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A new class of premature, partial latin squares

by
Reinhardt Euler

Bon anniversaire, Tudor!

Abstract

Combining two well-known types, we present a new class of partial latin squares which are not completable and minimal with respect to this property.

Key Words: Partial latin square, premature, completable.
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1 Introduction

A *partial latin square* P of order n is an $n \times n$-array partially filled with symbols from $N = \{1, \dots, n\}$ each symbol occurring at most once in each row and each column. If there are no empty cells we speak of a *latin square* L, and if for $r, s \in N$ the non-empty cells form a rectangle R of r rows and s columns, we speak of an $r \times s$ - *latin rectangle*. A partial latin square P is said to be *completable*, if there is a latin square L of the same order in which $P(i,j) = L(i,j)$ for each non-empty cell (i,j) of P. Characterizing for a given n those partial latin squares that are completable, is an open question. Partial results include the proof of Evan's conjecture [7] by Smetaniuk [10] and Andersen and Hilton[1], which states that a partial latin square containing at most $n-1$ filled cells is always completable. Hall [8] has shown that an $r \times n$ - latin rectangle can always be completed and Ryser's result [9] states that an $r \times s$ - latin rectangle R can be completed if and only if each symbol $i \in N$ appears at least $r + s - n$ times in R.

In Euler and Oleksik [6], we have given a complete answer to the question: when is a partial $3 \times n$ latin rectangle completable? To explain our approach, observe that any partial latin square P can be identified with a subset $E(P)$ of $E = \{e(i,j,k), 1 \leq i,j,k \leq n\}$ as follows: $e(i,j,k) \in E(P)$ if and only if $P(i,j) = k$. In particular, any latin square corresponds to a specific n^2-element subset of E, and the family $\mathcal{B}$ of all these subsets constitutes a clutter , say of *bases*, in the sense of matroid theory. Any such clutter induces a (unique) clutter $\mathcal{C}$ of *circuits*, i.e., subsets of E that are not contained in any member of $\mathcal{B}$ and minimal with respect to this property. Consequently, a complete knowledge of $\mathcal{C}$ would give an answer to the completability question in the following sense: a partial latin square P can be completed if and only if $E(P)$ does not contain any circuit. We may also think of a hypergraph $\mathcal{H} = (E, \mathcal{C})$ defined over the vertex set E, whose maximal independent sets are given by $\mathcal{B}$ and whose hyperedges are to be described explicitly.

Not much work has been done on describing the clutter $\mathcal{C}$ for latin squares of a given order, and by Colbourn's result [3] there is little hope to ever completely describe this clutter. There are, however, partial results which may be of interest beyond purely combinatorial aspects, in particular for related applications such as time-tabling, statistics or information theory.

Andersen and Hilton [1] have fully characterized those non-completable, partial latin squares which contain n symbols (see Figure 1 for two examples of order 4).

1			
	1		
		1	
			2

	2	3	
4			
1			

Figure 1:

In 1985, Euler et al. [4] have studied the clutter $\mathcal{C}$ in view of the application of linear programming techniques to the solution of the planar 3-dimensional assignment problem (3PAP), the solutions of which correspond to the latin squares of the given order. In this context, the members of $\mathcal{C}$ give rise to facet-defining inequalities for associated polyhedra.

In 2002, Brankovic et al. [2] have coined the term of *premature* partial latin squares, i.e., non-completable partial latin squares, which are completable after deletion of any of their symbols, and studied their *spectrum*, i.e., the set of integers t such that there exists a premature partial latin square of order n with exactly t non-empty cells.

Some types of premature partial latin squares, related to $r \times n$ - latin rectangles, are studied in Euler [5] and our work on $3 \times n$ - latin rectangles [6] contains a complete description of minimal, non-completable partial $3 \times n$ - latin rectangles. Moreover, we had generated the complete family of circuits for order 4, one type of which was not covered by this description, i.e., the associated partial latin square contains all 4 symbols and no empty row or column. This type can be generalized to a partial latin square of order n, say $\hat{P}$, as follows: $\hat{P}(1,j) = j$ for $j = 2, \ldots, p$, $\hat{P}(i,1) = n + 2 - i$ for $i = 2, \ldots, n - p + 1$, $\hat{P}(i,i) = 1$ for $i = 2, \ldots, n - 1$, $\hat{P}(n,n) = 2$ and all remaining cells being empty. It is the objective of this paper to show that this generalization to arbitrary n, illustrated in Figure 2, represents a premature, partial latin square of order n for all values p with $3 \le p \le n-1$. Our proof will also show that we hereby obtain a whole new class of such latin squares.

Quite obviously, $\hat{P}$ cannot be completed and what remains to be shown is its minimality. If for instance, we delete symbol 2 in cell (n,n), we can complete the resulting array to what is known as a *(forward) circulant latin square*, that we denote by CL (see Figure 3 for an illustration) and whose definition can be given as follows: $CL(1,j) = j$ for $j = 1, \ldots, n$, $CL(i,1) = n - i + 2$ for $i = 2, \ldots, n$ and $CL(i+1, j+1) = CL(i,j)$ for $i,j = 1, \ldots, n-1$. In a similar way, deleting symbol 1 in cell $(n-1, n-1)$ allows completability to CL with an interchange of the last two rows.

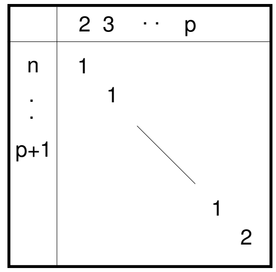

Figure 2: The partial latin square $\hat{P}$

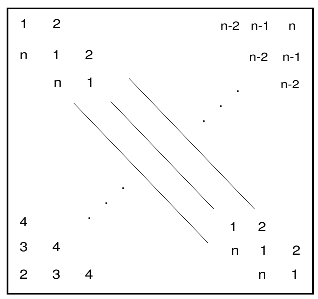

Figure 3: The circulant latin square CL

The completions that we are going to propose for a deletion of the remaining symbols of $\hat{P}$ will be constructed so as to be as close as possible to CL. For this we introduce the following definitions.

Definition 1.1. *For $i, j, k \in N$, a symbol k in cell (i, j) of a partial latin square P is said to be* in circulant position *if $CL(i, j) = k$. More generally, a completely filled row P_i or column P^j or a permutation Pk, i.e., n symbols of a same type k no two of which in a same row or column, is called* circulant, *if all of its symbols are in circulant position.*

Definition 1.2. *An (a, b)-*alternating path *p in a partial latin square P is given by a sequence $(i_1, j_1, a), (i_1, j_2, b), (i_2, j_2, a), (i_2, j_3, b), \ldots,$ up to $\alpha) : (i_l, j_l, a)$ or $\beta) : (i_l, j_{l+1}, b)$, where $a, b, l \in N$, $i_1, \ldots, i_l, j_1, \ldots, j_{l+1} \in N$ for all l, and $i_r \neq i_s, j_r \neq j_s$ for $r \neq s$. In case $\alpha)$, p is said to be* of odd length $2l - 1$, *in case $\beta)$* of even length $2l$. *Finally, if $j_{l+1} = j_1$, we speak of an (a, b)-alternating cycle (of length $2l$).*

Just observe, that the symbols a, b may start to appear in a same column, and that the notion of conjugacy gives us two more variations of such a path, i.e., all symbols a, b appear in two rows of P, different in each column, or in two columns of P and different in each row.

2 Our proof for the first row of $\hat{P}$

It will be sufficient to consider the first row of $\hat{P}$ since playing with the value of p allows to cover all cases in the first column of $\hat{P}$, except for $p = n-1$. Let q denote the symbol to be deleted in row 1 of $\hat{P}$ and $\hat{P}(q)$ the result of this deletion. The basic idea of our proof for most of the cases is to start with a partial latin square induced by $\hat{P}(q)$ that we call a *basic form* BF, in which symbol q is placed in cell $(1,1)$, replaced by symbol n in cell $(1,q)$ and symbol 2 in cell $(n-q+2,1)$, and which contains the completed permutations $\hat{P}1$ and $\hat{P}2$. Then we insert a number of circulant permutations in such a way that we obtain a completion to a full latin square by completing the 3 symbols of type q and n to a union of disjoint (q,n)-alternating cycles.

We will consider two cases, q odd and q even, and start with $q = 3,5,7,\ldots,n-1$ for n even and $q = 3,5,7,\ldots,n-2$ for n odd. The basic form BF_1 represented in Figure 4 illustrates the first step of our construction. In BF_1, we complete the permutations Pk for $k = 3,\ldots,n-1, k \neq q$ to become circulant. The remaining empty cells can then be filled along the disjoint union of (q,n)-alternating cycles.

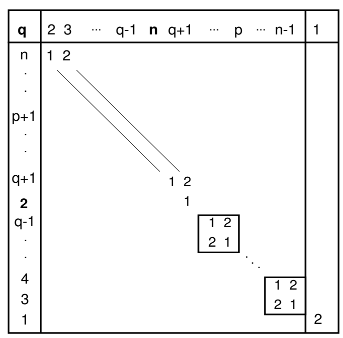

Figure 4: Basic form BF_1

Suppose now that q is even. For $q = 6,8,\ldots,n-2$ for n even and $q = 6,8,\ldots,n-3$ for n odd we use the basic form BF_2 represented in Figure 5, in which the $n-q$ symbols of type $n-1$ are placed in cells $(7,6),(8,7),(9,8),\ldots,(n-q+5,n-q+4)$ and $(n-q+6,5)$ for $n-q = 3,5,7,\ldots$. For $n-q = 2,4,6,\ldots$, the $n-q$ symbols of type $n-1$ are placed the same way. We then proceed as in BF_1 to end up with a disjoint union of (q,n)-alternating cycles providing a complete latin square.

Some cases remain to be covered:

For $q = 2$ and n even,, we complete BF_1 for $q = 2$ with circulant permutations Pk, $k = 3,\ldots,n-1$ to obtain a complete latin square by filling the remaining $n-2$ symbols of type n.

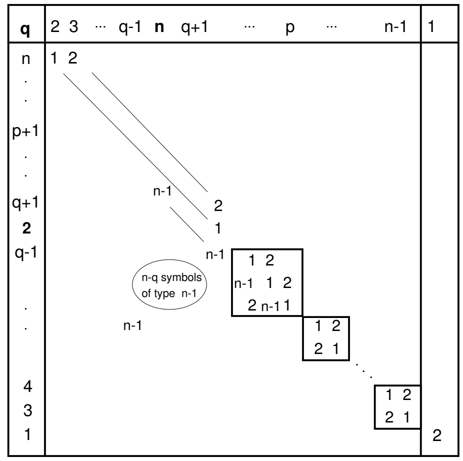

Figure 5: Basic form BF_2

For $q = 2$ and n odd, we complete BF_2 (the $n-q$ symbols of type $n-1$ being deleted) with circulant permutations Pk, $k = 3, \ldots, n-2$ to be left with a unique $(n-1, n)$-alternating cycle providing a complete latin square.

For $q = 4$ and n even or odd, we complete BF_1 for $q = 4$ by exchanging in column 1 symbol 2 against symbol 3, deleting symbol 2 in cell $(n-1, n-2)$ and placing symbol 3 in cells $(n-1, n)$ and $(n, 2)$. Then we complete symbols 3 as well as the permutations Pk, $k = 5, \ldots, n-1$ to become circulant. This will leave a disjoint union of $(4, n)$-alternating cycles allowing to obtain a complete latin square.

For $q = n$, i.e., the first symbol in $\hat{P}_2$, with n even and $q = n-1$, n odd, we complete BF_2 (without the $n-q$ symbols of type $n-1$) with circulant permutations Pk for $k = 3, \ldots, n-2$, which leaves a disjoint union of $(n-1, n)$-alternating cycles for an entire completion.

Finally, for the same $q = n$ and n odd, we complete BF_1 for $q = n$ with circulant permutations Pk, $k = 3, \ldots, n-1$, which leaves a disjoint union of $(n-1, n)$-alternating cycles for an entire completion.

3 Our proof for the diagonal of $\hat{P}$

For $\hat{P}$, let us denote with d the position of symbol 1 in cell (d, d) on the diagonal that is to be deleted and with $\hat{P}(d)$ the result of this deletion. Similar to our proof for the first row, we will consider two basic cases, namely d even and d odd, whose presentation we are now going to prepare. We start with $d = 2$ and consider the cases: n even and n odd. A completion to a full latin square $\hat{L}$ of $\hat{P}(2)$ can be obtained by completing to circulant form:

- columns $\hat{P}^k$ for $k = 3, 4, \ldots, n-1$,
- for n even, rows $\hat{P}_{n-k+1}$ as well as the permutations $\hat{P}k$ for $k = 4, 6, 8, \ldots, n-2, n$;
- for n odd, rows $\hat{P}_{n-k+1}$ as well as the permutations $\hat{P}k$ for $k = 3, 5, 7, \ldots, n-2, n$ (with a slight modification for $\hat{P}3$).

We also complete permutation $\hat{P}1$ in a unique way by placing symbol 1 in cells $(2, n)$ and $(n, 2)$, so that the remaining cells can now be uniquely filled, too. Our construction for both cases is illustrated in Figure 6.

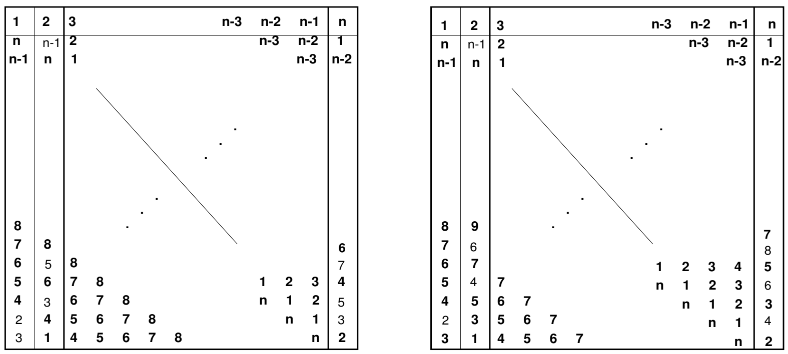

Figure 6: $d = 2$ for n even and for n odd

The question arises whether we could exploit some kind of symmetry to settle other cases. The answer is yes, as is shown by

Observation 3.1 (Symmetries)**.** *The following operation allows to obtain a completion for the case $d = n-2$ and n even:*

i) place the last column of $\hat{L}$ before the first;
ii) transpose the resulting latin square along its anti-diagonal;
iii) replace the first column after the last.

For n odd, a slight modification of step ii) will give the corresponding completion:

iia) exchange symbol 4 against symbol 5 along the alternating cycle of length 4 in rows $n-2, n-3$ and columns $2, 3$ and then symbol 3 against symbol 4 in such a cycle in rows $n-1, n-2$ and columns $1, 3$;

iib) transpose the resulting latin square along its anti-diagonal.

We should mention at this point that this operation will also be valid for the completions that we will construct for the two basic cases d even and d odd, i.e., when $d' = n - d$.

Let us now suppose that $d > 2$.

We first consider the case that d divides n with d an even number. The completion of $\hat{P}(d)$ is similar to the one we proposed for $d = 2$, i.e., we complete into circulant form:

- columns $\hat{P}^k$ for all $k \in N \setminus \{1, 2, d, n\}$
- rows $\hat{P}_{n-k+1}$ as well as the permutations $\hat{P}k$ for $k = 4, 6, 8, \ldots, d; d+2, d+3, ..., n$, except $k = 2d+1, \ldots, n-d+1$.

Finally, we complete permutation $\hat{P}1$ in a unique way by placing symbol 1 in cells $(1, 1)$, $(2, n)$ and $(n, 2)$ so that the remaining cells can now be uniquely filled, too. Figure 7 illustrates this completion.

If d is odd, we obtain a similar result if we complete into circulant form the same columns as before and

- rows $\hat{P}_{n-k+1}$ as well as the permutations $\hat{P}k$ for $k = 3, 5, 7, \ldots, d; d+2, d+3, ..., n$, except $k = 2d+1, \ldots, n-d+1$.

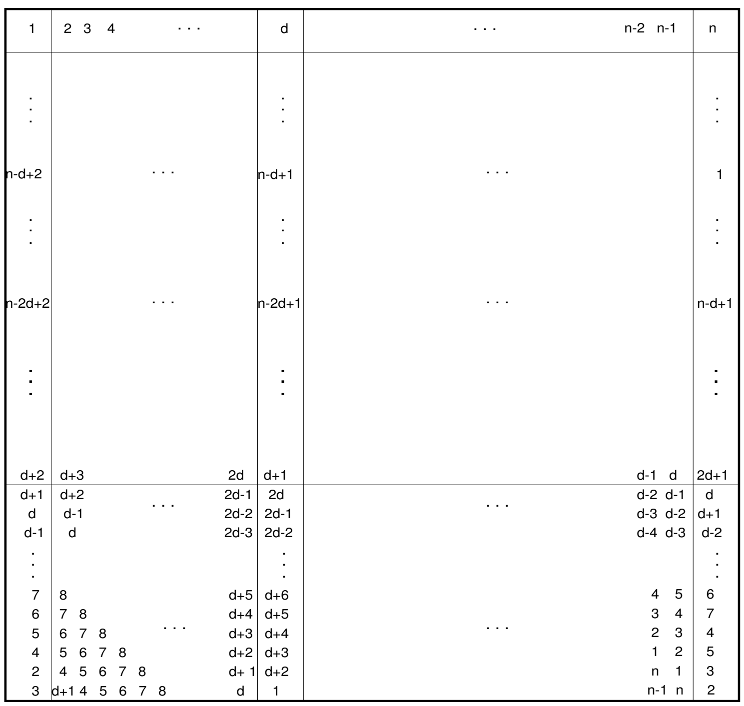

Figure 7: d divides n

We now turn to the case $n = md + r$ with $0 < r < d$. We may suppose that $m = 2$, both from our previous Observation 3.1 and since a completion for $n = 2d + r$ can easily be extended to $n = md + r$ for any $m > 2$ (see again Figure 7).

Case 1: d and r are both even.

Setting $d = mr + s$ with $0 \leq s < d$, it turns out that for $s = 2$ a completion of $\hat{P}(d)$ can be obtained by a $(d+1, d+r+1)$-alternating path starting in row $n-r$ and terminating in row $n-d+2$ (M' being empty in Figure 8). For the remaining cases and to stay as close as possible to CL we combine our path with a particular subarray M' that will result from a special calculation. The basic form BF_3 represented in Figure 8 illustrates all essential components from which a completion is obtained, if we insert circulant permutations $\hat{P}k$ for $k = 4, 6, 8, \ldots, d+r; d+r+2, d+r+3, \ldots, n$ and place all remaining symbols into circulant position.

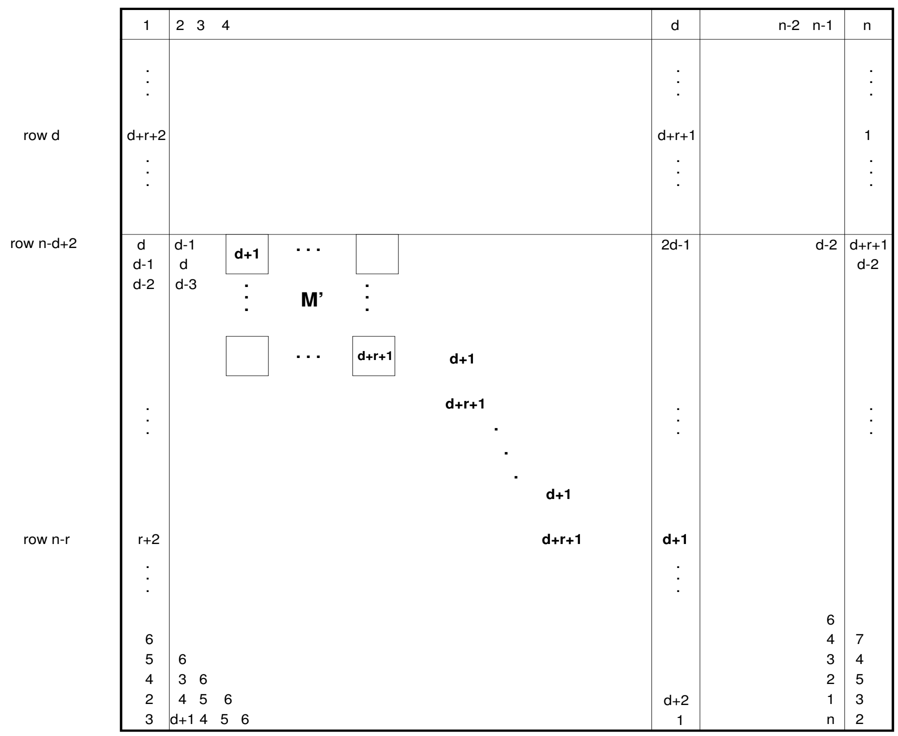

Figure 8: Basic form BF_3 for d and r even

How can we obtain an appropriate array M'?

We start by considering within the circulant latin square CL the subarray M of all odd symbols appearing in

- row $n-d+2$ up to row $n-d+r$ and column r up to column $r+2$, if $s = 0$,
- row $n-d+2$ up to row $n-d+s$ and column s up to column $r+2$, if $s > 2$.

M has the structure of a band-matrix BM of size, say $u \times v$, for which we may suppose that $u \le v$:

$$BM = \begin{bmatrix} 2u-1 & & \cdots & & 2v-1 & & & \cdots & 2(u+v) \\ \vdots & & & & & & & & \\ 7 & & & \ddots & & & \ddots & & \\ 5 & 7 & & & & & & & \\ 3 & 5 & 7 & & & & & & \\ 1 & 3 & 5 & 7 & \cdots & 2u-1 & & \cdots & 2v-1 \end{bmatrix}$$

and in which the entry 1 refers to symbol $d+1$ and $2(u+v)$ to $d+r+1$. To describe an array M' suitable to be combined with a $(d+1, d+r+1)$-alternating path, we need symbol $d+1$ to appear in the first and symbol $d+r+1$ in the last row of M'. This leads to the following problem for BM:

$$(P) = \begin{cases} \text{Transform } BM \text{ into } BM' \text{ in such a way that} \\ -BM_1'^1 = 1 \text{ and } BM_u'^v = 2(u+v) \\ -\text{in each row or column, the set of entries, up to 1 and } 2(u+v), \\ \ \text{remains unchanged and no entry appears twice.} \end{cases}$$

For matrix BM, and thus any array M, problem (P) can be solved as follows:

1. Define $BM(i)$ for $i = 3, 5, 7, \ldots, 2(u+v)-5$ to represent the unique square sub-matrix of BM induced by symbols $3, 5, 7, \ldots, 2(u+v)-5$ of order, respectively, $2, 3, 4, \ldots, u, \ldots, u, \ldots, 4, 3, 2$, order u appearing exactly $v-u+1$ times.

2. Start with an empty matrix BM' and set $BM_1'^1 = 1$ and $BM_u'^v = 2(u+v)$.

3. Insert a permutation Pi into sub-matrix $BM'(i)$ for $i = 3, 5, \ldots, 2(u+v)-5$.

For the validity of this procedure we just observe that at no step $i = 3, 5, \ldots, 2(u+v)-5$ there can exist a blocking sub-matrix within $BM'(i)$, i.e., an already filled sub-matrix of size $u' \times v'$ such that $u' + v' = \mathit{order}(BM'(i)) + 1$. Calculating the array M' for each value $s \neq 2$ thus provides a complete solution for Case 1.

Case 2: d is odd and r is even.

The characteristics for this case can be resumed as follows:

1. We use $(d+1, d+r-1)$-alternating paths starting in row $n-r+1$.

2. M' is empty for $s = 1$ and $s = r-1$, i.e., the alternating paths terminate in row $n-d+2$.

3. For $s \neq 1, r-1$, M' is a $u \times v$-array with $u = (s+1)/2$, $v = (r-s+1)/2$.

4. We insert circulant permutations $\hat{P}k$ for $k = 3, 5, \ldots, d, \ldots, d+r-2; d+r+2, d+r+3, \ldots, n$ and then place the remaining symbols in circulant position to obtain completion.

Case 3: r is odd and s is even.

1. We use $(d+1, d+r+1)$-alternating paths starting in row $n-r$.
2. M' is empty for
   - $s = 0$, i.e., the $(d+1, d+r+1)$-alternating path terminates in row $n-d$ if $r > 1$ and in row $n-d+2$ if $r = 1$,
   - $s = 2$, i.e., the $(d+1, d+r+1)$-alternating path terminates in row $n-d+2$.
3. For $s > 2$, M' is a $u \times v$-array with $u = s/2$, $v = (r+3-s)/2$.
4. We insert circulant permutations $\hat{P}k$ for $k = 4, 6, 8, \ldots, d+r-1; d+r+2, d+r+3, \ldots, n$ if d is even, and for $k = 3, 5, 7, \ldots, d+r-1; d+r+2, d+r+3, \ldots, n$ if d is odd and then place the remaining symbols in circulant position to obtain completion.

Case 4: r and s are both odd.

1. We use $(d+1, d+r)$-alternating paths starting in row $n-r+1$.
2. M' is empty for
   - $s = 1$ and $d = r+1$ even, case which requires a special solution whose basic form BF_4 is depicted in Figure 9. If we complete columns $2, d-2, d-1, d$ and n as indicated and insert all remaining symbols in circulant position, we obtain completion.
   - $s = 1$ and $d = mr+1$ with $m \geq 2$, i.e., the alternating paths terminate in row $n-d+2$.
3. For $s \neq 1$, M' is a $u \times v$-array with $u = (s+1)/2$, $v = (r+2-s)/2$.
4. We insert circulant permutations $\hat{P}k$ for $k = 4, 6, 8, \ldots, d+r-1; d+r+2, d+r+3, \ldots, n$ if d is even, and for $k = 3, 5, 7, \ldots, d+r-1; d+r+2, d+r+3, \ldots, n$ if d is odd and then place the remaining symbols into circulant position to obtain completion.

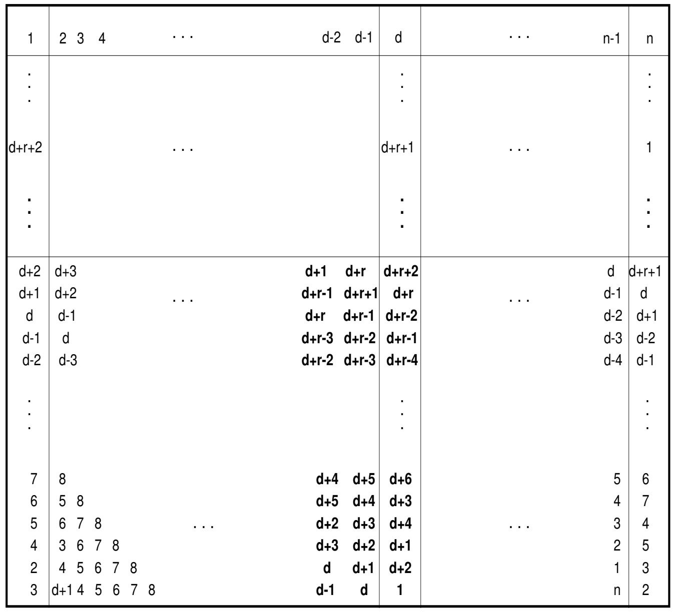

Figure 9: Basic form BF_4 for $d = r + 1$ in Case 4

Altogether, we have shown

Theorem 3.2. *$\hat{P}$ represents a premature, partial latin square for all p with $3 \leq p \leq n - 1$.*

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Lab-STICC, CNRS, UMR 6285, University of Brest,
B. P. 809, 20 Avenue Le Gorgeu, 29285 Brest Cedex, France
E-mail: `Reinhardt.Euler@univ-brest.fr`