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A q-supercongruence modulo the fourth power of a cyclotomic polynomial

by

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Abstract

Recently, Liu provided several nice supercongruences. Inspired by his work, in this paper, we establish a new q-supercongruence with two free parameters modulo the fourth power of a cyclotomic polynomial. By taking suitable parameter substitutions in this q-supercongruence, we derive some new results including a partial q-analogue of Liu's supercongruence. Our main auxiliary tools are Watson's ${}_8\phi_7$ transformation formula for basic hypergeometric series, the 'creative microscoping' method introduced by Guo and Zudilin and the Chinese remainder theorem for coprime polynomials.

Key Words: Basic hypergeometric series, Watson's ${}_8\phi_7$ transformation, creative microscoping, Chinese remainder theorem.
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1 Introduction

In 1997, Van Hamme [17] conjectured 13 Ramanujan-type supercongruences which were labeled as (A.2)–(M.2). The supercongruences (C.2) and (D.2) can be stated as follows:

$$\text{(C.2)} \qquad \sum_{k=0}^{(p-1)/2}(4k+1)\frac{(1/2)_k^4}{k!^4} \equiv p \pmod{p^3}, \quad p \neq 2;$$

$$\text{(D.2)} \qquad \sum_{k=0}^{(p-1)/3}(6k+1)\frac{(1/3)_k^6}{k!^6} \equiv -p\Gamma_p(1/3)^9 \pmod{p^4}, \quad p \equiv 1 \pmod{6}.$$

Here and throughout the paper, p is a prime, $(x)_0 = 1$, $(x)_n = x(x+1)\cdots(x+n-1)$ stands for the the Pochhammer symbol and $\Gamma_p(x)$ is the p-adic Gamma function. In 2006, making use of Dougall's formula, Long and Ramakrishna [14] gave an extension of Van Hamme's (D.2):

$$\sum_{k=0}^{p-1}(6k+1)\frac{(1/3)_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p(1/3)^9 \pmod{p^6}, & \text{if} \quad p \equiv 1 \pmod{6}, \\ -\dfrac{10}{27}p^4\Gamma_p(1/3)^9 \pmod{p^6}, & \text{if} \quad p \equiv 5 \pmod{6}. \end{cases}$$

Similarly, Liu [11] established a new supercongruence: for $p \geq 5$,

$$\sum_{k=0}^{p-1}(6k-1)\frac{(-1/3)_k^6}{k!^6} \equiv \begin{cases} 140p^4\Gamma_p(2/3)^9 \pmod{p^5}, & \text{if} \quad p \equiv 1 \pmod{6}, \\ 378p\Gamma_p(2/3)^9 \pmod{p^5}, & \text{if} \quad p \equiv 5 \pmod{6}. \end{cases} \tag{1.1}$$

Also, Guo and Schlosser [4] proposed one conjecture as follows: for $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(p+1)/3} (6k-1) \frac{(-1/3)_k^4 (1)_{2k}}{(1)_k^4 (-2/3)_{2k}} \equiv p \pmod{p^3}. \tag{1.2}$$

By using the hypergeometric series identities and p-adic Gamma functions, Jana and Kalita [8] first confirmed the supercongruence (1.2). Later, based on combinatorial identities arising from symbolic summation, Liu [10] provided a stronger version of (1.2): for odd primes $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(p+1)/3} (6k-1) \frac{(-1/3)_k^4 (1)_{2k}}{(1)_k^4 (-2/3)_{2k}} \equiv p - p^3 \left(\frac{1}{9} B_{p-2}(1/3) - 2 \right) \pmod{p^4}, \tag{1.3}$$

where the *Bernoulli polynomials* are given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

During the past few years, there has been an increasing attention to the issue of finding q-analogues of congruences and supercongruences. The reader may be referred to [3, 6, 7, 9, 13, 16, 18, 19, 20, 21] for some of their work. Recently, in [4], Guo and Schlosser gave a partial q-analogue of supercongruence (1.2): for integers $n > 2$ with $n \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(n+1)/3} [6k-1] \frac{\left(q^{-1}; q^3\right)_k^4 \left(q^3; q^3\right)_{2k}}{\left(q^3; q^3\right)_k^4 \left(q^{-2}; q^3\right)_{2k}} \equiv 0 \pmod{\Phi_n(q)}. \tag{1.4}$$

Here and throughout the paper, the *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right)$ with $n \in \mathbb{Z}^+$. For brevity, its product form can be written as $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$. And $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$ denotes the *q-integer*. Moreover, $\Phi_n(q)$ represents the n-th *cyclotomic polynomial* in q.

Motivated by the work just mentioned, in this paper, we shall establish a new q-supercongruence with two free parameters c and e, from which we can deduce a partial q-analogue of Liu's congruence (1.1).

The rest of this paper is arranged as follows. Our main results will be shown in the next section. Then the proof of our q-supercongruence will be presented in Section 3, where the 'creative microscoping' method introduced by Guo and Zudilin [5] and the Chinese remainder theorem for coprime polynomials will be used.

2 Main results

Theorem 1. *Let $n \equiv 2 \pmod 3$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\begin{aligned} &\sum_{k=0}^{M} [6k-1] \frac{\left(q^{-1}; q^3\right)_k^4 \left(cq^{-1}, eq^{-1}; q^3\right)_k}{\left(q^3; q^3\right)_k^4 \left(q^3/c, q^3/e; q^3\right)_k} \left(\frac{q^9}{ce}\right)^k \\ &\equiv [n] q^{(n+1)/3} \frac{\left(q^{-2}; q^3\right)_{(n+1)/3}}{\left(q^3; q^3\right)_{(n+1)/3}} \left(1 - [n]^2 \sum_{i=1}^{(n+1)/3} \frac{q^{3i}}{[3i]^2}\right) \sum_{k=0}^{(n+1)/3} \frac{\left(q^4/ce; q^3\right)_k \left(q^{-1}; q^3\right)_k^3}{\left(q^3/c, q^3/e, q^3, q^{-2}; q^3\right)_k} q^{3k}, \end{aligned}$$

where here and in what follows $M=(n+1)/3$ or $n-1$.

Setting $c\to 1$, $e\to 1$ in Theorem 1, we obtain a partial q-analogue of Liu's congruence (1.1) as follows.

Corollary 1. *Let $n\equiv 2 \pmod 3$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[6k-1]\frac{\left(q^{-1};q^3\right)_k^6}{(q^3;q^3)_k^6}q^{9k} \equiv [n]q^{(n+1)/3}\frac{\left(q^{-2};q^3\right)_{(n+1)/3}}{(q^3;q^3)_{(n+1)/3}}\left(1-[n]^2\sum_{i=1}^{(n+1)/3}\frac{q^{3i}}{[3i]^2}\right)\sum_{k=0}^{(n+1)/3}\frac{\left(q^4;q^3\right)_k\left(q^{-1};q^3\right)_k^3}{(q^3;q^3)_k^3\left(q^{-2};q^3\right)_k}q^{3k}.$$

Furthermore, letting $q\to q^2$ and $c=e=q^7$ in Theorem 1, we obtain a new result as follows.

Corollary 2. *Let $n\equiv 2 \pmod 3$ be a positive integer. Then, modulo $[n]_{q^2}\Phi_n(q^2)^3$,*

$$\sum_{k=0}^{M}[6k-1]_{q^2}[6k-1]^2\frac{\left(q^{-2};q^6\right)_k^4}{(q^6;q^6)_k^4}q^{4k}\equiv\frac{-2[n]_{q^2}q^{\frac{2n-7}{3}}\left(q^{-4};q^6\right)_{(n+1)/3}}{\left(1+q^{-2}\right)\left(q^6;q^6\right)_{(n+1)/3}}\left(1-[n]_{q^2}^2\sum_{i=1}^{(n+1)/3}\frac{q^{6i}}{[3i]_{q^2}^2}\right).$$

By using the following congruence from [15]: for primes $p>5$, $\lfloor x\rfloor$ denotes the integral part of x,

$$\sum_{k=1}^{\lfloor p/3\rfloor}\frac{1}{k^2}\equiv\frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}(1/3)\pmod p, \tag{2.1}$$

and letting $n=p$ with $p\equiv 2 \pmod 3$ and $p>5$, $q\to 1$ in Corollary 2, we get the supercongruence: for primes $p\equiv 2 \pmod 3$ with $p>5$, modulo p^4,

$$\sum_{k=0}^{(p+1)/3}(6k-1)^3\frac{(-1/3)_k^4}{k!^4}\equiv(-1)^{(p-2)/3}p\Gamma_p^2(2/3)\left(1-p^2-\frac{p^2}{18}\left(\frac{p}{3}\right)B_{p-2}(1/3)\right), \tag{2.2}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Moreover, taking $ce=q^4$ in Theorem 1, we get the following q-supercongruence.

Corollary 3. *Let $n\equiv 2 \pmod 3$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{M}[6k-1]\frac{\left(q^{-1};q^3\right)_k^4}{(q^3;q^3)_k^4}q^{5k}\equiv[n]q^{(n+1)/3}\frac{\left(q^{-2};q^3\right)_{(n+1)/3}}{(q^3;q^3)_{(n+1)/3}}\left(1-[n]^2\sum_{i=1}^{(n+1)/3}\frac{q^{3i}}{[3i]^2}\right).$$

Letting $n=p$ with $p\equiv 2 \pmod 3$ and $p>5$, $q\to 1$ in Corollary 3, we obtain a new congruence: for primes $p\equiv 2 \pmod 3$ with $p>5$, modulo p^4,

$$\sum_{k=0}^{(p+1)/3}(6k-1)\frac{(-1/3)_k^4}{k!^4}\equiv(-1)^{(p+1)/3}p\Gamma_p^2(2/3)\left(1-p^2-\frac{p^2}{18}\left(\frac{p}{3}\right)B_{p-2}(1/3)\right). \tag{2.3}$$

Combining (2.2) and (2.3), we get a new and rare supercongruence: for primes $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{(p+1)/3} (6k-1)(18k^2-6k+1)\frac{(-1/3)_k^4}{k!^4} \equiv 0 \pmod{p^4}. \tag{2.4}$$

3 Proof of Theorem 1

In fact, the proof of Theorem 1 can be transformed into confirming the following generalized theorem.

Theorem 2. *Let $n>1$, $d\geq 2$ be integers with $n\equiv r \pmod d$ and $r\in\{1,-1\}$. Then, modulo $[n]\Phi_n(q)^3$,*

$$\begin{aligned}&\sum_{k=0}^{W}[2dk+r]\frac{\left(q^r;q^d\right)_k^4\left(cq^r,eq^r;q^d\right)_k}{(q^d;q^d)_k^4\left(q^d/c,q^d/e;q^d\right)_k}(ce)^{-k}q^{(2d-3r)k}\\ &\equiv [n]\frac{q^{r(r-n)/d}}{(q^d;q^d)_{(n-r)/d}}\left(1-[n]^2\sum_{i=1}^{(n-r)/d}\frac{q^{di}}{[di]^2}\right)\\ &\quad\times\sum_{k=0}^{(n-r)/d}\frac{\left(q^{2r+dk};q^d\right)_{(n-r)/d-k}\left(q^{d-r}/ce;q^d\right)_k\left(q^r;q^d\right)_k^3}{\left(q^d/c,q^d/e,q^d;q^d\right)_k}q^{dk},\end{aligned} \tag{3.1}$$

where here and in what follows $W=(n-r)/d$ or $n-1$.

Clearly, when $d=3$, $r=-1$, Theorem 2 reduces to Theorem 1. Actually, by making appropriate parameter substitutions in Theorem 2, more results can be obtained. For example, letting $d=3, r=1$, $c\to 1, e\to 1$ and $q\to 1$ in Theorem 2, we reprove Van Hamme's (D.2). In addition, setting $d=2, r=1$ and $c=e=q^{1/2}$ in Theorem 2, we get a new q-analogue of Van Hamme's (C.2) modulo p^4 as follows: for positive odd integers n, modulo $[n]\Phi_n(q^3)$,

$$\sum_{k=0}^{N}[4k+1]\frac{\left(q;q^2\right)_k^4}{\left(q^2;q^2\right)_k^4} \equiv [n]q^{(1-n)/2}-[n]^3q^{(1-n)/2}\sum_{k=0}^{(n-1)/2}\frac{q^{2k}}{[2k]^2},$$

where $N=(n-1)/2$ or $n-1$. It should be point out that Guo [2] gave another q-analogue of Van Hamme's (C.2) modulo p^4: for positive odd integers n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{\left(q;q^2\right)_k^4}{\left(q^2;q^2\right)_k^4} \equiv q^{(1-n)/2}[n]+\frac{\left(n^2-1\right)(1-q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}.$$

In the process of proving Theorem 2, we shall utilize Watson's ${}_8\phi_7$ transformation formula [1]:

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q,\ \frac{a^2q^{n+2}}{bcde}\right]
$$

$$
= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a \end{matrix}; q,\ q\right]. \tag{3.2}
$$

Here, the basic hypergeometric series ${}_{r+1}\phi_r$, following Gasper and Rahman[1], is defined as

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k z^k}{(q, b_1, \ldots, b_r; q)_k}, \qquad \text{for} \quad 0 < |q| < 1.
$$

Before proving Theorem 2, we first list the following two related results, which have been proved in [12].

Lemma 1. *Let d, n be positive integers with* $\gcd(d,n) = 1$. *Let r be an integer and a, b, c, e be indeterminates. Then, modulo $[n]$,*

$$
\sum_{k=0}^{m_1} [2dk+r] \frac{\left(q^r, cq^r, eq^r, q^r/b, aq^r, q^r/a; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, bq^d, q^d/a, aq^d; q^d\right)_k} \left(\frac{b}{ce}\right)^k q^{(2d-3r)k} \equiv 0,
$$

$$
\sum_{k=0}^{n-1} [2dk+r] \frac{\left(q^r, cq^r, eq^r, q^r/b, aq^r, q^r/a; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, bq^d, q^d/a, aq^d; q^d\right)_k} \left(\frac{b}{ce}\right)^k q^{(2d-3r)k} \equiv 0,
$$

where $0 \le m_1 \le n-1$ and $dm_1 \equiv -r \pmod{n}$.

Lemma 2. *Let $n > 1$, $d \ge 2$, r be integers with* $\gcd(r,d) = 1$ *and $n \equiv r \pmod{d}$ such that $n + d - nd \le r \le n$. Then, modulo $\Phi_n(q)\,(1 - aq^n)\,(a - q^n)$,*

$$
\sum_{k=0}^{(n-r)/d} [2dk+r] \frac{\left(q^r, cq^r, eq^r, q^r/b, aq^r, q^r/a; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, bq^d, q^d/a, aq^d; q^d\right)_k} \left(\frac{b}{ce}\right)^k q^{(2d-3r)k}
$$

$$
\equiv [n] \left(\frac{b}{q^r}\right)^{(n-r)/d} \frac{\left(q^{2r}/b; q^d\right)_{(n-r)/d}}{\left(bq^d; q^d\right)_{(n-r)/d}} \sum_{k=0}^{(n-r)/d} \frac{\left(q^{d-r}/ce, q^r/b, aq^r, q^r/a; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, q^{2r}/b; q^d\right)_k} q^{dk}. \tag{3.3}
$$

In order to complete our proof of Theorem 2, we still need the following lemma.

Lemma 3. *Let $n > 1$, $d \ge 2$ be integers with $n \equiv r \pmod{d}$ and $r \in \{1, -1\}$. Then, modulo $b - q^n$,*

$$
\sum_{k=0}^{W} [2dk+r] \frac{\left(q^r, cq^r, eq^r, q^r/b, aq^r, q^r/a; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, bq^d, q^d/a, aq^d; q^d\right)_k} \left(\frac{b}{ce}\right)^k q^{(2d-3r)k}
$$

$$
\equiv [n] \frac{\left(q^r, q^{d-r}; q^d\right)_{(n-r)/d}}{\left(q^d/a, aq^d; q^d\right)_{(n-r)/d}} \sum_{k=0}^{(n-r)/d} \frac{\left(q^{d-r}/ce, aq^r, q^r/a, q^r/b; q^d\right)_k}{\left(q^d, q^d/c, q^d/e, q^{2r}/b; q^d\right)_k} q^{dk}, \tag{3.4}
$$

where $W = (n-r)/d$ or $n-1$.

Proof. Letting $q \to q^d$, $n \to (n-r)/d$, $a = q^r$, $b = cq^r$, $c = eq^r$, $d = aq^r$ and $e = q^r/a$ in Watson's ${}_8\phi_7$ transformation formula (3.2), we have

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{\left(q^r,cq^r,eq^r,q^{r-n},aq^r,q^r/a;q^d\right)_k}{\left(q^d,q^d/c,q^d/e,q^{d+n},q^d/a,aq^d;q^d\right)_k}\left(\frac{q^{2d+n-3r}}{ce}\right)^k$$
$$=[n]\frac{\left(q^r,q^{d-r};q^d\right)_{(n-r)/d}}{\left(q^d/a,aq^d;q^d\right)_{(n-r)/d}}\sum_{k=0}^{(n-r)/d}\frac{\left(q^{d-r}/ce,aq^r,q^r/a,q^{r-n};q^d\right)_k}{\left(q^d,q^d/c,q^d/e,q^{2r-n};q^d\right)_k}q^{dk}.$$

In light of the fact that $\left(q^{r-n};q^d\right)_k=0$ for $n-1\geq k>(n-r)/d$, we confirm the correctness of (3.4). ∎

Now, we present a parametric generalization of Theorem 2.

Theorem 3. *Let $n>1$, $d\geq 2$ be integers with $n\equiv r \pmod d$ and $r\in\{1,-1\}$. Then, modulo $\Phi_n(q)^2\,(1-aq^n)\,(a-q^n)$,*

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{\left(q^r;q^d\right)_k^2\left(cq^r,eq^r,aq^r,q^r/a;q^d\right)_k}{\left(q^d;q^d\right)_k^2\left(q^d/c,q^d/e,q^d/a,aq^d;q^d\right)_k}\left(\frac{q^{2d-3r}}{ce}\right)^k$$
$$\equiv[n]Q_q(a,n)\sum_{k=0}^{(n-r)/d}\frac{\left(q^{2r+dk};q^d\right)_{(n-r)/d-k}\left(q^{d-r}/ce,aq^r,q^r/a,q^r;q^d\right)_k}{\left(q^d,q^d/c,q^d/e;q^d\right)_k}q^{dk}, \tag{3.5}$$

where

$$Q_q(a,n)=\frac{q^{r(r-n)/d}\,(1-aq^n)\,(a-q^n)}{(1-a)^2}\left\{\frac{1}{\left(q^d;q^d\right)_{(n-r)/d}}-\frac{\left(q^d;q^d\right)_{(n-r)/d}}{\left(q^d/a,aq^d;q^d\right)_{(n-r)/d}}\right\}$$
$$+\frac{q^{r(r-n)/d}}{\left(q^d;q^d\right)_{(n-r)/d}}.$$

Proof. It is easy to see that $\Phi_n(q)\,(1-aq^n)\,(a-q^n)$ and $b-q^n$ are relatively prime polynomials. Noting the relations

$$\frac{(b-q^n)\left(ab-1-a^2+aq^n\right)}{(a-b)(1-ab)}\equiv 1 \pmod{(1-aq^n)\,(a-q^n)},$$
$$\frac{(1-aq^n)\,(a-q^n)}{(a-b)(1-ab)}\equiv 1 \pmod{b-q^n},$$

and employing the Chinese remainder theorem for coprime polynomials, we arrive at the following result from Lemma 2 and Lemma 3: modulo $\Phi_n(q)\,(1-aq^n)\,(a-q^n)\,(b-q^n)$,

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{\left(q^r,cq^r,eq^r,q^r/b,aq^r,q^r/a;q^d\right)_k}{\left(q^d,q^d/c,q^d/e,bq^d,q^d/a,aq^d;q^d\right)_k}\left(\frac{b}{ce}\right)^k q^{(2d-3r)k}$$
$$\equiv[n]\theta_q(a,b,n)\sum_{k=0}^{(n-r)/d}\frac{\left(q^{d-r}/ce,aq^r,q^r/a,q^r/b;q^d\right)_k}{\left(q^d,q^d/c,q^d/e,q^{2r}/b;q^d\right)_k}q^{dk}, \tag{3.6}$$

where the notation $\theta_q(a,b,n)$ on the right-hand side denotes

$$\theta_q(a,b,n)=\frac{(b-q^n)\left(ab-1-a^2+aq^n\right)}{(a-b)(1-ab)}\frac{(b/q^r)^{(n-r)/d}\left(q^{2r}/b;q^d\right)_{(n-r)/d}}{(bq^d;q^d)_{(n-r)/d}}+\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\frac{\left(q^r,q^{d-r};q^d\right)_{(n-r)/d}}{\left(q^d/a,aq^d;q^d\right)_{(n-r)/d}}.$$

It is not difficult to see that

$$\begin{aligned}\left(q^{d-r};q^d\right)_{(n-r)/d}&=\left(1-q^{d-r}\right)\left(1-q^{2d-r}\right)\cdots\left(1-q^{n-2r}\right)\\&\equiv\left(1-bq^{d-r-n}\right)\left(1-bq^{2d-r-n}\right)\cdots\left(1-bq^{-2r}\right)\\&\equiv(-1)^{(n-r)/d}b^{(n-r)/d}q^{\frac{(n-r)(d-n-3r)}{2d}}\left(q^{2r}/b;q^d\right)_{(n-r)/d}\pmod{b-q^n},\\\left(q^r;q^d\right)_{(n-r)/d}&=(1-q^r)\left(1-q^{d+r}\right)\cdots\left(1-q^{n-d}\right)\\&\equiv\left(1-bq^{r-n}\right)\left(1-bq^{d+r-n}\right)\cdots\left(1-bq^{-d}\right)\\&\equiv(-1)^{(n-r)/d}q^{\frac{(n-r)(n-d+r)}{2d}}\left(q^d/b;q^d\right)_{(n-r)/d}\pmod{b-q^n}.\end{aligned}$$

Therefore, we can rewrite (3.6) as, modulo $\Phi_n(q)\,(1-aq^n)\,(a-q^n)\,(b-q^n)$,

$$\begin{aligned}&\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{\left(q^r,cq^r,eq^r,q^r/b,aq^r,q^r/a;q^d\right)_k}{\left(q^d,q^d/c,q^d/e,bq^d,q^d/a,aq^d;q^d\right)_k}\left(\frac{b}{ce}\right)^k q^{(2d-3r)k}\\&\equiv[n]\Omega_q(a,b,n)\sum_{k=0}^{(n-r)/d}\frac{\left(q^{2r+dk}/b;q^d\right)_{(n-r)/d-k}\left(q^{d-r}/ce,aq^r,q^r/a,q^r/b;q^d\right)_k}{\left(q^d,q^d/c,q^d/e;q^d\right)_k}q^{dk},\end{aligned}\tag{3.7}$$

where the notation $\Omega_q(a,b,n)$ on the right-hand side denotes

$$\Omega_q(a,b,n)=\frac{(b-q^n)\left(ab-1-a^2+aq^n\right)(b/q^r)^{(n-r)/d}}{(a-b)(1-ab)\left(bq^d;q^d\right)_{(n-r)/d}}+\frac{(1-aq^n)(a-q^n)(b/q^r)^{(n-r)/d}\left(q^d/b;q^d\right)_{(n-r)/d}}{(a-b)(1-ab)\left(q^d/a,aq^d;q^d\right)_{(n-r)/d}}.$$

It is easy to say that the limit of $b-q^n$ as $b\to1$ has the factor $\Phi_n(q)$. Meanwhile, since $n\equiv r \pmod d$, i.e., $\gcd(d,n)=1$, the factor $\left(bq^d;q^d\right)_{(n-r)/d}$ in the denominator of the left-hand side of (3.7) as $b\to1$ is relatively prime to $\Phi_n(q)$. Thus, letting $b\to1$ in (3.7), we conclude that (3.5) is true modulo $\Phi_n(q)^2\,(1-aq^n)\,(a-q^n)$ with the relation:

$$(1-q^n)\left(1+a^2-a-aq^n\right)=(1-a)^2+(1-aq^n)(a-q^n).$$

□

Proof of Theorem 2. By the L'Hospital rule, we have

$$\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{\frac{1}{(q^d;q^d)_{(n-r)/d}}-\frac{\left(q^d;q^d\right)_{(n-r)/d}}{(aq^d,q^d/a;q^d)_{(n-r)/d}}\right\}$$
$$=-\frac{[n]^2}{(q^d;q^d)_{(n-r)/d}}\sum_{i=1}^{(n-r)/d}\frac{q^{di}}{[di]^2}.$$

Letting $a\to 1$ in Theorem 3 and utilizing the above limit, we deduce that (3.1) is true modulo $\Phi_n(q)^4$ by noticing that $\left(q^r;q^d\right)_k^4\equiv 0 \pmod{\Phi_n(q)^4}$ for $(n-r)/d<k\le n-1$. From Lemma 1 with $r\in\{1,-1\}$ and $a=b=1$, we conclude that the congruence (3.1) holds modulo $[n]$. Since the least common multiple of $[n]$ and $\Phi_n(q)^4$ is $[n]\Phi_n(q)^3$, we obtain the desired result. □

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