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On the Ţiţeica-Johnson theorem

by

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Dedicated to Tudor Zamfirescu on his eightieth birthday

Abstract

We use planar circular inversions to show that the Ţiţeica-Johnson theorem, Euler's triangle theorem, and the porism of triangles for two circles are all equivalent.

Keywords: Three-circles theorem, Euler's triangle theorem, Poncelet's porism.
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1 Introduction

Many years ago, T. Zamfirescu informed one of the authors that the *three-circles theorem*, which has many interesting applications, goes back to the Romanian mathematician G. Ţiţeica (see, e.g., [11]) who derived it in 1908. It was rediscovered in 1916 by R. A. Johnson, cf. [3].

(i) Ţiţeica-Johnson theorem: *If three pairwise different circles of equal radii intersect at a point, then the circle passing through the remaining three points of intersection also has the same radius, see Figure 1.*

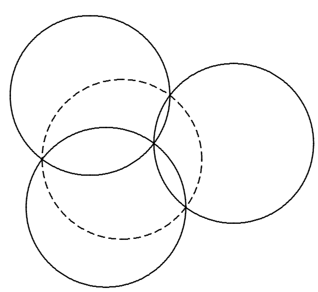

Figure 1: The Ţiţeica-Johnson theorem

Johnson says that "singularly enough, this remarkable theorem appears to be new. A rather cursory search in several of the treatises on modern elementary geometry fails to disclose it, and the author has not yet found any person to whom it was known." He observes that the four intersection points of the whole system form an *orthocentric system* (i.e., each of these points is the orthocenter of the triangle created by the other three), and he presents the three-circles theorem also in his book [4] (see page 75). Generalizations in Euclidean elementary geometry were, for example, obtained by Popescu [10] via the known *propeller theorem*, and (omitting the concurrence of the three given circles) by Mackenzie [5], getting the so-called triquetra theorem which shows interesting relations to *Poncelet's porism*.

More recent extensions of the theorem **(i)** refer to *normed planes*, also called *Minkowski planes*. Namely, Asplund and Grünbaum [1] confirmed that **(i)** even holds in all smooth, strictly convex normed planes obtaining from it further results which are basic for developing elementary geometry in normed planes. Martini and Spirova [6] continue these studies by showing immediate connections to *orthogonality concepts* and notions like the orthocenter and the Feuerbach (or nine-point) circle of triangles in such planes (also deleting the smoothness assumption). In the papers [7] and [8] the authors directly extend the three-circles theorem to *Clifford's chain of theorems* for congruent Minkowskian circles, and to a deeper study of the concept of *orthocentricity* with respect to suitable orthogonality concepts in normed planes.

By a *tangent circle* of a triangle in the plane we mean a circle that is tangent to the three lines determined by the three sides of the triangle. The unique tangent circle that is containd in the triangle is the *inscribed circle*, and other (three) tangent circles are called *escribed circles* of the triangle. Figure 3 below shows an escribed circle.

(ii) Euler's triangle theorem: *Let R, r denote the radii of the circumscribed circle and a tangent circle of a triangle in the plane, and let d denote the distance between the centers of these two circles. Then*

$$d^2 = \begin{cases} R^2 - 2rR & \text{if the tangent circle is the inscribed circle,} \\ R^2 + 2rR & \text{if the tangent circle is an escribed circle.} \end{cases}$$

Let K, C be two circles in the plane such that either C lies in the interior of K, or they cross each other. Note that if K is the circumscribed circle and C is an escribed circle of one and the same triangle, then K and C cross each other. When K and C cross, there are exactly two common tangents of K and C. In this case, the contact points of these common tangents with the circle K are denoted by a_1, a_2, and the intersection points of K, C are denoted by b_1, b_2 as shown in Figure 2.

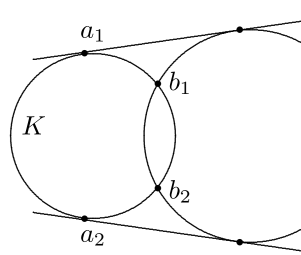

Figure 2: The crossing case

(iii) The porism of triangles for two circles: *Suppose that* (K, C) *is the pair consisting of the circumscribed circle and a tangent circle of a triangle* Δ*.*

(1) *If* C *is the inscribed circle of* Δ*, then for every point* $x \in K$ *there are* $y, z \in K$ *such that* C *is also the inscribed circle of the triangle* $\triangle xyz$*.*

(2) *If* C *is an escribed circle of* Δ*, then for any point* $x \in K \setminus \{a_1, a_2\}$ *that lies in the exterior of* C*, there are* $y, z \in K$ *such that* C *is also an escribed circle of the triangle* xyz*.*

Poncelet's porism for a pair of conics in general position is derived in [2] as a consequence of Pascal's theorem.

From (2) of **(iii)**, we have the following corollary.

Corollary. *If there is a triangle with circumscribed circle* K *and escribed circle* C*, then the line* a_1b_2 *is tangent to* C*.*

For a circle in the plane $\mathbb{R}^2$ with center p and radius r, the inversion φ of the plane $\mathbb{R}^2$ with respect to this circle is the map

$$\varphi : \mathbb{R}^2 \setminus \{p\} \to \mathbb{R}^2 \setminus \{p\}$$

such that for every $x \in \mathbb{R}^2 \setminus \{p\}$, $\varphi(x)$ lies on the ray $\overrightarrow{px}$ and satisfies $\|x - p\| \|\varphi(x) - p\| = r^2$. Note that $\varphi(p)$ is not defined. For properties of inversions of the plane see, e.g., [9]. In the following, we show that, under inversions of the plane, **(i)**,**(ii)** and **(iii)** are all equivalent.

2 (i) ⇒ (ii)

Let (K, C) be a pair consisting of the circumscribed circle and a tangent circle of a triangle $\triangle xyz$, and let R, r be the radii of K and C, respectively. We consider here only the case that the tangent circle C is an escribed circle tangent to the side yz. Let p be the center of C and q be the center of K. Hence $d = \|q - p\|$. Since the line xy meets C at a point on the prolongation of xy beyond y, and the line xz meets C at a point on the prolongation of xz beyond z, we have $\angle xyp > \pi/2$ and $\angle xzp > \pi/2$. Hence the center p of C must lie in the exterior of K. Let s, t be the intersection points of the line pq and the circle K, where s lies between q and p, as shown in Figure 3.

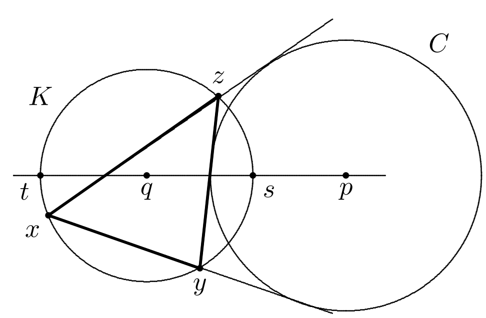

Figure 3: C is an escribed circle of $\triangle xyz$

Let φ be the inversion of the plane with respect to the circle C. The three lines xy, yz, zx are sent by φ to the circles of diameter r intersecting at p. Since $x, y, z \in K$, the image

$\varphi(K)$ is the circle passing through the three intersection points $\varphi(x), \varphi(y), \varphi(z)$ of the circles $\varphi(xy), \varphi(yz), \varphi(zx)$. Hence by **(i)**, $\varphi(K)$ has diameter r. Since $\|t-p\| = d+R$ and $\|s-p\| = d-R$, we have

$$\|\varphi(t)-p\| = \frac{r^2}{d+R}, \quad \|\varphi(s)-p\| = \frac{r^2}{d-R}.$$

Thus

$$\|\varphi(s)-\varphi(t)\| = \frac{r^2}{(d-R)} - \frac{r^2}{d+R} = \frac{2r^2R}{d^2-R^2}.$$

Since $\|\varphi(s)-\varphi(t)\|$ is equal to the diameter of $\varphi(K)$, we have

$$r = \frac{2r^2R}{d^2-R^2},$$

and hence $d^2 = R^2 + 2rR$. ∎

3 (ii) ⇒ (i)

Suppose that three circles B_1, B_2, B_3 of the same *diameter* r intersect at the point p. Let w_1, w_2, w_3 be the remaining three intersection points of B_1, B_2, B_3, and let D be the circle that passes through w_1, w_2, w_3. Let us show that the diameter of D is also r.

Let C be the circle with center p and *radius* r. Then, for each $i = 1, 2, 3$, the circle B_i is internally tangent to C. Let φ be the inversion of the plane with respect to the circle C. Then the images $\varphi(B_i), i = 1, 2, 3$, are lines tangent to C. Hence, C is a tangent circle of the triangle xyz, where $x = \varphi(w_1), y = \varphi(w_2), z = \varphi(w_3)$. Let us consider the case that C is the escribed circle of the triangle xyz tangent to the side yz. (The case that C is the inscribed circle of the triangle xyz follows similarly.) The circle $K := \varphi(D)$ is the circumscribed circle of the triangle xyz. Let q be the center of K and R be the radius of K. Put $d = \|q-p\|$, the distance between the centers of C and K. By **(ii)**, we have

$$d^2 = R^2 + 2rR.$$

Let s, t be the points where the line pq intersects K, as shown in Figure 3. Then

$$\|t-p\| = d+R, \quad \|s-p\| = d-R.$$

Hence

$$\|\varphi(t)-p\| = \frac{r^2}{d+R}, \quad \|\varphi(s)-p\| = \frac{r^2}{d-R},$$

and therefore

$$\|\varphi(s)-\varphi(t)\| = \frac{r^2}{d-R} - \frac{r^2}{d+R} = \frac{2r^2R}{d^2-R^2},$$

which is equal to r by **(ii)**. Since $\|\varphi(s)-\varphi(t)\|$ is the diameter of $\varphi(K) = D$, we are done. ∎

4 (i) ⇒ (iii)

Let us derive only the case (2) of **(iii)**. Let r be the radius of C, and p be the center of C. Denote by φ the inversion of the plane with respect to the circle C. Since the three lines l_1, l_2, l_3 determined by the three sides of Δ are at distance r from the center p of C, the images $\varphi(l_1), \varphi(l_2), \varphi(l_3)$ are circles of diameter r passing through p. Since the circle K passes through the three intersections of the lines l_1, l_2, l_3, the circle $\varphi(K)$ passes through the three intersection points of the circles $\varphi(l_1), \varphi(l_2), \varphi(l_3)$ other than p. Hence, by **(i)**, $\varphi(K)$ is a circle of diameter r.

Let $x \in K \setminus \{a_1, a_2\}$ lie in the exterior of C. Then there are two tangent lines of C that pass through x. And since x is different from a_1, a_2, these two tangent lines intersect K at two points $y, z \in K \setminus \{x\}$, that is, the lines xy, xz are tangent to C. Then the images $\varphi(xy), \varphi(xz)$ are circles passing through p and tangent to C. Hence they are circles of diameter r. Note that the three circles $\varphi(xy), \varphi(xy)$ and $\varphi(K)$ are all circles of diameter r, passing through $\varphi(x)$, and hence the circle, say D, determined by the three points $\varphi(y), \varphi(z), p$ is also a circle of diameter r by **(i)**. Hence D is tangent to C, and $\varphi^{-1}(D) = \varphi(D)$ is a line tangent to C. Since $\varphi(D)$ passes through y, z, it is the line yz. Thus the line yz is also tangent to C, that is, C is an escribed circle of the triangle xyz. □

Proof of the Corollary. Suppose that there is a triangle with circumscribed circle K, escribed circle C, and so that the line a_1b_2 is not tangent to C. Since the line a_2b_2 is also not tangent to C (it clearly cuts C), there is a point $x \in K \setminus \{a_1, a_2\}$ such that the line xb_2 is tangent to C. Then x lies in the exterior of C. Hence, by **(iii)**, there is a triangle xb_2z that has the circumscribed circle K and the escribed circle C. However, since the line xb_2 is tangent to C at b_2, the line zb_2 is never tangent to C, a contradiction. Therefore, the line a_1b_2 must be tangent to C. □

5 (iii)⇒(i)

Let α, β, γ be three pairwise distinct unit circles intersecting at a point p. We show that the circle τ passing through the remaining three intersection points is also a unit circle. Let C be a circle with radius 2 and center p. Then three circles α, β, γ are all internally tangent to C. Let φ denote the inversion of the plane with respect to the circle C. Then $\varphi(C) = C$, the lines $\varphi(\alpha), \varphi(\beta), \varphi(\gamma)$ are tangent to the circle C, $K := \varphi(\tau)$ is the circumscribed circle of the triangle determined by the three lines $\varphi(\alpha), \varphi(\beta), \varphi(\gamma)$, and C is a tangent circle of this triangle. Our task is to show that $\varphi(K) = \tau$ is a unit circle.

Let q be the center of K, and $\bar{x} \in K$ be the point lying on the prolongation of pq beyond q. By **(iii)**, there are $\bar{y}, \bar{z} \in K$ such that C is also a tangent circle of the triangle $\triangle\bar{x}\bar{y}\bar{z}$. Note that $\triangle\bar{x}\bar{y}\bar{z}$ is an isosceles triangle with base $\bar{y}\bar{z}$. Let $x = \varphi(\bar{x}), y = \varphi(\bar{y}), z = \varphi(\bar{z})$, and $X = \varphi(\bar{y}\bar{z}), Y = \varphi(\bar{z}\bar{x}), Z = \varphi(\bar{x}\bar{y})$. Then X, Y, Z are unit circles internally tangent to $C = \varphi(C)$. Let u, v, w be the centers of X, Y, Z, respectively, see Figure 4. Let ps be the diameter of X. Then

(∗) the quadrilateral $yszx$ is a rhombus.

This can be seen as follows. Since $ywpu$ is a rhombus, the lines yu and wp are parallel. Hence $\angle yws = \angle uyw = \angle upw$, and since $\angle upw = \angle yux$ (the triangle $\triangle uxp$ is an isosceles triangle), we have $\angle yws = \angle yux$. Thus the two isosceles triangles $\triangle yws$ and $\triangle yux$ are congruent, and hence $\|y - s\| = \|y - x\|$. Therefore $yszx$ is a rhombus. Since the circle X passing through y, s, z is a unit circle, the circle $\tau = \varphi(K)$ that passes through x, y, z is also a unit circle. ∎

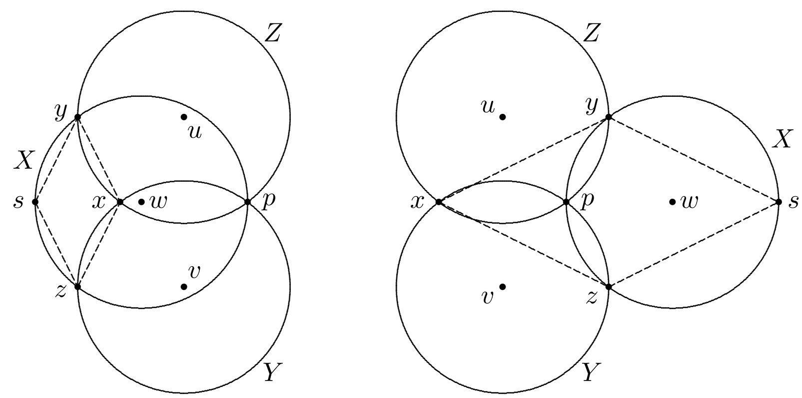

Figure 4: Two cases for the location of X

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