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Six problems in Intuitive Geometry

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Dedicated to Professor Tudor Zamfirescu, an exceptional problem solver and poser, on the occasion of his 80th birthday

Abstract
In this note we present several open, yet easy to state, geometric problems.

Key Words: Intuitive geometry, wrapping of a polyhedral surface, isoperimetric inequality, mean curvature, Möbius band.
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1 Introduction

What is "Intuitive Geometry"? There is no such mathematical field. The phrase is known as the title of the books [4], [1] and from a conference named "Intuitive Geometry" which was held in Hungary. Gábor Fejes Tóth, the organizer, said that "intuitive geometry" is a study that deals with geometric problems that can be explained and understood by passersby. The third author holds a small meeting under this title every year in Japan. In this note, we will discuss six open problems which were proposed at this meeting.

2 Wrapping convex bodies by square-shaped paper

It is known that the efficient[1] wrapping of polyhedral surfaces by paper assumes using any thin strip [2], [3]. Here, we focus on using square-shaped paper to cover convex bodies without cutting. It is also known that the smallest square-shaped paper that wraps the unit cube has a side length of $2\sqrt{2}$, see [3] and Figure 1.

For the surface (i.e., boundary) S of a compact convex body, its radius $r(S)$ is defined by $r(S) := \min_{x\in S} \max_{y\in S} d(x, y)$, where $d(\cdot, \cdot)$ denotes the intrinsic distance on S. Using this radius, it follows easily that if a convex body is wrapped with square-shaped paper, then its radius is at most half of the diagonals, and the radius of the unit cube is 2.

The second author showed at the meeting "Intuitive Geometry 2023" that the regular tetrahedron T with its edge length 1 can be wrapped by a square-shaped paper with side length $\sqrt{3 - x^2}$, see Figure 2, where x is satisfying the following equations:

$$\sqrt{3 - x^2} = 2\sqrt{y^2 + (\frac{1}{2})^2} + x \text{ and } y^2 = \frac{x^2}{4(3 - x^2)}.$$

[1]Efficient means the area of the wrapping strip is close enough to the area of polyhedral surface.

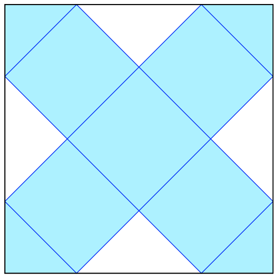

Figure 1: A square which can wrap the unit cube

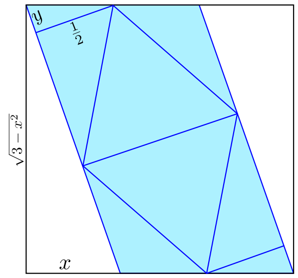

Figure 2: A square which can wrap the unit regular tetrahedron.

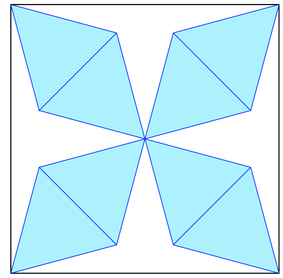

Figure 3: A square which can wrap the unit regular octahedron.

The side length of this square is approximately 1.63411, while $r(T) = 1$ [7].

He also showed that the regular octahedron with edge length 1 can be wrapped by a square-shaped paper with side length $\sqrt{6}$, see Figure 3.

It seems that the side lengths provided in the above two examples are minimal, but we don't have a complete proof. Notice that the unit sphere has its radius equal to π, and the second author can prove that a square with side length $\sqrt{2}\pi$, with infinitely many wrinkles, can wrap it; see Figure 4. Imagine also a wrapped chocolate candy.

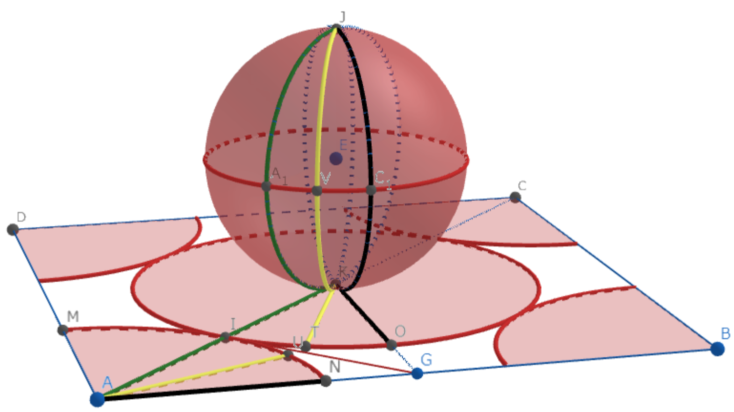

Figure 4: How to wrap the unit sphere by the square with diagonal 2π.

Question 2.1. *What is the smallest square-shaped paper that can wrap the regular tetrahedron, respectively the regular octahedron, without cutting?*

Question 2.2. *What is the convex body of maximal volume which can be wrapped by a square-shaped paper with side length* 1 *without cutting?*

Let K be a convex body. If we consider K to be an extended cube or a square frustum (truncated square pyramid), the second author can prove that the maximal volume is $\frac{2}{27}$, and it is attained by the extended cube with edge lengths in the ratio of $2:2:1$, see Figure

5. In the case K is a round sphere, the maximal volume is not more than $\frac{4}{3}\pi(\frac{\sqrt{r}}{2\pi})^3$, which is smaller than for the respective extended cube.

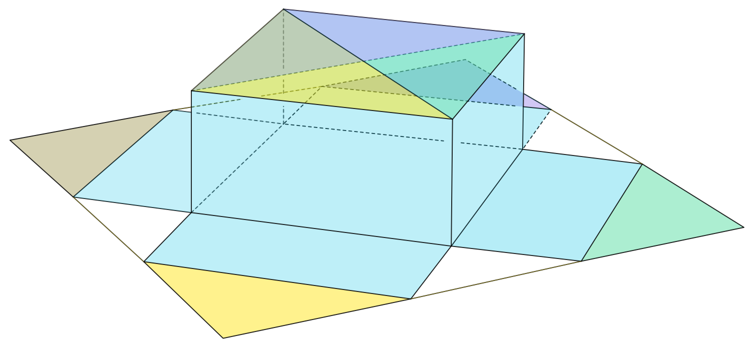

Figure 5: Extended cube of maximal volume, wrapped by the unit squre.

3 Special isoperimetric inequalities

Many results are known about isoperimetric inequality, the oldest dating back to the Ancient Greece. However, there are fewer results when restricting to certain shapes. In the case of vertical cylinders with lids, the following can easily be shown, by using the inequality of arithmetic and geometric means [6]. Under fixing the surface area, the volume becomes maximal when the ratio of the base area to the side area is $1 : 4$. Y. Matsuda [6] showed that, under fixing the sum of the side area and the bottom base area (excluding the top base area), the volume becomes maximal when the ratio of the base area to the side area is $1 : 2$. This is commonly referred to as the "material-saving container problem". In the case of tetrahedra, the regular one attains the maximal volume, as shown by Zalgaller [8].

The forth author showed[2] the following, by using Lagrange's undetermined multiplier method. In the case of the regular cone (where the base is a disc or a regular n-gon and the apex projects orthogonally on the base center), under the constraint of fixing the surface area, the volume attains its maximum when the ratio of the base area to the side area is $1 : 3$. Moreover, she showed that if only the side area is fixed, the volume becomes maximal when the ratio of the base area and the side area is $1 : \sqrt{3}$.

In the case of a cone with a regular n-gon base, let h be its height and r be the radius of circumscribed circle of the base. If we fix the surface area, the volume attains its maximum value when $r : h = 1 : 2\sqrt{1 + \cos\frac{2\pi}{n}}$. For example, in the case of the cone over the equilateral triangle, the maximum volume is attained when $r : h = 1 : \sqrt{2}$ (i.e., the regular tetrahedron), whereas in the case of the cone over the square, that happens when $r : h = 1 : 2$. For the cone with a circular base, under fixing the total surface area, the volume is maximal when the ratio of the radius of the base disc to the height is $1 : 2\sqrt{2}$. See Figures 6, 7 and 8.

[2] Graduation thesis at Sugiyama Jogakuen University, 2024.

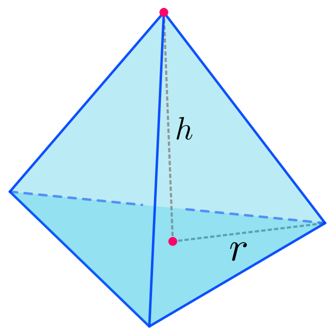

Figure 6: The triangular cone of maximal volume fixing the whole surface area ($r : h = 1 : \sqrt{2}$, hence it is a regular tetrahedron).

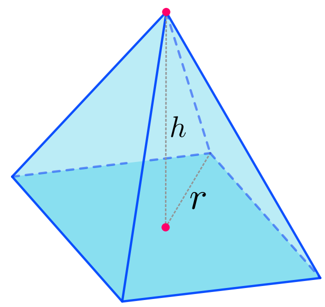

Figure 7: The regular cone of maximal volume over square, fixing the whole surface area ($r : h = 1 : 2$).

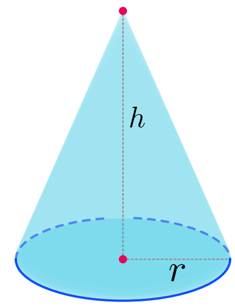

Figure 8: The cone of maximal volume fixing the whole surface area ($r : h = 1 : 2\sqrt{2}$).

If we fix only the side area, for a cone with a regular n-gon base, the volume becomes maximal when $r : h = 1 : \sqrt{1 + \cos\frac{2\pi}{n}}$. Specifically, in the case when the base is an equilateral triangle or a square, the optimal ratios are $1 : \frac{\sqrt{2}}{2}$, and $1 : 1$ respectively. The optimal ratio for a cone with a circular base is $1 : \sqrt{2}$.

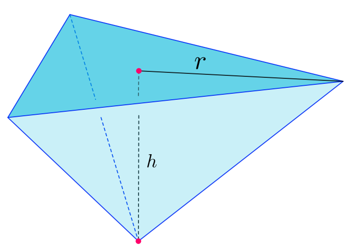

Figure 9: The triangular base container with maximal volume fixing the side area ($r : h = 1 : \frac{1}{\sqrt{2}}$).

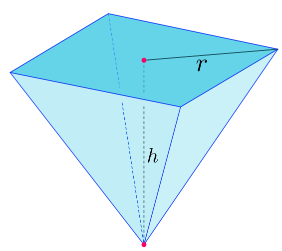

Figure 10: The square base container with maximal volume fixing the side area ($r : h = 1 : 1$).

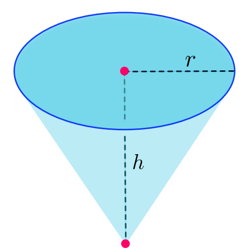

Figure 11: The cone container with maximal volume fixing the side area ($r : h = 1 : \sqrt{2}$).

In the case of a circular frustum which is a truncated cone (pedestal cone), the authors couldn't determine the critical shape.

Question 3.1. *Which shape of a circular frustum (or frustum with a regular n-gonal base) attains the maximal volume, under fixing the total surface area?*

Of course, considering the shape of a bucket (pudding container) is also an interesting problem.

Question 3.2. *Which shape of a circular frustum attains the maximal volume, under fixing the sum of the side surface area and one of the base areas?*

4 Mean curvature on developable Möbius bands

In 1930, M. Sadowsky constructed a developable Möbius band by using two cylinders with radius 1 and one cylinder with radius 2, see [5] p.3-6, and Figure 12, where developable means that all points on the surface have zero Gauss curvature, i.e., it can be made of flat paper.

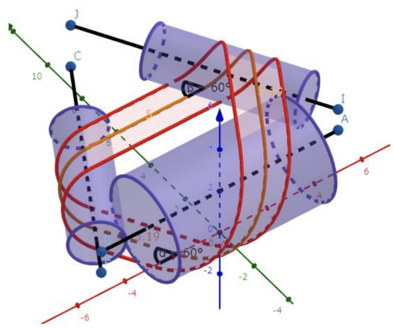

Figure 12: M. Sadowsky's developable Möbius band.

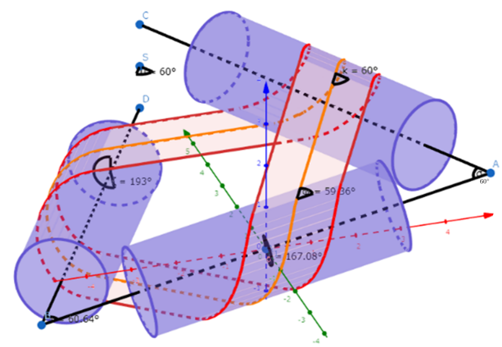

Figure 13: S. Honda's developable Möbius band.

The first author constructed[3] a developable Möbius band by using three cylinders with the same radius, see Figure 13.

The width of these Möbius bands in Figure 12 and 13 is equal to 2. In M. Sadowsky's example, the integral of their squared mean curvature is approximately 5.88, while in S. Honda's example, it is approximately 5.08.

Question 4.1 (M. Sadowsky)**.** *Which developable Möbius band attains the minimal value of the integral of its squared mean curvature?*

As the radii of all cylinders are gradually increased, keeping their ratio while fixing the axes of the three cylinders, the planar[4] part of the Möbius band becomes smaller, and finally, the width of the Möbius band becomes 0. In this state, we compare the value of the integrals of the squared mean curvature. In M. Sadowsky's example, $\int_0^l H^2 ds = \frac{15\pi^2}{4l} \approx \frac{37.06}{l}$, and in S. Honda's example, $\int_0^l H^2 ds \approx \frac{31.92}{l}$, where l is the length of the curve (i.e., the extremal state of the respective Möbius band).

In S. Honda's example of this extreme state, there is one cylinder whose axis intersects the axes of the two adjacent cylinders. She calculated the angle to be approximately $62.88°$ by using Newton's method, see Figure 15.

Furthermore, she considered a developable Möbius band twisted twice, thrice, four times, and five times, see e.g. Figures 16-17. Notice that the Möbius bands in Figures 12-13 are twisted only once.

Question 4.2. *Which developable Möbius band twisted n-times attains the minimal value of the integral of its squared mean curvature?*

[3]Graduation thesis at Sugiyama Jogakuen University, 2021.

[4]I.e., both the Gauss curvature and the mean curvature are 0.

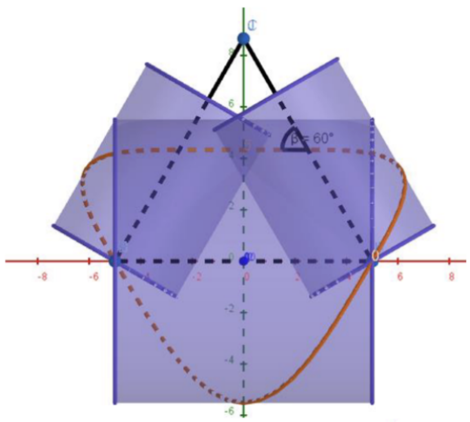

Figure 14: Extreme state of M. Sadowsky's developable Möbius band.

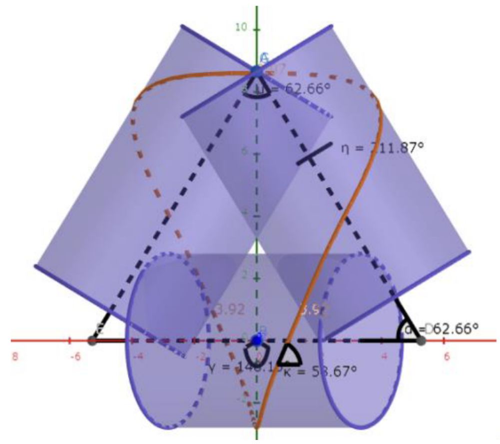

Figure 15: Extreme state of S. Honda's developable Möbius band.

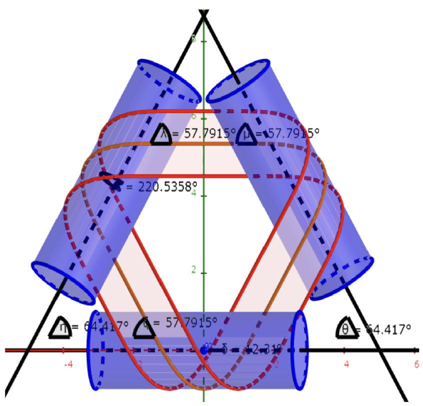

Figure 16: A developable Möbius band twisted thrice, viewed from the top.

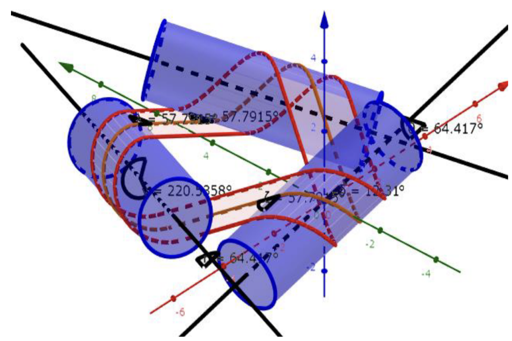

Figure 17: A developable Möbius band twisted thrice, viewed from an angle.

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