1 is the largest divisor of k that has the same prime factors with k^*. If $(n,k) = 1$, then we have

$$G(\chi,n) = \bar{\chi}^*(n)\chi^*(\frac{k}{k^*})\mu(\frac{k}{k^*})\tau(\chi^*).$$

If χ be a primitive character modulo k, then

$$G(\chi,n) = \bar{\chi}(n)\tau(\chi).$$

Proof See Lemma 3 of [14]. □

Lemma 2.7 *Let $q > 1$be a square-full number, for any non-primitive character χ mod q, we have the identity*

$$\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right) = 0.$$

Proof See Lemma 5 of [10]. □

Lemma 2.8 *Let q be an odd number and χ be a Dirichlet character modulo q such that $\chi(-1) = -1$. Then we have the identity*

$$\sum_{b=1}^{[\frac{q}{4}]}\chi(b) = \frac{\bar{\chi}(4)-\bar{\chi}(2)-2}{2q}\sum_{b=1}^{q} b\chi(b).$$

Proof See Lemma 3 of [11]. □

Lemma 2.9 *Let $q > 3$ be an integer and $3 \nmid q$. For odd primitive character χ modulo q. Then we have the identity*

$$\sum_{b<\frac{q}{3}} \chi(b) = \frac{3\tau(\chi)}{2\pi i} L(1, \bar{\chi}\chi_3^o).$$

Proof Taking $p = 3$ in Lemma 4 of [7], we have

$$\sum_{b<\frac{q}{3}} \chi(b) = \frac{\tau(\chi)}{\pi i} \left(\left(1 - \frac{\bar{\chi}(3)}{3}\right) L(1, \bar{\chi}) + \frac{L(1, \bar{\chi}\chi_3^o)}{2} \right).$$

Furthermore,

$$\begin{aligned} L(1, \bar{\chi}) =& \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} = \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\bar{\chi}(n)}{n} + \sum_{\substack{n=1 \\ 3|n}}^{\infty} \frac{\bar{\chi}(n)}{n} = L(1, \bar{\chi}\chi_3^o) + \frac{\bar{\chi}(3)}{3} L(1, \bar{\chi}) \\ =& L(1, \bar{\chi}\chi_3^o) \left(1 + \frac{\bar{\chi}(3)}{3} + \left(\frac{\bar{\chi}(3)}{3}\right)^2 + \cdots \right) = \frac{3}{3 - \bar{\chi}(3)} L(1, \bar{\chi}\chi_3^o). \end{aligned}$$

Therefore,

$$\sum_{b<\frac{q}{3}} \chi(b) = \frac{3\tau(\chi)}{2\pi i} L(1, \bar{\chi}\chi_3^o).$$

□

Lemma 2.10 *Let s, t be positive odd integers with $t > s$. For $\sigma_\delta(n)$ is a divisor function and $\sigma_0(n) = \tau(n)$, $\rho(n) = \sum_{d|n} \chi_3^o(d)$, we have*

(i) if $q \geq 3$ is a positive integer, then

$$\sum_{n=1}^{\infty}{}' \frac{\tau(n)\sigma_{s-t}(n)}{n^{s+1}} = \xi(t, s, q); \tag{1}$$

(ii) if $q \geq 3$ is an odd integer, then

$$\sum_{n=1}^{\infty}{}' \frac{\tau(n)\sigma_{s-t}(2^l n)}{n^{s+1}} = \frac{2^{s+1}(2^{t+1}-1)^2 - 2^{t+1} \cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)} \xi(t, s, q); \tag{2}$$

(iii) if $q > 3$ is an odd integer and $3 \nmid q$, then

$$\begin{aligned} \sum_{n=1}^{\infty}{}' \frac{\rho(n)\sigma_{s-t}(2^l n)}{n^{s+1}} =& \frac{2^{s+1}(2^{t+1}-1)^2 - 2^{t+1} \cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)} \\ & \times \frac{(3^{t+1}-1)(3^{s+1}-1)}{3^{s+t+2}-1} \xi(t, s, q). \end{aligned} \tag{3}$$

Proof We give the proof of identity (2.2) as an example here and the others can be proved similarly. Let $\Phi(n)=\tau(n)\sigma_{s-t}(n)$, it is clear that $\Phi(n)$ is a multiplicative function and we can write

$$\sideset{}{'}\sum_{n=1}^{\infty}\frac{\tau(n)\sigma_{s-t}(2^l n)}{n^{s+1}}=\sum_{i=0}^{\infty}\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau(2^i n)\sigma_{s-t}(2^{l+i}n)}{2^{i(s+1)}n^{s+1}}$$
$$=\sum_{i=0}^{\infty}\frac{\tau(2^i)\sigma_{s-t}(2^{l+i})}{2^{i(s+1)}}\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau(n)\sigma_{s-t}(n)}{n^{s+1}}.$$

From Euler's product, we arrive at

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau(n)\sigma_{s-t}(n)}{n^{s+1}}$$
$$=\prod_{p\nmid 2q}\left(1+\frac{\Phi(p)}{p^{s+1}}+\frac{\Phi(p^2)}{p^{2(s+1)}}+\cdots+\frac{\Phi(p^n)}{p^{n(s+1)}}+\cdots\right)$$
$$=\prod_{p\nmid 2q}\left(1+\frac{2(1-p^{2(s-t)})}{p^{s+1}(1-p^{s-t})}+\frac{3(1-p^{3(s-t)})}{p^{2(s+1)}(1-p^{s-t})}+\cdots\right.$$
$$\left.+\frac{(n+1)(1-p^{(n+1)(s-t)})}{p^{n(s+1)}(1-p^{s-t})}+\cdots\right)$$
$$=\prod_{p\nmid 2q}\frac{p^{s+t+2}(p^{s+t+2}-1)}{(p^{t+1}-1)^2(p^{s+1}-1)^2}.$$

Note that $\zeta(s)=\prod_p(1-\frac{1}{p^s})^{-1}$, hence

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\tau(n)\sigma_{s-t}(n)}{n^{s+1}}=\xi(t,s,2q).$$

In addition,

$$\sum_{i=0}^{\infty}\frac{\tau(2^i)\sigma_{s-t}(2^{l+i})}{2^{i(s+1)}}=\sum_{i=0}^{\infty}\frac{i+1}{2^{i(s+1)}}\frac{1-2^{(s-t)(l+i+1)}}{1-2^{s-t}}$$
$$=\frac{2^{s+t+2}[2^{s+1}(2^{t+1}-1)^2-2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2]}{(2^{t+1}-2^{s+1})(2^{s+1}-1)^2(2^{t+1}-1)^2}.$$

Therefore,

$$\sideset{}{'}\sum_{n=1}^{\infty}\frac{\tau(n)\sigma_{s-t}(2^l n)}{n^{s+1}}=\frac{2^{s+1}(2^{t+1}-1)^2-2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)}\xi(t,s,q).$$

□

Lemma 2.11 *Let s be positive odd integer, $\tau(n)$ and $\rho(n)$ be the same as Lemma 2.10,* *(i) if $q \geq 3$ is a positive integer, then we have*

$$\sum_{n=1}^{\infty}{}' \frac{\tau^2(n)}{n^{s+1}} = \xi(s,s,q); \tag{4}$$

(ii) if $q \geq 3$ is an odd integer, then we have

$$\sum_{n=1}^{\infty}{}' \frac{\tau(n)\tau(2^l n)}{n^{s+1}} = \frac{l(2^{s+1}-1)+2^{s+1}+1}{2^{s+1}+1}\xi(s,s,q); \tag{5}$$

(iii) if $q > 3$ is an odd integer and $3 \nmid q$, then we have

$$\sum_{n=1}^{\infty}{}' \frac{\rho(n)\tau(2^l n)}{n^{s+1}} = \frac{l(2^{s+1}-1)+2^{s+1}+1}{2^{s+1}+1} \cdot \frac{3^{s+1}-1}{3^{s+1}+1}\xi(s,s,q). \tag{6}$$

Proof Similar to the proof of Lemma 2.10, we have

$$\sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{\tau^2(n)}{n^{s+1}} = \prod_{p \nmid 2q} \frac{1+\frac{1}{p^{s+1}}}{(1-\frac{1}{p^{s+1}})^3} = \xi(s,s,2q).$$

Moreover,

$$\sum_{i=0}^{\infty} \frac{\tau(2^i)\tau(2^{l+i})}{2^{i(s+1)}} = \sum_{i=0}^{\infty} \frac{(i+1)(l+i+1)}{2^{i(s+1)}} = \frac{(1+l)2^{3(s+1)}+(1-l)2^{2(s+1)}}{(2^{s+1}-1)^3}.$$

Therefore,

$$\sum_{n=1}^{\infty}{}' \frac{\tau(n)\tau(2^l n)}{n^{s+1}} = \frac{l(2^{s+1}-1)+2^{s+1}+1}{2^{s+1}+1}\xi(s,s,q).$$

This proves identity (2.5), the rest of the identities can be proved similarly. □

Lemma 2.12 *Let s,t be positive odd integers with $t > s$, we have the asymptotic formulas,*

(i) if $q \geq 3$ is a positive integer,

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L^2(1,\chi)L(s,\bar{\chi})L(t,\bar{\chi}) = \frac{J(q)}{2}\xi(t,s,q) + O(q^{\varepsilon}); \tag{7}$$

(ii) if $q \geq 3$ is an odd integer,

$$\begin{aligned} &\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2^l)L^2(1,\chi)L(s,\bar{\chi})L(t,\bar{\chi}) \\ =&\frac{J(q)}{2^{ls+1}} \cdot \frac{2^{s+1}(2^{t+1}-1)^2 - 2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)}\xi(t,s,q) + O(q^{\varepsilon}); \end{aligned} \tag{8}$$

(iii) if $q>3$ is a integer and $3\nmid q$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} L^2(1,\chi\chi_3^o)L(s,\bar{\chi})L(t,\bar{\chi}) = \frac{J(q)}{2}\xi(t,s,3q) + O\big(q^{\varepsilon}\big); \tag{9}$$

(iv) if $q>3$ is an odd integer and $3\nmid q$,

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} \chi(2^l)L(1,\chi)L(1,\chi\chi_3^o)L(s,\bar{\chi})L(t,\bar{\chi})\\ =&\frac{J(q)}{2^{ls+1}}\cdot\frac{2^{s+1}(2^{t+1}-1)^2-2^{t+1}2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)}\\ &\times\frac{(3^{t+1}-1)(3^{s+1}-1)}{3^{s+t+2}-1}\xi(t,s,q)+O\big(q^{\varepsilon}\big). \end{aligned}\tag{10}$$

Proof Let

$$A(y,\chi,\delta)=\sum_{N<n\le y}\chi(n)\sigma_\delta(n)$$

where N is a parameter with $N>q$, $\sigma_\delta(n)$ were defined in Lemma 2.10. Then from Abel's identity, we have

$$L^2(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)\tau(n)}{n}=\sum_{1\le n\le N}\frac{\chi(n)\tau(n)}{n}+\int_N^{\infty}\frac{A(y,\chi,0)}{y^2}dy,$$

$$L(s,\bar{\chi})L(t,\bar{\chi})=\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\sigma_{s-t}(n)}{n^s}=\sum_{1\le n\le N}\frac{\bar{\chi}(n)\sigma_{s-t}(n)}{n^s}+s\int_N^{\infty}\frac{A(y,\bar{\chi},s-t)}{y^{s+1}}dy.$$

Hence, we can wrtie

$$\begin{aligned}&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} \chi(2^l)L^2(1,\chi)L(s,\bar{\chi})L(t,\bar{\chi})\\ =&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} \chi(2^l)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)\tau(n_1)}{n_1}+\int_N^{\infty}\frac{A(y,\chi,0)}{y^2}dy\right)\\ &\times\left(\sum_{1\le n_2\le N}\frac{\bar{\chi}(n_2)\sigma_{s-t}(n_2)}{n_2^s}+s\int_N^{\infty}\frac{A(y,\bar{\chi},s-t)}{y^{s+1}}dy\right)\\ =&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} \chi(2^l)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)\tau(n_1)}{n_1}\right)\left(\sum_{1\le n_2\le N}\frac{\bar{\chi}(n_2)\sigma_{s-t}(n_2)}{n_2^s}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l) \left(\sum_{1\le n_1\le N} \frac{\chi(n_1)\tau(n_1)}{n_1} \right) \left(s\int_N^\infty \frac{A(y,\bar{\chi},s-t)}{y^{s+1}} dy \right) \\
&+ \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l) \left(\sum_{1\le n_2\le N} \frac{\bar{\chi}(n_2)\sigma_{s-t}(n_2)}{n_2^s} \right) \left(\int_N^\infty \frac{A(y,\chi,0)}{y^2} dy \right) \\
&+ \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l) \left(\int_N^\infty \frac{A(y,\chi,0)}{y^2} dy \right) \left(s\int_N^\infty \frac{A(y,\bar{\chi},s-t)}{y^{s+1}} dy \right) \\
=\!:& B_1 + B_2 + B_3 + B_4. \qquad (11)
\end{aligned}
$$

Now we calculate each term in the expression (2.11).

For B_1, from Lemma 2.4, we know that

$$
\begin{aligned}
B_1 =& \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l) \left(\sum_{1\le n_1\le N} \frac{\chi(n_1)\tau(n_1)}{n_1} \right) \left(\sum_{1\le n_2\le N} \frac{\bar{\chi}(n_2)\sigma_{s-t}(n_2)}{n_2^s} \right) \\
=& \sum_{1\le n_1\le N} \sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l)\chi(n_1)\bar{\chi}(n_2) \\
=& \frac{1}{2} \sum_{1\le n_1\le N} \sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \sideset{}{^*}\sum_{\chi \bmod q} \chi(2^l)\chi(n_1)\bar{\chi}(n_2) \\
&- \frac{1}{2} \sum_{1\le n_1\le N} \sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \sideset{}{^*}\sum_{\chi \bmod q} \chi(-1)\chi(2^l)\chi(n_1)\bar{\chi}(n_2) \\
=& \frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \sum_{d|(q,2^l n_1\bar{n}_2-1)} \mu\left(\frac{q}{d}\right)\phi(d) \\
&- \frac{1}{2} \sideset{}{'}\sum_{1\le n_1\le N} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \sum_{d|(q,2^l n_1\bar{n}_2+1)} \mu\left(\frac{q}{d}\right)\phi(d) \\
=& \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^l n_1\equiv n_2 (mod\ d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \\
&- \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right)\phi(d) \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^l n_1\equiv -n_2 (mod\ d)}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \qquad (12)
\end{aligned}
$$

where $\sum'_{1\le n\le N}$ denotes the summation over n from 1 to N such that $(n,q)=1$.

For calculation convenience, we split the sum over n_1 or n_2 into following cases:

i) $\frac{d}{2^l} \le n_1 \le N,\ d \le n_2 \le N;$ *ii)* $1 \le n_1 \le \frac{d}{2^l} - 1,\ d \le n_2 \le N;$

iii) $\frac{d}{2^l} \le n_1 \le N,\ 1 \le n_2 \le d-1;$ *iv)* $1 \le n_1 \le \frac{d}{2^l} - 1,\ 1 \le n_2 \le d-1.$

So we have

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{d/2^l \le n_1 \le N \\ 2^l n_1 \equiv n_2 (mod\ d)}} \sideset{}{'}\sum_{d \le n_2 \le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \\
\ll &\sum_{d|q} \phi(d) \sum_{1 \le r_1 \le 2^l N/d} \sum_{1 \le r_2 \le N/d} \sideset{}{'}\sum_{\substack{l_1=1 \\ l_1 \equiv l_2 (mod\ d)}}^{d-1} \sideset{}{'}\sum_{l_2=1}^{d-1} \frac{\tau(r_1 d + l_1)\sigma_{s-t}(r_2 d + l_2)}{(r_1 d + l_1)(r_2 d + l_2)^s} \\
\ll &\sum_{d|q} \phi(d) \sum_{1 \le r_1 \le 2^l N/d} \sum_{1 \le r_2 \le N/d} \sum_{l_1=1}^{d-1} \frac{(r_1 d + l_1)^\varepsilon (r_2 d + l_1)^\varepsilon}{(r_1 d + l_1)(r_2 d + l_1)^s} \\
\ll &\sum_{d|q} \frac{\phi(d)}{d^s} \sum_{1 \le r_1 \le 2^l N/d} \sum_{1 \le r_2 \le N/d} \frac{(r_1 d + 1)^\varepsilon (r_2 d + 1)^\varepsilon}{r_1 r_2^s} \ll q^\varepsilon, \\
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{1 \le n_1 \le d/2^l - 1 \\ 2^l n_1 \equiv n_2 (mod\ d)}} \sideset{}{'}\sum_{d \le n_2 \le N} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \\
\ll &\sum_{d|q} \phi(d) \sum_{1 \le n_1 \le d/2^l - 1} \sum_{1 \le r_2 \le N/d} \frac{(n_1(r_2 d + n_1))^\varepsilon}{n_1 (r_2 d + n_1)^s} \ll q^\varepsilon
\end{aligned}$$

and

$$\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{d/2^l \le n_1 \le N \\ 2^l n_1 \equiv n_2 (mod\ d)}} \sideset{}{'}\sum_{1 \le n_2 \le d-1} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \ll q^\varepsilon,$$

where we have used the estimate $\tau(n) \ll n^\varepsilon$.

For the case (iv), the solution of the congruence $2^l n_1 \equiv n_2 (mod\ d)$ is $2^l n_1 = n_2$. Hence,

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{\substack{1 \le n_1 \le d/2^l \\ 2^l n_1 \equiv n_2 (mod\ d)}} \sideset{}{'}\sum_{1 \le n_2 \le d-1} \frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s} \\
= &\sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{1 \le n_1 \le d/2^l} \frac{\tau(n_1)\sigma_{s-t}(2^l n_2)}{2^{ls} n_1^{s+1}} \\
= &\frac{1}{2^{ls}} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sideset{}{'}\sum_{n_1=1}^{\infty} \frac{\tau(n_1)\sigma_{s-t}(2^l n_2)}{n_1^{s+1}} + O(q^\varepsilon).
\end{aligned}$$

Now from Lemma 2.10, we immediately get

$$\begin{aligned}
&\frac{1}{2}\sum_{d\mid q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le d/2^l\\ 2^l n_1\equiv n_2 (mod\ d)}}\sideset{}{'}\sum_{1\le n_2\le d-1}\frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s}\\
=&\frac{J(q)}{2^{ls+1}}\cdot\frac{2^{s+1}(2^{t+1}-1)^2-2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)}\xi(t,s,q)+O\big(q^{\varepsilon}\big). \qquad (13)
\end{aligned}$$

Similarly, we can also get the estimate

$$\frac{1}{2}\sum_{d\mid q}\mu\left(\frac{q}{d}\right)\phi(d)\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^l n_1\equiv -n_2 (mod\ d)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{\tau(n_1)\sigma_{s-t}(n_2)}{n_1 n_2^s}\ll q^{\varepsilon}. \qquad (14)$$

Then from (2.12), (2.13) and (2.14), we have

$$B_1=\frac{J(q)}{2^{ls+1}}\cdot\frac{2^{s+1}(2^{t+1}-1)^2-2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)}\xi(t,s,q)+O\big(q^{\varepsilon}\big). \qquad (15)$$

If $\delta<0$, noting that the partition identity

$$\begin{aligned}
A(y,\bar\chi,\delta)=&\sum_{n\le\sqrt{y}}\bar\chi(n)n^{\delta}\sum_{m\le y/n}\bar\chi(m)+\sum_{m\le\sqrt{y}}\bar\chi(m)\sum_{n\le y/m}\bar\chi(n)n^{\delta}\\
&-\left(\sum_{n\le\sqrt{y}}\bar\chi(n)n^{\delta}\right)\left(\sum_{n\le\sqrt{y}}\bar\chi(n)\right)-\sum_{n\le\sqrt{N}}\bar\chi(n)n^{\delta}\sum_{m\le N/n}\bar\chi(m)\\
&-\sum_{m\le\sqrt{N}}\bar\chi(m)\sum_{n\le N/m}\bar\chi(n)n^{\delta}+\left(\sum_{n\le\sqrt{N}}\bar\chi(n)n^{\delta}\right)\left(\sum_{n\le\sqrt{N}}\bar\chi(n)\right)
\end{aligned}$$

and from the Pólya$-$Vinogradov inequality, we have

$$\sideset{}{^*}\sum_{\substack{\chi\ mod\ q\\ \chi(-1)=-1}}|A(y,\bar\chi,\delta)|\ll\sqrt{yq}q\ln q.$$

Then we have

$$\begin{aligned}
B_2&\ll\sum_{1\le n_1\le N}\frac{n_1^{\varepsilon}}{n_1}\int_N^{\infty}\frac{1}{y^{s+1}}\left(\sideset{}{^*}\sum_{\substack{\chi mod\ q\\ \chi(-1)=-1}}|A(y,\bar\chi,s-t)|\right)dy\\
&\ll N^{\varepsilon-1}\int_N^{\infty}\frac{\sqrt{y}}{y^{s+1}}q^{\frac{3}{2}}\ln q dy\ll\frac{q^{\frac{3}{2}}\ln q}{N^{s-\varepsilon+\frac{1}{2}}}. \qquad (16)
\end{aligned}$$

$$B_3 \ll \sum_{1\le n_2\le N} \frac{n_2^{\varepsilon}}{n_2^{s}} \int_N^{\infty} \frac{1}{y^2} \left(\sideset{}{^*}\sum_{\substack{\chi mod\ q\\ \chi(-1)=-1}} |A(y,\chi,0)| \right) dy$$
$$\ll N^{\varepsilon-s} \int_N^{\infty} \frac{1}{y^{\frac{3}{2}}} q^{\frac{3}{2}} \ln q dy \ll \frac{q^{\frac{3}{2}} \ln q}{N^{s-\varepsilon+\frac{1}{2}}}. \tag{17}$$

For M_4, from [15], we can obtain the estimate

$$\sum_{\chi\neq\chi_0} |A(y,\chi,0)|^2 \ll y^{1+\varepsilon}\phi^2(q),$$

where χ_0 denotes the principal character modulo q. Hence,

$$B_4 \ll \int_N^{\infty} \frac{\sum\limits_{\chi\neq\chi_0} |A(y,\chi,0)|^2\, dy}{y^4} \ll \phi^2(q) \int_N^{\infty} \frac{1}{y^{3-\varepsilon}} dy \ll \frac{\phi^2(q)}{N^{2-\varepsilon}}. \tag{18}$$

Now, taking $N=q^2$, combining (2.11) with (2.15)-(2.18), we obtain the asymptotic formula

$$\sideset{}{^*}\sum_{\substack{\chi\ mod\ q\\ \chi(-1)=-1}} \chi(2^l)L^2(1,\chi)L(s,\bar{\chi})L(t,\bar{\chi})$$
$$=\frac{J(q)}{2^{ls+1}} \cdot \frac{2^{s+1}(2^{t+1}-1)^2 - 2^{t+1}\cdot 2^{l(s-t)}(2^{s+1}-1)^2}{(2^{t+1}-2^{s+1})(2^{s+t+2}-1)} \xi(t,s,q) + O\big(q^{\varepsilon}\big).$$

This proves formula (2.8), other formulas can be determined in the same way. □

Lemma 2.13 *Let s be a positive odd integer, we have*

(i) if $q\ge 3$ is a positive integer, then

$$\sideset{}{^*}\sum_{\substack{\chi\ mod\ q\\ \chi(-1)=-1}} L^2(1,\chi)L^2(s,\bar{\chi}) = \frac{J(q)}{2}\xi(s,s,q) + O\big(q^{\varepsilon}\big);$$

(ii) if $q\ge 3$ is an odd integer, then

$$\sideset{}{^*}\sum_{\substack{\chi\ mod\ q\\ \chi(-1)=-1}} \chi(2^l)L^2(1,\chi)L^2(s,\bar{\chi})$$
$$=\frac{J(q)}{2^{ls+1}} \cdot \frac{l(2^{s+1}-1)+2^{s+1}+1}{2^{s+1}+1} \xi(s,s,q) + O\big(q^{\varepsilon}\big);$$

(iii) if $q>3$ is a integer and $3\nmid q$, then

$$\sideset{}{^*}\sum_{\substack{\chi\ mod\ q\\ \chi(-1)=-1}} L^2(1,\chi\chi_3^o)L^2(s,\bar{\chi}) = \frac{J(q)}{2}\xi(s,s,3q) + O\big(q^{\varepsilon}\big);$$

(iv) if $q > 3$ is an odd integer and $3 \nmid q$, then

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^l)L(1,\chi)L(1,\chi\chi_3^o)L^2(s,\bar{\chi})$$
$$=\frac{J(q)}{2^{ls+1}}\cdot\frac{l(2^{s+1}-1)+2^{s+1}+1}{2^{s+1}+1}\cdot\frac{3^{s+1}-1}{3^{s+1}+1}\xi(s,s,q)+O\big(q^{\varepsilon}\big).$$

Proof Similar to Lemma 2.12, combining with Lemma 2.11, we obtain the results. □

3 Proof of theorems

In this section, we will complete the proof of the theorems. First, form the definition of Ramanujan sum and the properties of Gauss sums, we have

$$\sideset{}{'}\sum_{a=1}^{q}\bar{\chi}(a)R_p(a+1) = \sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\bar{\chi}(a)e\left(\frac{b(a+1)}{q}\right)$$
$$= \sideset{}{'}\sum_{b=1}^{q}\chi(b)e\left(\frac{b}{q}\right)\sideset{}{'}\sum_{a=1}^{q}\bar{\chi}(a)e\left(\frac{a}{q}\right) = \tau(\chi)\tau(\bar{\chi}). \tag{19}$$

Then from (3.1), Lemmas 2.1, 2.3, 2.7, 2.12 and 2.13, we know that

$$\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{d=1}^{q} bdW_q(a,bd,1,m,n)$$
$$=\frac{4m!n!}{(2\pi i)^{m+n}\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left(\sideset{}{'}\sum_{a=1}^{q}\bar{\chi}(a)R_q(a+1)\right)\left(\sideset{}{'}\sum_{b=1}^{q}b\bar{\chi}(b)\right)^2$$
$$\times\left(\sum_{r=1}^{\infty}\frac{G(\chi,r)}{r^m}\right)\left(\sum_{s=1}^{\infty}\frac{G(\chi,s)}{s^n}\right)$$
$$=\frac{4m!n!}{(2\pi i)^{m+n}\phi(q)}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\tau(\chi)\tau(\bar{\chi})\left(\sideset{}{'}\sum_{b=1}^{q}b\bar{\chi}(b)\right)^2\left(\sum_{r=1}^{\infty}\frac{G(\chi,r)}{r^m}\right)\left(\sum_{s=1}^{\infty}\frac{G(\chi,s)}{s^n}\right)$$
$$=\frac{-4m!n!q}{(2\pi i)^{m+n}\phi(q)}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left(\frac{qi}{\pi}\tau(\bar{\chi})L(1,\chi)\right)^2\tau^2(\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$
$$=\frac{4m!n!q^5}{(2\pi i)^{m+n}\phi(q)\pi^2}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}L^2(1,\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$
$$=\frac{2m!n!\phi(q)q^4}{(2\pi i)^{m+n}\pi^2}\xi(m,n,q)+O\big(q^{4+\varepsilon}\big).$$

By the same method, from Lemmas 2.8 and 2.9, we obtain other expressions in Theorem 1.1.

For any primitive character χ modulo p^β and $\beta < \alpha$, from Lemma 2.6, we have

$$\sum_{r=1}^{\infty} \frac{G(\chi\chi_{p^\alpha}^o, r)}{r^m} = \sum_{\substack{r=1 \\ (r,p^\alpha)=p^{\alpha-\beta}}}^{\infty} \frac{\bar{\chi}\left(\frac{r}{(r,p^\alpha)}\right)\chi^*\left(\frac{p^\alpha}{p^\beta (r,p^\alpha)}\right)\mu\left(\frac{p^\alpha}{p^\beta (r,p^\alpha)}\right)\phi(p^\alpha)\tau(\chi)}{r^m \phi\left(\frac{p^\alpha}{(r,p^\alpha)}\right)}$$

$$= \frac{\phi(p^\alpha)\tau(\chi)}{\phi(p^\beta)p^{m(\alpha-\beta)}} \sum_{\substack{r=1 \\ (r,p^\beta)=1}}^{\infty} \frac{\bar{\chi}(r)}{r^m} = \frac{\phi(p^\alpha)\tau(\chi)L(m,\bar{\chi})}{\phi(p^\beta)p^{m(\alpha-\beta)}}. \tag{20}$$

By Lemmas 2.3 and 2.5, we know that

$$\sum_{b=1}^{p^\alpha}{}' b\bar{\chi}\chi_{p^\alpha}^o(b) = p^{\alpha-\beta}\left(\prod_{\substack{p \mid p^\alpha \\ p \nmid p^\beta}} (1-\chi(p))\right)\left(\sum_{b=1}^{p^\beta} b\chi(b)\right) = \frac{p^\alpha i}{\pi}\tau(\bar{\chi})L(1,\chi). \tag{21}$$

Hence, if $\alpha \geq 2$, from(3.2), (3.3), Lemma 2.1, 2.2, 2.12 and 2.13, we have

$$\sum_{b=1}^{p^\alpha}{}' \sum_{d=1}^{p^\alpha}{}' bdW_{p^\alpha}(a, bd, 2, m, n)$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p^\alpha)} \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \left(\sum_{a=1}^{p^\alpha}{}' \bar{\chi}(a)R_{p^\alpha}^2(a+1)\right)\left(\sum_{b=1}^{p^\alpha}{}' b\bar{\chi}(b)\right)^2$$

$$\times \left(\sum_{r=1}^{\infty} \frac{G(\chi, r)}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{G(\chi, s)}{s^n}\right)$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p^\alpha)} \sum_{\beta=1}^{\alpha} \sideset{}{^*}\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=-1}} \left(\sum_{a=1}^{p^\alpha}{}' \bar{\chi}\chi_{p^\alpha}^o(a)R_{p^\alpha}^2(a+1)\right)\left(\sum_{b=1}^{p^\alpha}{}' b\bar{\chi}\chi_{p^\alpha}^o(b)\right)^2$$

$$\times \left(\sum_{r=1}^{\infty} \frac{G(\chi\chi_{p^\alpha}^o, r)}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{G(\chi\chi_{p^\alpha}^o, s)}{s^n}\right)$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p^\alpha)} \sideset{}{^*}\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \left(\sum_{a=1}^{p^\alpha}{}' \bar{\chi}(a)R_{p^\alpha}^2(a+1)\right)\left(\sum_{b=1}^{p^\alpha}{}' b\bar{\chi}(b)\right)^2\left(\sum_{r=1}^{\infty} \frac{G(\chi, r)}{r^m}\right)$$

$$\times \left(\sum_{s=1}^{\infty} \frac{G(\chi, s)}{s^n}\right) + \frac{4m!n!}{(2\pi i)^{m+n}\phi(p^\alpha)} \sum_{\beta=1}^{\alpha-1} \sideset{}{^*}\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=-1}} \left(\sum_{a=1}^{p^\alpha}{}' \bar{\chi}\chi_{p^\alpha}^o(a)R_{p^\alpha}^2(a+1)\right)$$

$$\times \left(\sum_{b=1}^{p^\alpha}{}' b\bar{\chi}\chi_{p^\alpha}^o(b)\right)^2 \left(\sum_{r=1}^{\infty} \frac{G(\chi\chi_{p^\alpha}^o, r)}{r^m}\right) \left(\sum_{s=1}^{\infty} \frac{G(\chi\chi_{p^\alpha}^o, s)}{s^n}\right)$$

$$= \frac{4m!n!p^{6\alpha}}{(2\pi i)^{m+n}\pi^2\phi(p^\alpha)} \left(1 - \frac{2}{p}\right) \sideset{}{^*}\sum_{\substack{\chi \ mod \ p^\alpha \\ \chi(-1)=-1}} L^2(1,\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$

$$+ \frac{4m!n!p^{6\alpha}\phi(p^\alpha)}{(2\pi i)^{m+n}\pi^2} \left(1 - \frac{1}{p}\right) \sum_{\beta=1}^{\alpha-1} \frac{1}{\phi^2(p^\beta)p^{(m+n)(\alpha-\beta)}} \sideset{}{^*}\sum_{\substack{\chi \ mod \ p^\beta \\ \chi(-1)=-1}} L^2(1,\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$

$$= \frac{2m!n!p^{5\alpha}\phi(p^\alpha)}{(2\pi i)^{m+n}\pi^2} \cdot \frac{2p^{m+n-1} - p^{m+n} - 1}{p - p^{m+n}} \xi(m,n,p^\alpha) + O\big(p^{5\alpha+\varepsilon}\big).$$

Furthermore, if $\alpha = 1$, we have

$$\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} bdW_p(a,bd,2,m,n)$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p)} \sideset{}{^*}\sum_{\substack{\chi \ mod \ p \\ \chi(-1)=-1}} \left(\sum_{a=1}^{p-1} \bar{\chi}(a)R_p^2(a+1)\right) \left(\sum_{b=1}^{p-1} b\bar{\chi}(b)\right)^2$$

$$\times \left(\sum_{r=1}^{\infty} \frac{G(\chi, r)}{r^m}\right) \left(\sum_{s=1}^{\infty} \frac{G(\chi, s)}{s^n}\right)$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p)} \sideset{}{^*}\sum_{\substack{\chi \ mod \ p \\ \chi(-1)=-1}} -p(p-2) \left(\frac{pi}{\pi}\tau(\bar{\chi})L(1,\chi)\right)^2 \tau(\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$

$$= \frac{4m!n!}{(2\pi i)^{m+n}\phi(p)} \frac{p^5(p-2)}{\pi^2} \sideset{}{^*}\sum_{\substack{\chi \ mod \ p \\ \chi(-1)=-1}} L^2(1,\chi)L(m,\bar{\chi})L(n,\bar{\chi})$$

$$= \frac{2m!n!p^6}{(2\pi i)^{m+n}\pi^2} \xi(m,n,p) + O\big(p^{5+\varepsilon}\big).$$

Hence,

$$\sum_{b=1}^{p^\alpha}{}' \sum_{d=1}^{p^\alpha}{}' bdW_{p^\alpha}(a,bd,2,m,n)$$

$$= \frac{2m!n!p^{5\alpha}\phi(p^\alpha)}{(2\pi i)^{m+n}\pi^2} \cdot \frac{2p^{m+n-1} - p^{m+n} - 1}{p - p^{m+n}} \xi(m,n,p^\alpha) + O\big(p^{5\alpha+\varepsilon}\big).$$

From Lemmas 2.3, 2.5, and 2.8, we arrive at

$$\sum_{b<\frac{q}{4}}{}' \bar{\chi}(b) = \frac{2 + \chi(2) - \chi(4)}{2\pi i} \tau(\bar{\chi})L(1,\chi). \tag{22}$$

Similarly, we obtain other expressions in Theorem 1.2.

Acknowledgement *This work is supported by National Natural Science Foundation of China (11971381).*

References

[1] K. Bibak, B. M. Kapron, V. Srinivasan, R. Tauraso, L. Tóth, Restricted linear congruences, *J. Number Theory* **171** (2017), 128–144.

[2] M. C. Dağli, New relations concerning a mean value of some Hardy sums and Ramanujan sum, *Fen Derg.* **15** (2020), 148–157.

[3] M. C. Dağli, On the hybrid mean value of generalized Dedekind sums, generalized Hardy sums and Ramanujan sum, *Bull. Math. Soc. Sci. Math. Roumanie* **63 (111)** (2020), 325–333.

[4] D. Han, W. Zhang, Some new identities involving Dedekind sums and the Ramanujan sum, *Ramanujan J.* **35** (2014), 253–262.

[5] H. Liu, W. Zhang, Some identities involving certain Hardy sums and Ramanujan sum, *Acta Math. Sin. (Engl. Ser.)* **21** (2005), 109–116.

[6] H. Liu, W. Zhang, Generalized Cochrane sums and Cochrane-Hardy sums, *J. Number Theory* **122** (2007), 415–428.

[7] L. Liu, Z. Xu, Mean value of the general Dedekind sums over interval $[1, \frac{q}{p})$, *Symmetry* **2079** (2020).

[8] K. V. Namboothiri, On the number of solutions of a restricted linear congruence, *J. Number Theory* **188** (2018), 324–334.

[9] M. Toyoizumi, On certain character sums, *Acta Arith.* **55** (1990), 229–232.

[10] W. Wang, D. Han, An identity involving certain Hardy sums and Ramanujan's sum, *Adv. Difference Equ.* **261** (2013).

[11] Z. Xu, A mean value of Cochrane sum, *Topol. Methods Nonlinear Anal.* **25** (2009), 223–234.

[12] W. Zhang, A problem of D. H. Lehmer and its generalization (II), *Compositio Math.* **91** (1994), 47–56.

[13] W. Zhang, On a hybrid mean value of certain Hardy sums and Ramanujan sum, *Osaka J. Math.* **40** (2003), 365–373.

[14] W. Zhang, H. Liu, A note on the Cochrane sum and its hybrid mean value formula, *J. Math. Anal. Appl.* **288** (2003), 646–659.

[15] W. Zhang, X. Wang, On the fourth power mean of the character sums over short intervals, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), 153–164.

Received: 28.07.2022
Revised: 19.11.2022
Accepted: 21.11.2022

(1) Research Center for Number Theory and its Applications,
Northwest University, Xi'an 710127, P. R. China
E-mail: `wangxiao2@stumail.nwu.edu.cn`

(2) Research Center for Number Theory and its Applications,
Northwest University, Xi'an 710127, P. R. China
E-mail: `wangjiankang@stumail.nwu.edu.cn`

(3) Research Center for Number Theory and its Applications,
Northwest University, Xi'an 710127, P. R. China
E-mail: `zfxu@nwu.edu.cn`