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Minimizing vertex-degree function index for k-generalized quasi-trees

by

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Dedicated to my old friend Tudor on the occasion of his 80^{th} *birthday*

Abstract

In this paper the problem of minimizing the vertex-degree function index $H_f(G)$ for k-generalized quasi-trees of order n is solved for $k \geq 1$ and $n \geq 3k$ if the function f is strictly increasing and strictly convex. The extremal graph is a cycle C_n for $k = 1$ and $n \geq 3$. For $k = 2$ and $n \geq 6$ there are two families of extremal graphs depending upon the case when the inequality $f(3) + 3f(1) < 4f(2)$ is fulfilled or not. For $k \geq 3$ and $n \geq 3k$ there is a single family of extremal graphs and the number of pairwise non-isomorphic graphs of this family equals $1 + \lfloor (n - 3k)/2 \rfloor$.

Key Words: Vertex-degree function index, k-generalized quasi-tree, Jensen's inequality, majorization, Muirhead's Lemma.
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1 Introduction

Let G be a simple graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x, i.e., the number of neighbors of x in G. If the graph G is clear from the context, then we use $d(x)$ instead of $d_G(x)$. A vertex with degree one will also be referred as a pendant vertex. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the degree of vertex v_i equals d_i for $i = 1, 2, \ldots, n$, then $\pi = (d_1, d_2, \ldots, d_n)$ is called the *degree sequence* of G. We always will enumerate the degrees in non-increasing order, i.e., $d_1 \geq d_2 \geq \ldots \geq d_n$. P_n and C_n will denote the path and cycle on n vertices.

For $S \subset V(G)$, the subgraph induced by S is denoted $G[S]$. For a graph G and a subset X of $V(G)$, $G - X$ is the graph obtained from G by removing the vertices of X and all edges incident to any of them. In particular, when X consists of only one vertex v, $G - \{v\}$ is always abbreviated to $G - v$. Similar notation is $G - uv$, where $uv \in E(G)$.

A unicyclic graph G of order n is connected and has n edges. It consists of a cycle C_r, where $3 \leq r \leq n$ and some vertex-disjoint trees having each a vertex common with C_r. A bicyclic graph G of order n is a connected graph of size $|E(G)| = n + 1$. It has two linearly independent cycles which have a common vertex or a common path P_a with $a \geq 2$ or they are connected by a path P_b with $b \geq 2$. The *quasi-tree* is a connected graph G in which there exists a vertex $v \in V(G)$ such that $G - v$ is a tree. Xu et al. generalized in [14] the concept of quasi-tree to *k-generalized quasi-tree* as:

For any integer $k \geq 1$, the connected graph G is called a k-generalized quasi-tree, if there exists a subset $V_k \subseteq V(G)$ with cardinality k such that $G - V_k$ is a tree but for every subset V_{k-1} of cardinality $k-1$ of $V(G)$, the graph $G - V_{k-1}$ is not a tree. In [14], the authors pointed out that any tree is a quasi-tree since the deletion of any pendant vertex will produce another tree. Thus, they called any tree a trivial quasi-tree, and other quasi-tree graphs as non-trivial quasi-trees. In what follows, we call the vertex set V_k as a *k-quasi-vertex set*, and we use the symbol $\mathcal{T}_n^k$ to denote the class of k-generalized quasi-trees with n vertices.

For other notations and definitions in graph theory, we refer to [12].

The first Zagreb index $M_1(G)$ [3] is defined as $M_1(G) = \sum_{v \in V(G)} d^2(v)$. The general first Zagreb index (sometimes referred as "zeroth-order general Randić index" [4]), denoted by ${}^0R_\alpha(G)$ was defined [5] as ${}^0R_\alpha(G) = \sum_{v \in V(G)} d(v)^\alpha$, where α is a real number, $\alpha \notin \{0, 1\}$. For $\alpha = 2$ it is the first Zagreb index $M_1(G)$.

Todeschini et al. [7] introduced a variant of Zagreb indices which are called the first and second multiplicative Zagreb indices, and they are defined as:

$$\Pi_1(G) = \prod_{u \in V(G)} d(u)^2, \ \ \Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{u \in V(G)} d(u)^{d(u)}.$$

A generalized form of multiplicative Zagreb indices, which are called the first and second general multiplicative Zagreb indices was proposed by Vetrík and Balachandran [9]. For a graph G, they are defined as:

$$P_1^\alpha(G) = \prod_{u \in V(G)} d(u)^\alpha, \ \ P_2^\alpha(G) = \prod_{uv \in E(G)} (d(u)d(v))^\alpha = \prod_{u \in V(G)} d(u)^{\alpha d(u)},$$

where $\alpha \in \mathbb{R} \backslash \{0\}$.

In [9] the minimum and maximum general multiplicative Zagreb indices of trees with given order and number of branching vertices, pendant vertices or segments were obtained. Extremal results concerning general multiplicative Zagreb indices for unicyclic graphs were obtained in [1], for trees and unicyclic graphs with given matching number in [10], for trees and quasi-trees with perfect matchings and with given order and number of pendant vertices in [2].

The sum lordeg index is one of the Adriatic indices introduced in [11] and it is defined by $SL(G) = \sum_{v \in V(G)} d(v)\sqrt{\ln d(v)} = \sum\limits_{v \in V(G): d(v) \geq 2} d(v)\sqrt{\ln d(v)}$.

The vertex-degree function index $H_f(G)$ was defined in [15] as

$$H_f(G) = \sum_{v \in V(G)} f(d(v))$$

for a function $f(x)$ defined on positive real numbers. The problem of minimizing the vertex-degree function index $H_f(G)$ for k-generalized quasi-unicyclic graphs of given order was solved in [8] for functions $f(x)$ which are strictly increasing and strictly convex.

Several topological indices mentioned above are related to vertex-degree function index $H_f(G)$, where $f(x)$ is strictly convex and strictly increasing in the following cases:

1) ${}^0R_\alpha(G) = \sum_{v \in V(G)} d(v)^\alpha$, corresponds to $f(x) = x^\alpha$, where $x \geq 1$. This function is strictly convex and strictly increasing for $\alpha > 1$. Inequality (1) is fulfilled if and only if

$0 < \alpha < x_0$, where $x_0 \approx 3.21066$ is the positive solution of the equation $3^x - 4 \cdot 2^x + 3 = 0$ (see [13] for example).

2) The second general multiplicative Zagreb index is $P_2^\alpha(G) = \prod_{u \in V(G)} d(u)^{\alpha d(u)}$. We have $\ln P_2^\alpha(G) = \alpha \sum_{u \in V(G)} d(u) \ln d(u)$ and $f(x) = \alpha x \ln x$, where $x \geq 1$. This function is strictly increasing and strictly convex for $\alpha > 0$ and strictly decreasing and strictly concave for $\alpha < 0$. Inequality (1) is verified if and only if $\alpha > 0$; its reverse holds for $\alpha < 0$.

3) The sum lordeg index $SL(G) = \sum_{v \in V(G)} d(v)\sqrt{\ln d(v)}$. We get $f(x) = x\sqrt{\ln x}$, which is strictly increasing and strictly convex for $x \geq 2$ and (1) is true.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we solve the problem of minimizing the vertex-degree function index $H_f(G)$ for k-generalized quasi-trees G with given order if $f(x)$ is strictly increasing and strictly convex and characterize the extremal graphs.

2 Preliminary results

In what follows we shall suppose that the function $f(x)$ is strictly increasing and strictly convex. The following lemmas will be used in our proofs.

Lemma 1([8]). *Assume x and y are real numbers such that $y > 0$ and $x \geq y + 2$. Then*

$$f(x) + f(y) > f(x-1) + f(y+1).$$

For integers n, p such that $n \geq 1$ and $p \geq n$ denote by $D_{n,p}$ the set of n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of integers such that $x_1 \geq x_2 \geq \ldots \geq x_n \geq 1$ and $\sum_{i=1}^n x_i = p$. Consider the function $F(\mathbf{x}) = \sum_{i=1}^n f(x_i)$. By Lemma 1 the minimum of $F(\mathbf{x})$ is reached if and only if $|x_i - x_j| \leq 1$ for every $1 \leq i < j \leq n$, or equivalently, if and only if $x_1 + x_2 + \ldots + x_n$ is an equipartition of p, having almost equal parts. It follows that the point of minimum of $F(\mathbf{x})$ on $D_{n,p}$ is unique. Denote this point of minimum by $\mathbf{x}_{\min}$.

Let $\pi = (x_1, \ldots, x_n)$ and $\pi' = (x'_1, \ldots, x'_n)$ be two non-increasing integer sequences; we write $\pi \lhd \pi'$ if $\pi \neq \pi', \sum_{i=1}^n x_i = \sum_{i=1}^n x'_i$ and $\sum_{i=1}^j x_i \leq \sum_{i=1}^j x'_i$ for all $j = 1, \ldots, n-1$. Such an ordering is sometimes called majorization. For this partial order relation in $D_{n,p}$ the smallest n-tuple is $\mathbf{x}_{\min}$ and the greatest is $(p-n+1, 1, \ldots, 1)$, which will be also denoted by $(p-n+1, 1^{n-1})$, where the exponent indicates multiplicity. If $\pi = (x_1, \ldots, x_n)$ is a non-increasing integer sequence such that $i < j$ and $x_j \geq 2$, then the following operation is called a unit transformation from j to i on π: subtract 1 from x_j, add 1 to x_i and set in non-increasing order the sequence deduced in this way. If this transformation is denoted by $T(\pi)$ and $\pi \in D_{n,p}$ then $T(\pi) \in D_{n,p}$ and by Lemma 1 we get $F(T(\pi)) > F(\pi)$ since $f(x)$ is strictly convex. The following lemma on majorization of integer sequences is due to Muirhead (see [6]):

Lemma 2. *If π and π' are two non-increasing integer sequences and $\pi \lhd \pi'$, then π' can be obtained from π by a finite sequence of unit transformations.*

If $\pi \in D_{n,p}$ denote by $\mathcal{M}(\pi)$ the set of sequences in $D_{n,p}$ which can be obtained from

π by a single unit transformation. Since every sequence $\pi \in D_{n,p}$ which is different from $\mathbf{x}_{\min}$ can be obtained by a sequence of unit transformations and these transformations strictly increase the value of F, it follows that the minimum of F in the set $D_{n,p} \setminus \{\mathbf{x}_{\min}\}$ can be reached only in the set $\mathcal{M}(\mathbf{x}_{\min})$ and so on.

Lemma 3([8]). *If $q > p \geq n$ then*

$$\min_{\mathbf{x} \in D_{n,q}} F(\mathbf{x}) > \min_{\mathbf{x} \in D_{n,p}} F(\mathbf{x}).$$

For a natural number s, $1 \leq s \leq n-1$, denote by $D^s_{n,p} \subset D_{n,p}$ the set of n-tuples $(x_1, x_2, \ldots, x_n) \in D_{n,p}$ such that the last s components are equal to 1: $x_{n-s+1} = x_{n-s+2} = \ldots = x_n = 1$. The following property also holds by Lemma 1:

Lemma 4([8]). *If $s < t \leq n-1$ and $p \geq 2n-t+1$ then*

$$\min_{\mathbf{x} \in D^s_{n,p}} F(\mathbf{x}) < \min_{\mathbf{x} \in D^t_{n,p}} F(\mathbf{x}).$$

Since $\sum_{i=1}^n d_i = 2|E(G)|$ we get:

Lemma 5([8]). *We have*

$$H_f(G) \geq \min_{\mathbf{x} \in D_{n,2|E(G)|}} F(\mathbf{x}).$$

Equality may hold only if the point of minimum $(x_1^*, x_2^*, \ldots, x_n^*)$ of $F(x_1, x_2, \ldots, x_n)$ in $D_{n,2|E(G)|}$ is graphical, i.e., if there exists a graph G with degrees $d_i = x_i^*$ for $i = 1, \ldots, n$.

3 Main results

First we shall consider the case of quasi-trees (k=1).

Theorem 1. *Let G be a quasi-tree of order $n \geq 3$. If $f(x)$ is strictly convex and strictly increasing, then the minimum of $H_f(G)$ equals $2f(1)+(n-2)f(2)$ if G is a tree or $nf(2)$ if G is a non-trivial quasi-tree. The extremal graph is the path P_n or the cycle C_n, respectively.*

Proof. Let $G \in \mathcal{T}_n^1$ such that $H_f(G)$ is minimum. We shall distinguish two cases: Case 1: G is a tree and Case 2: G is a non-trivial quasi-tree.

Case 1. In this case G has $n-1$ edges and the minimum of the function $F(\mathbf{x})$ in $D_{n,2n-2}$ is attained only for the n-tuple $(2, 2, \ldots, 2, 1, 1)$. This degree sequence is graphical and has a unique realization, namely P_n.

Case 2. By definition, there exists a vertex v_0 such that $G - v_0$ is a tree having $n-1$ vertices and $n-2$ edges and G is not a tree. We will show that $d_G(v_0) = 2$. Because G is not a tree we get $d_G(v_0) \geq 2$. Suppose that $d_G(v_0) \geq 3$. If $v_0 u \in E(G)$, let $G_1 = G - v_0 u$. It follows that $F = G - v_0$ is a tree, but G_1 is not a tree since $|E(G_1)| = |E(F)| + d_G(v_0) - 1 \geq n$. We get $G_1 \in \mathcal{T}_n^1$. Since $f(x)$ is strictly increasing we obtain $f(d_{G_1}(v_0)) = f(d_G(v_0) - 1) < f(d_G(v_0))$ and $f(d_{G_1}(u)) = f(d_G(u) - 1) < f(d_G(u))$, which implies $H_f(G_1) < H_f(G)$, a contradiction. We deduce that $d_G(v_0) = 2$, hence G has $n - 2 + 2 = n$ edges. The minimum of the function $F(\mathbf{x})$ in $D_{n,2n}$ is attained only for the n-tuple $(2, 2, \ldots, 2)$. This degree sequence is graphical and has a unique connected realization, namely C_n. □

For $k \geq 2$ and $n \geq 3k$ denote by $\mathcal{QT}_{n,k}$ the set of graphs consisting of K_3 with $V(K_3) = \{x_1, y_1, z_1\}$ and three vertex-disjoint paths $x_1, \ldots, x_k; y_1, \ldots, y_p$ and $z_1, \ldots, z_q$, where $p, q \geq k$ and $p + q = n - k$. It is not difficult to see that the number of pairwise non-isomorphic graphs from $\mathcal{QT}_{n,k}$ equals $1 + \lfloor (n - 3k)/2 \rfloor$. Indeed, this number equals the number of representations of $n - 3k$ as $n - 3k = n_1 + n_2$, where $n_1 \geq n_2 \geq 0$ and this number is equal to $1 + \lfloor (n - 3k)/2 \rfloor$. If $G \in \mathcal{QT}_{n,k}$ then $H_f(G) = 3f(3) + (n - 6)f(2) + 3f(1)$. Denote

$$\varphi(n) = 3f(3) + (n - 6)f(2) + 3f(1).$$

For $n \geq 6$ denote by $\mathcal{B}_n$ the set of bicyclic graphs of order n consisting of two vertex disjoint cycles C_p and C_q, where $p, q \geq 3$ and $p + q \leq n$ joined by a path. If $G \in \mathcal{B}_n$ then $G \in \mathcal{T}_n^2$ and $H_f(G) = 2f(3) + (n - 2)f(2)$. We have $\varphi(n) < 2f(3) + (n - 2)f(2)$ if and only if the following inequality holds:

$$f(3) + 3f(1) < 4f(2). \tag{1}$$

Theorem 2. *Let $f(x)$ be a strictly convex and strictly increasing function and G be a k-generalized quasi-tree of order $n \geq 3k$. Then the following properties hold:*
A. *If $k = 2$ and $n \geq 6$ the minimum of $H_f(G)$ is equal to*:
i) *$\varphi(n)$ if (1) is true; G reaches this minimum if and only if $G \in \mathcal{QT}_{n,2}$.*
ii) *$\varphi(n)$ if $f(3) + 3f(1) = 4f(2)$ and the set of extremal graphs is $\mathcal{QT}_{n,2} \cup \mathcal{B}_n$;*
iii) *$2f(3) + (n - 2)f(2)$ if $f(3) + 3f(1) > 4f(2)$ and the set of extremal graphs is $\mathcal{B}_n$.*
B. *For $k \geq 3$ the minimum of $H_f(G)$ is equal to $3f(3) + (n - 6)f(2) + 3f(1)$. G reaches this minimum if and only if $G \in \mathcal{QT}_{n,k}$.*

Proof. Suppose that $k \geq 2, n \geq 3k$ and $G \in \mathcal{T}_n^k$ such that $H_f(G)$ is minimum. There exists a k-quasi-vertex set $V_k \subseteq V(G), |V_k| = k$ such that $G - V_k$ is a tree, but for every subset V_{k-1} of cardinality $k - 1$ of $V(G)$, the graph $G - V_{k-1}$ is not a tree, hence it has cycles. Denote $W_{n-k} = G - V_k$. We have seen that W_{n-k} is a tree. If there exists a vertex $x \in V_k$ which is adjacent to a single vertex from W_{n-k}, then $V_{k-1} = V_k - v$ has the property that $G - V_{k-1}$ is a tree, a contradiction. It follows that every vertex of V_k is not adjacent to any vertex of W_{n-k} or is adjacent to at least two vertices from W_{n-k}. Suppose that the subgraph induced by V_k has $r \geq 1$ connected components $A_1, A_2, \ldots, A_r$ and in each component there is at least one vertex which is adjacent with at least two vertices in W_{n-k}. Any component A_i has at least $|A_i| - 1$ edges and equality holds if and only if this component is a tree. Therefore the number of edges of G with at least one end in V_k is at least $2r + \sum_{i=1}^{r}(|A_i| - 1) = 2r + k - r = k + r \geq k + 1$ because $\sum_{i=1}^{r} |A_i| = |V_k| = k$. Since W_{n-k} is a tree it has $n - k - 1$ edges, which implies that $|E(G)| \geq k + 1 + n - k - 1 = n$ and equality holds only if $r = 1$. Consequently, we shall consider the following cases: Case 1. $|E(G)| = n$, therefore G is unicyclic; Case 2. $|E(G)| = n + 1$, hence G is bicyclic and Case 3. $|E(G)| \geq n + 2$.

Case 1. In this case $r = 1$ and $G[V_k]$ has $|V_k| - 1 = k - 1$ edges, i.e., $G[V_k]$ is a tree with $k \geq 2$ vertices and exactly one vertex denoted by w of this tree is adjacent with two vertices of W_{n-k}. This implies that G has at least one pendant vertex. Four subcases may hold: Subcase 1.i. ($1 \leq i \leq 4$): G has i pendant vertices and subcase 1.5. G has at least five pendant vertices.

Subcase 1.1. G being a connected unicyclic graph of size n with one pendant vertex it follows that $G[V_k]$ is a path $w, y_1, y_2, \ldots, y_{k-1}$ such that $d_G(w) = 3, d_G(y_1) = \ldots = d_G(y_{k-2}) = 2$ and $d_G(y_{k-1}) = 1$. We get that G consists of a cycle C_s and a path P_t having a common vertex such that $s \geq 3, t \geq 2$ and $s + t = n + 1$. It is not difficult to see that in this subcase $G \notin \mathcal{T}_n^k$, which contradicts the hypothesis.

Subcase 1.2. In this case G has two pendant vertices. We consider other two subcases: Subcase 1.2.1. $G[V_k]$ is a path; Subcase 1.2.2. The tree $G[V_k]$ has two pendant vertices different from w. In both subcases the vertex w is adjacent with two vertices of W_{n-k}.

Subcase 1.2.1. Since $W_{n-k} = G - V_k$ is a tree it follows that G is a unicyclic graph consisting of a cycle C_s with $s \geq 3$ and two vertex disjoint paths having each a vertex v_1 and v_2 respectively, common with C_s. We can choose a subset $V_{k-1} \subset V(G)$, $|V_{k-1}| = k-1$ containing at least a vertex of C_s different from v_1 and v_2, having degree equal to two such that $G - V_{k-1}$ is a tree, a contradiction.

Subcase 1.2.2. In this case G is a unicyclic graph containing a cycle C_s and a tree with two pendant vertices having a vertex common with C_s. As in the previous subcase we find that $G \notin \mathcal{T}_n^k$.

Subcase 1.3. In this case G has three pendant vertices. We further prove that in this case $G \in \mathcal{QT}_{n,k}$. We shall distinguish the subcases 1.3.1, 1.3.2 and 1.3.3 which correspond to the cases when $G[V_k]$ has one, two or three pendant vertices different of w, respectively. In all these subcases G has a unique cycle C_s with $s \geq 3$.

Subcase 1.3.1. In this case G consists of C_s and three vertex disjoint paths having each a vertex x_1, y_1 and z_1, respectively common with C_s or one path and a tree with two pendant vertices which are vertex disjoint and have each a vertex common with C_s. In the first case since $G \in \mathcal{T}_n^k$ it is necessary that $s = 3$ and from the paths with ends in x_1, y_1, z_1 at least one must have k vertices and the other at least k vertices each. It follows that $G \in \mathcal{QT}_{n,k}$. In all other subcases we can find a vertex $v \in V(C_s)$ such that $d_G(v) = 2$ and a subset $V_{k-1} \subset V(G)$ with $k-1$ vertices such that $v \in V_{k-1}$ and $G - V_{k-1}$ is a tree, a contradiction.

Subcase 1.4. Because $k \geq 2$ we get $n \geq 3k \geq 6$. We shall consider the cases $n = 6, n = 7$ and $n \geq 8$.

If $n = 6$ then $k = 2$ and $\min_{\mathbf{x} \in D^4_{6,12}} F(\mathbf{x})$ is reached for $\mathbf{x}^* = (4^2, 1^4)$ by Lemma 1 since G has four pendant vertices. But the degree sequence $(4^2, 1^4)$ is not graphical. We have $\mathcal{M}(4^2, 1^4) = \{(5, 3, 1^4)\}$, therefore the next point of minimum of F is $(5, 3, 1^4)$, which implies that $H_f(G) \geq f(5) + f(3) + 4f(1)$. We have $f(5) + f(3) + 4f(1) > \varphi(6) = 3f(3) + 3f(1)$ since this is equivalent to $f(5) + f(1) > 2f(3)$, which is true by Jensen's inequality.

For $n = 7$ we have $k = 2$ and $H_f(G) \geq \min_{\mathbf{x} \in D^4_{7,14}} F(\mathbf{x})$. This minimum is reached for $\mathbf{x}^* = (4, 3^2, 1^4)$ and $F(\mathbf{x}^*) = f(4) + 2f(3) + 4f(1)$. We get $f(4) + 2f(3) + 4f(1) > \varphi(7) = 3f(3) + f(2) + 3f(1)$, a contradiction, since this is equivalent to $f(4) + f(1) > f(3) + f(2)$ which holds by Lemma 1.

If $n \geq 8$ then $\min_{\mathbf{x} \in D^4_{n,2n}} F(\mathbf{x})$ is reached for $\mathbf{x}^* = (3^4, 2^{n-8}, 1^4)$, which implies that $H_f(G) \geq F(\mathbf{x}^*) = 4f(3) + (n-8)f(2) + 4f(1)$ and $4f(3) + (n-8)f(2) + 4f(1) > \varphi(n)$ since this inequality is equivalent to $f(3)+f(1)>2f(2)$, which follows by Jensen's inequality.

Subcase 1.5. If G has $s \geq 5$ pendant vertices, by Lemma 4 we get $\min_{\mathbf{x}\in D^s_{n,2n}} F(\mathbf{x}^*) > \min_{\mathbf{x}\in D^4_{n,2n}} F(\mathbf{x}^*)$, which implies that $H_f(G) > \varphi(n)$, which is a contradiction.

Case 2. If $|E(G)| = n+1$ then G is bicyclic and $\min_{\mathbf{x}\in D_{n,2n+2}} F(\mathbf{x})$ is reached for $\mathbf{x}^* = (3^2, 2^{n-2})$. This degree sequence has graphical realizations consisting of two cycles having in common a path P_t with $t \geq 2$ or two cycles joined by a path P_t with $t \geq 2$. These bicyclic graphs do not belong to $\mathcal{T}_n^k$ for $k \geq 3$ and $n \geq 3k$. For $k=2$ if G consists of two cycles having in common a path P_t with $t \geq 2$ then $G \notin \mathcal{T}_n^2$, but if G is composed of two cycles joined by a path P_t with $t \geq 2$, which means that $G \in \mathcal{B}_n$, then $G \in \mathcal{T}_n^2$ and $H_f(G) = 2f(3)+(n-2)f(2)$. If (1) holds, or $\varphi(n) < 2f(3)+(n-2)f(2)$, and $G \in \mathcal{B}_n$, this contradicts the minimality of G. If $f(3)+3f(1) = 4f(2)$ then graphs from $\mathcal{QT}_{n,2}$ and graphs from $\mathcal{B}_n$ have the same vertex-degree function index and if $\varphi(n) > 2f(3)+(n-2)f(2)$ and $G \in \mathcal{QT}_{n,2}$ then again contradicts the minimality of G. Further we get $\mathcal{M}(3^2, 2^{n-2}) = \{(4,3,2^{n-3},1),(4,2^{n-1})\}$. We have $F(4,3,2^{n-3},1) = f(4)+f(3)+(n-3)f(2)+f(1) > \varphi(n)$ since this is equivalent to $f(4)+3f(2) > 2f(3)+2f(1)$, which is true since $f(4)+f(2) > 2f(3)$ by Jensen's inequality and $f(2) > f(1)$, $f(x)$ being strictly increasing, which contradicts the minimality of G. The sequence $(4, 2^{n-1})$ has graphical realizations consisting of two cycles having a vertex in common, but none of these bicyclic graphs belong to $\mathcal{T}_n^k$ for $k \geq 3$ and $n \geq 3k$. For $k=2$ if G consists of two cycles having only one vertex in common then $G \in \mathcal{T}_n^2$ and $H_f(G) = f(4)+(n-1)f(2)$. We get $f(4)+(n-1)f(2) > 2f(3)+(n-2)f(2)$ since this is equivalent to $f(4)+f(2) > 2f(3)$ and this holds by Jensen's inequality. We obtain that in this case G cannot be extremal. By Muirhead's Lemma to find the next minimum points of $F(\mathbf{x})$ for $k \geq 3$ it is necessary to study the points in $\mathcal{M}((4,2^{n-1}) = \{(5,2^{n-2},1),(4,3,2^{n-3},1)\}$. For the second n-tuple we have seen that the value of F is greater than $\varphi(n)$ and all realizations of the first n-tuple consist of two cycles having a unique vertex v in common and a path P_t with $t \geq 2$ ending in v, which is the single vertex of P_t common with the cycles. These bicyclic graphs do not belong to $\mathcal{T}_n^k$ for $k \geq 3$ and $n \geq 3k$. Further $\mathcal{M}((5,2^{n-2},1)) = \{(6,2^{n-3},1,1),(5,3,2^{n-4},1,1)\}$. We deduce that $F(6,2^{n-3},1,1) > \varphi(n)$ or $f(6)+(n-3)f(2)+2f(1) > 3f(3)+(n-6)f(2)+3f(1)$ since this is equivalent to $f(6)+3f(2) > 3f(3)+f(1)$. Since $f(x)$ is strictly convex and strictly increasing we obtain $f(6)+3f(2) > 2f(4)+2f(2) > 4f(3) > 3f(3)+f(1)$. Also $F(5,3,2^{n-4},1,1) > \varphi(n)$ or $f(5)+f(3)+(n-4)f(2)+2f(1) > 3f(3)+(n-6)f(2)+3f(1)$, which is equivalent to $f(5)+2f(2) > 2f(3)+f(1)$. The last inequality holds since $f(5)+f(2) > 2f(3.5) > 2f(3)$ and $f(2) > f(1)$.

Case 3. If $|E(G)| \geq n+2$ by Lemma 5 we deduce that $H_f(G) \geq \min_{\mathbf{x}\in D_{n,2n+4}} F(\mathbf{x})$. This minimum is reached for $\mathbf{x}^* = (3^4, 2^{n-4})$ and $F(\mathbf{x}^*) = 4f(3)+(n-4)f(2) > \varphi(n)$ since this is equivalent to $f(3)+2f(2) > 3f(1)$. This inequality holds since $f(x)$ is strictly increasing and contradicts the hypothesis about the minimality of G. □

Concluding remarks. In this paper we have solved an optimization problem concerning the vertex-degree function index $H_f(G)$ in the case when G is a k-generalized quasi-tree of order $n \geq 3k$ and $k \geq 1$.

Note that by replacing minimum by maximum, strictly increasing by strictly decreasing, strictly convex by strictly concave and by reversing the inequality in (1), Theorems 1 and 2 remain true.

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