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Continuous folding of the surface of a regular simplex onto its facet

by
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Dedicated to Professor Tudor Zamfirescu on his 80th birthday

Abstract

Whether the surface of a polyhedron made of a flexible material such as paper can be flattened without cutting or stretching is a problem that has been investigated. This problem has been solved for any 3-dimensional convex polyhedron using moving (rolling) creases, and has been extended to higher dimensional polytopes. We refer the set of facets for a polytope as surface. In this paper we focus on a 4-dimensional regular simplex (a regular 5-cell) whose surface consists of five regular tetrahedra (facets). We provide a continuously folding motion of its surface onto one facet such that the moving creases of this motion occupy one sixth of the surface volume. Note that if we allow moving creases in the major part of the surface, such a continuous motion has been given by the authors together with Abel et al., with creases whose total volume is four fifths of the surface's. Hence, in this paper the ratio of rigid portions (not occupied by any moving creases) to the surface volume is increased from one fifth to five sixths.

Key Words: Regular simplex, continuous folding, rigidity, crease.
2020 Mathematics Subject Classification: Primary 52B99; Secondary 52C25.

1 Introduction

Whether the surface of a polyhedron made of a flexible material such as paper can be flattened without cutting or stretching is a problem that has been investigated (see [2], p.279). This problem was solved in [1, 6] for any convex polyhedron using moving creases to change the shapes of some faces, which follows from Cauchy's rigidity theorem. The flattening motions are described by using straight skeletons in [1], and cut locus and Alexandrov's gluing theorem in [6]. The portions of the moving creases occupy a major part of its surface. For example, for a regular tetrahedron the portions of the moving creases by the method in [1] occupy three fourths of the entire surface, see Fig. 1 (a), and by the method in [6] occupy almost all of the surface. On the other hand, by the so called *kite method*, described in [3, 8], the portions of the moving creases occupy one twelfth of the entire surface, that is, eleven twelfth are rigid, see Fig. 1 (b); for details, see Section 3. Moreover, the number of faces in each folded state is less than or equal to eight. Note that it was proved in [7] that we can reduce the area of moving creases as small as we want, where the number of faces in each folded state increases.

We have previously proved in a joint paper [1] that any n-dimensional convex polytope can be continously folded in any $(n-1)$-dimensional face (called facet). However, the entire surface except at most two facets is occupied by the moving creases. Here, we focus on a

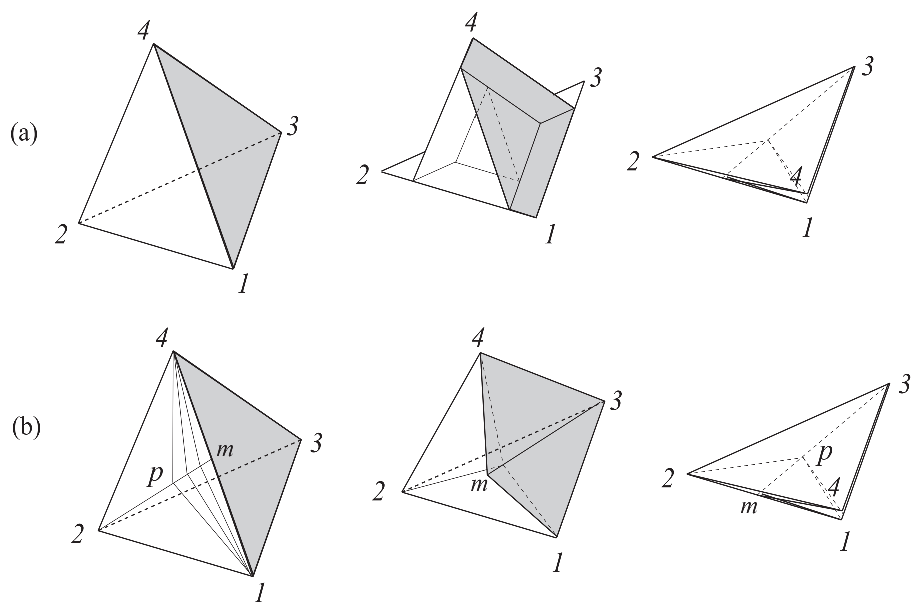

Figure 1: Continuous flattening of the surface of a regular tetrahedron using two methods; (a) The method shown in [1]; (b) The kite method shown in [3, 8], where $m = (14)$ and $p = (123)$ are the midpoint of the edge [14] and the center of gravity of the triangular face [123], respectively.

regular 4-dimensional simplex (called a regular 5-cell) and provide a continuous folding motion onto any of its facets. The moving creases of this motion occupy one sixth of the surface volume, which is much smaller than the value (four fifths) required by the method in [1]; that is, the rigid portions (not occupied by any moving creases) is increased from one fifth to five sixths by our method proposed here. We analyze those creases and give their concrete figures. The rigid portions are important for constructing some figures, when we consider applications to origami-based engineering (see e.g., [9]). We prove the following theorem.

Theorem 1.1. *The surface of a 4-dimensional regular simplex can be continuously folded onto any of its facets such that the moving creases of this motion occupy one sixth of the surface volume, that is, five sixths of the surface are rigid (not occupied by any moving creases).*

In section 2, a definition and notation are given. In section 3, we analize a continuous flattening motions in a 3-dimensional regular simplex (tetrahedron). In section 4, we prove Theorem 1.1 by extending the kite method from the 3- dimensional space to a 4-dimensional space.

2 Definition and notation

We define a *continuous folding motion* of the surface Q of a 4-dimensional polytope as a sequence of polyhedral manifolds $\{Q(t) : 0 \leq t \leq 1\}$ satisfying the following conditions, and we call each $Q(t)$ the *folded state* of Q for t in the motion.

Condition 1. For each $t\,(0 \leq t \leq 1)$, there is an intrinsically isometric mapping from $Q(t)$ onto Q, that is, there are polyhedral subdivisions of $Q(t)$ and Q into the same number of pieces such that there is a one-to-one correspondence between those sets and that every two corresponding pieces have neighbors congruent to each other.

Condition 2. The mapping from $t\,(0 \leq t \leq 1)$ to $Q(t)$ is continuous.

Condition 3. $Q(0) = Q$.

For $k \geq 2$ points $p_1, p_2, ..., p_k$ in n-space ($n = 3, 4$) we denote by $[p_1p_2...p_k]$ their covex hull and by $(p_1p_2...p_k)$ their center (of gravity). So (p_1p_2) means the midpoint of $[p_1p_2]$. Let P be an n-dimensional regular simplex with $n+1$ vertices $v_1, v_2, , \cdots , v_{n+1}$ in n-space, which are denoted more briefly as $1, 2, \cdots , n + 1$, respectively. We denote by l the edge length of P, and by Q_i the facet of P with all vertices except i, for $i = 1, 2, \cdots , n + 1$. We denote by $S(r; u)$ the $(k - 1)$-dimensional sphere of radius r with center u in a given k-space.

3 Analyzing the motions in a 3-dimensional simplex

We prove the theorem by using a kite method extended to 4-dimensional case in some sense, applying a similar motion to the one shown in Fig. 1(b) for the 3-dimensional simplex (tetrahedron). Thus, we analyze the continuous motion by kite method used in the proof for the regular tetrahedron. We derive several facts, given in the following as criteria for further motions.

In this section P is a regular tetrahedron [1234] with edge length l as shown in Fig. 1(b). Let m be the midpoint (14) of the edge [14], and p and g the centers of gravity of the face $Q_4 = [123]$, and $P = [1234]$, respectively.

Criterion 1. The face Q_4 is fixed.

Criterion 2. The face Q_1 rotates about the edge [23] and overlaps on Q_4. At that time, the edge [14] is folded in half at the midpoint (14) and vertex 4 moves to vertex 1 along the shorter circular arc in the intersection of two spheres of radius l with centers 2 and 3, and so the position $(v_4)_t$ of v_4 for $t\,(0 \le t \le 1)$ satisfies

$$(v_4)_t \in S(l; v_2) \bigcap S(l; v_3).$$

Criterion 3. The face Q_2 is folded with the crease [(14)3], and located between Q_4 and Q_1. The triangle $[1m4]$, a half of Q_2, rotates about the edge [13]. The midpoint m moves to the midpoint (12) along the shorter circular arc in the intersection of two spheres of radius $l/2$ and $(\sqrt{3}/2)\,l = \sqrt{3}\,l/2$ and with centers 1 and 3, respectively. Hence, the position m_t of m for $t\,(0 \le t \le 1)$ satisfies

$$m_t \in S(l/2; v_1) \bigcap S(\sqrt{3}\,l/2; v_3).$$

Criterion 4. If the midpoint $m = (14)$ is moved onto (13) instead of (12), the face Q_3 can be continuously folded onto [13(12)] similarly to the motion of Q_2. However, since m is moved onto (12), the line segment $[m2]$ should be folded at some point q in $[m(124)]$, and $[q1]$ and $[q4]$ are attached to Q_2 (see Fig. 1). The existence of such q is guaranteed by the intersection of two line segments $[m2]$ and $[m3]$ (see [4, 8] for details). The point q traces $[mp]$ from the midpoint $m = (14)$ of the edge [14] to the center of gravity $p = (124)$ of the triangular face [124].

Criterion 5. The centers of gravity of all faces move onto (123).

For such a motion of Q_3 verifying Criteria 1-5, we say that "Q_3 follows Q_2 by a *priority rule for faces*."

4 Proof of Theorem

Hereafter through the paper, $P = [12345]$ and Q is the surface of P as shown in Fig. 2, where in the left figure the vertex 5 is added to the facet $Q_5 = [1234]$ in another direction as a 4-dimensional figure. We will apply a motion similar to the surface of a regular tetrahedron when using the kite method, and described below.

4.1 Motions of vertices, edges, and faces

We show the motions of vertices, edges, and faces of the surface Q sequentially.

4.1.1 Vertices

Four vertices in the facet [1234] are fixed. So the facet [1234] is also fixed. We fold only one edge [15] while other edges are not folded, that is, rigid through the motions. The vertex 5 moves to the vertex 1 along the shorter circular arc in the intersection of three

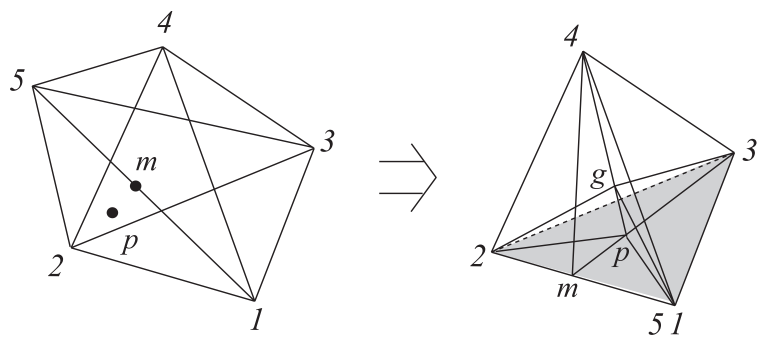

Figure 2: A continuous folding of the surface of a 4-dimensional regular tetrahedron [12345] onto its facet [1234]: m is the midpoint of the edge [15], p is the center of gravity of the triangular face [125], and g is the center of gravity the facet [1234].

3-dimensional spheres of the centers $2, 3$, and 4 with radius l, and so the position $(v_5)_t$ of v_5 for $t\,(0 \leq t \leq 1)$ satisfies

$$(v_5)_t \in S(l; v_2) \bigcap S(l; v_3) \bigcap S(l; v_4).$$

See Fig. 3.

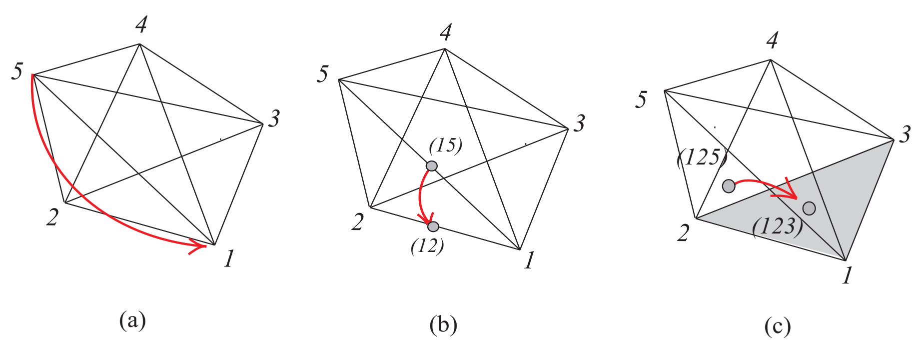

Figure 3: Motions of vertex 5, the midpoint (15), and the center of gravity (125) of the face [125] are shown in (a), (b), and (c), respectively.

4.1.2 Edges

All edges except [15] are rigid. The edge [15] is folded in half at the midpoint $m = (15)$ and moves to the midpoint (12) along the shorter circular arc in the intersection of three spheres two of which are centered at 3 and 4 and have the same radius $(\sqrt{3}/2)\,l$ while the third has center at 1 and radius $l/2$. The motion is such that for each moment the point $m = (15)$ is located in the hyperplane bisecting two points v_1 and $(v_5)_t$ in the folded state

Q_t for $t\,(0 \le t \le 1)$. So the position m_t of $m = (15)$ for $t\,(0 \le t \le 1)$ satisfies

$$m_t \in S(l/2; v_1) \bigcap S(\sqrt{3}\,l/2); v_3) \bigcap S(\sqrt{3}\,l/2; v_4).$$

4.1.3 Faces

All faces except the three faces attaching to the edge [15] are rigid, that is, not folded anywhere through the motion. The other three faces are [152], [153], and [154]. Two faces [153] and [154] are folded in half and overlap onto [1(12)3] and [1(12)4], respectively. The face [152] is folded onto [12(123)] with moving creases, similarly to the face [124] of a regular tetrahedron [1234] shown in Fig. 1. Note that we move (125) onto (123). See Fig. 3 and Fig. 4.

The above motions are the same as the motions described in [5] for the 2-skeleton of P. In the next subsection, we determine the motion of the facets of P.

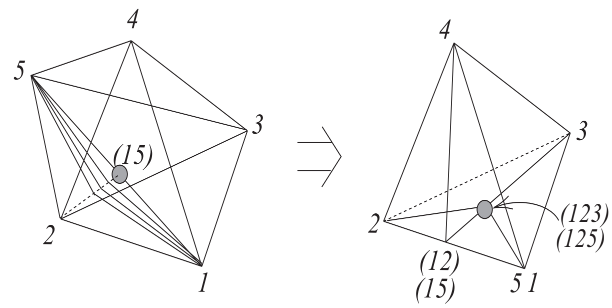

Figure 4: Motion of the 2-skelerton of Q.

4.2 Motion of facets

Basically, we apply a motion similar to the one for the continuous flattening of the surface of a regular tetrahedron by the kite method. Two facets $Q_5 = [1234]$ and $Q_1 = [2345]$ are rigid and other three facets are folded in half with modification to manage collisions among those three facets. We define a rule for those three facets as follows. Q_3 follows Q_2, and Q_4 follows both Q_2 and Q_3, which is called a "*priority rule for facets.*"

4.2.1 Facet Q_5 and Q_1

The facet Q_5 is fixed. The facet Q_4 is rotated about the face [123] and overlapped onto Q_5. The vertex 5 moves along the shorter circular arc of the intersection of three 3-dimensional spheres all of radius l (the edge length) with centers $2, 3$, and 4.

4.2.2 Facet Q_2

The facet Q_2 is folded in half by the triangular face $T_2 = [(15)34]$. We call T_2 a *medial face* of Q_2. The half part [(15)134] is rotated about the face [134] and overlapped (multilayered) onto [(12)134], and located between Q_5 and Q_1 (see Fig. 5). The face T_2 moves to the medial face [(12)34] of Q_5. The point (15) moves along the shorter circular arc of the intersection of three spheres; two of them have radius $(\sqrt{3}/2)\,l$ with centers 3 and 4, and one of them has radius $1/2\,l$ with the center 1.

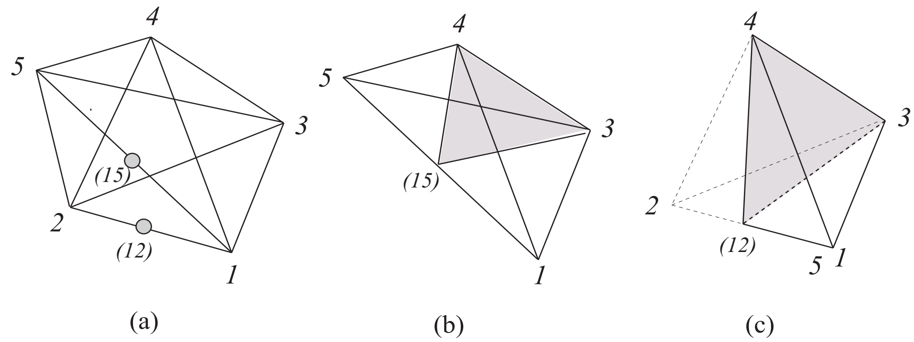

Figure 5: Motion of the facet $Q_2 = [1345]$; (a) The surface Q of the regular simplex $P = [12345]$; (b) The facet Q_2 of P with the medial face [(15)34]; (c) The final folded state of Q_2 in Q_5.

4.2.3 Facet Q_3

If the midpoint (15) is moved onto (13) instead of (12), ignoring the motion Q_2, the facet Q_3 can be continuously folded in half by the medial face $T_3 = [(15)24]$ and overlaped onto the half [(13)124] of Q_5 as similar to the motion of Q_2 (see Fig.6 (a),(b), and (c)). However, since (15) is moved onto (12), the medial face T_3 should be folded along the line segment [(123)4] at $t = 1$ as shown in Fig. 6 (d) and (e). Since for each t ($0 \le t \le 1$) the medial face T_3 intersects with T_2 on the line segment $[q4]$ for some q in the line segment [(15)(125)], T_3 should be folded along $[q4]$ with crease and attached to T_2 by the priority rule for facets.

More precisely, consider the rotation of [(15)134] (a half of Q_2) about the face [134] and rotation of [(15)124] (a half of Q_3) about the face [124] simultanuously for t ($0 \le t \le 1$). For each t the intersection of medial faces $T_2 = [(15)34]$ and $T_3 = [(15)24]$ is the line segment $[q4]$ for some q in [(15)(125)] where $q = (15)$ at $t = 0$ and $q = (125)$ at $t = 1$. See Fig. 6 (f) where Q_3 is described in a different way from Fig 6 (b). Therefore, the set $S(Q_3)$ of moving creases in Q_3 is an infinite set of triangles $[q14]$ and $[q45]$ where q moves from (15) to (125), that is,

$$S(Q_3) = \{[q14], [q45] : q \in [(15)(125)]\},$$

and the final folded state of Q_3 is a multilayed triangular pyramid onto [124(123)] in Q_5 (see Fig. 6 (e)).

4.2.4 Facet Q_4

If the midpoint (15) moves onto (14) instead of (12), the facet Q_4 can be continuously folded in half onto [(14)123] similarly to the motion of Q_2 (see Fig. 7 (a), (b), and (c)).

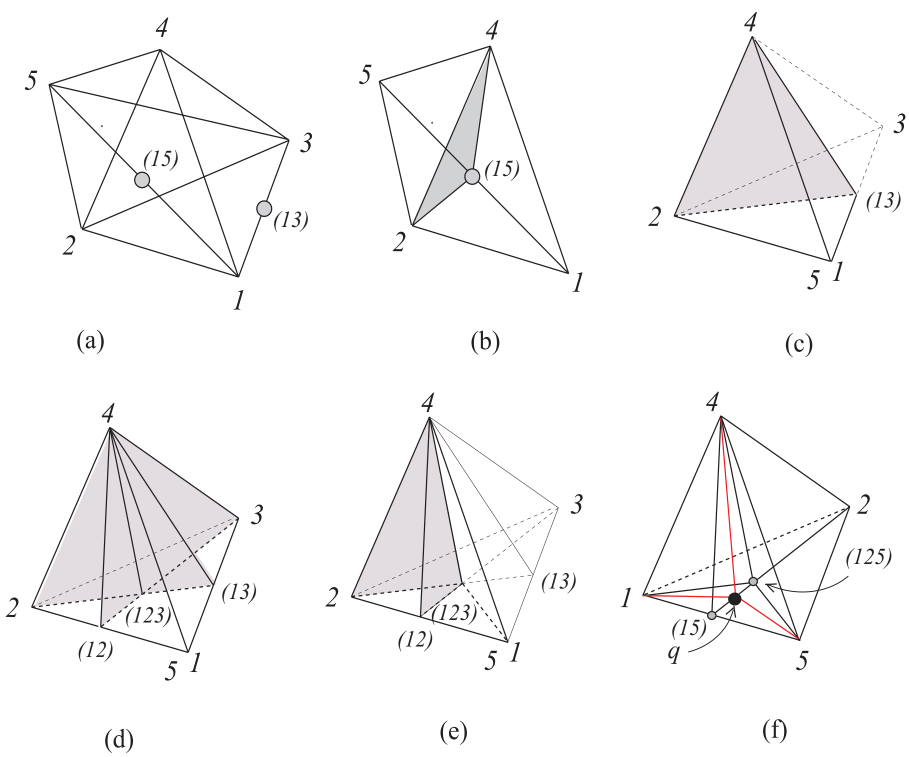

Figure 6: Motion of the facet $Q_3 = [1245]$; (a) The surface Q of the regular simplex $P = [12345]$; (b) The facet Q_3 with the medial face $T_3 = [(15)24]$; (c) A folded state of Q_3 in Q_5 with the assumption that (15) moves onto (13); (d) Folded state of T_2 and T_3 if they can move separately; (e) The final folded state of Q_3 with T_3 shaded; (f) The moving creases $[q14]$ and $[q45]$ in Q_3 for $q \in [(15)(125)]$.

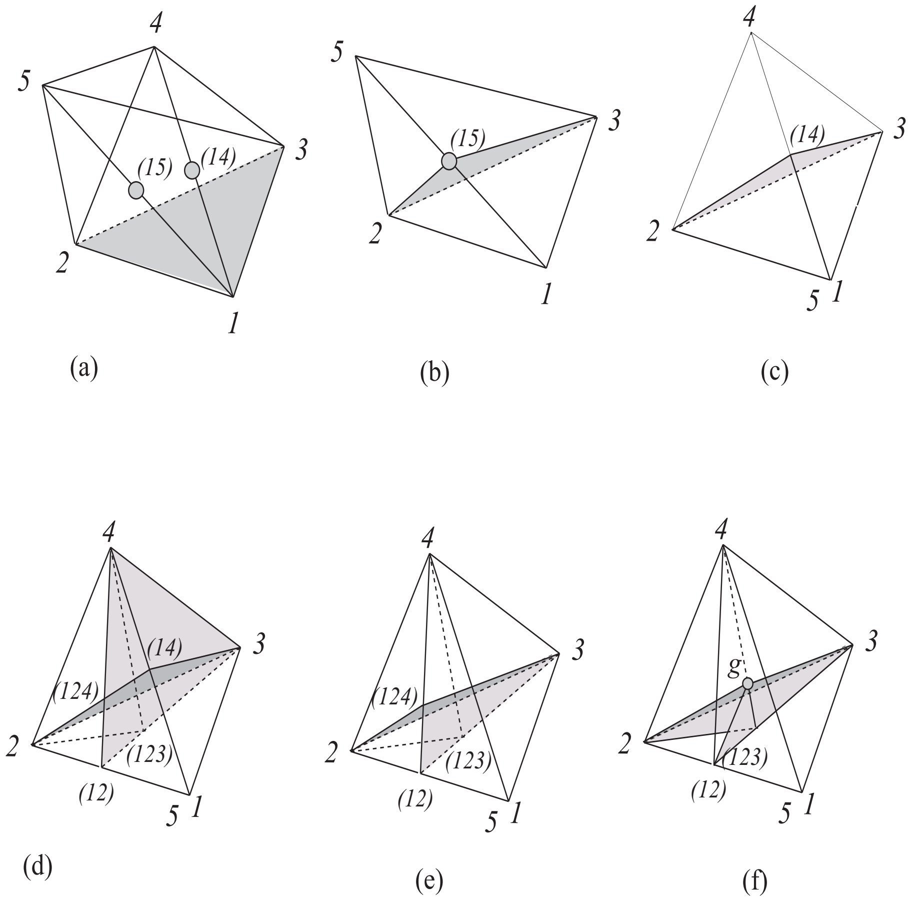

Figure 7: Motion of the facet $Q_4 = [1235]$; (a) The surface Q of the regular simplex $P = [1235]$; (b) The facet Q_4 of P with the medial face $T_4 = [(15)23]$; (c) The final folded state of T_4 in Q_5 if (15) moves onto (14); (d) The intersection of the folded state of T_2 and T_4; (c) Modified folded state of T_4 by T_2; (f) Modified folded state of T_4 by T_2 and T_3.

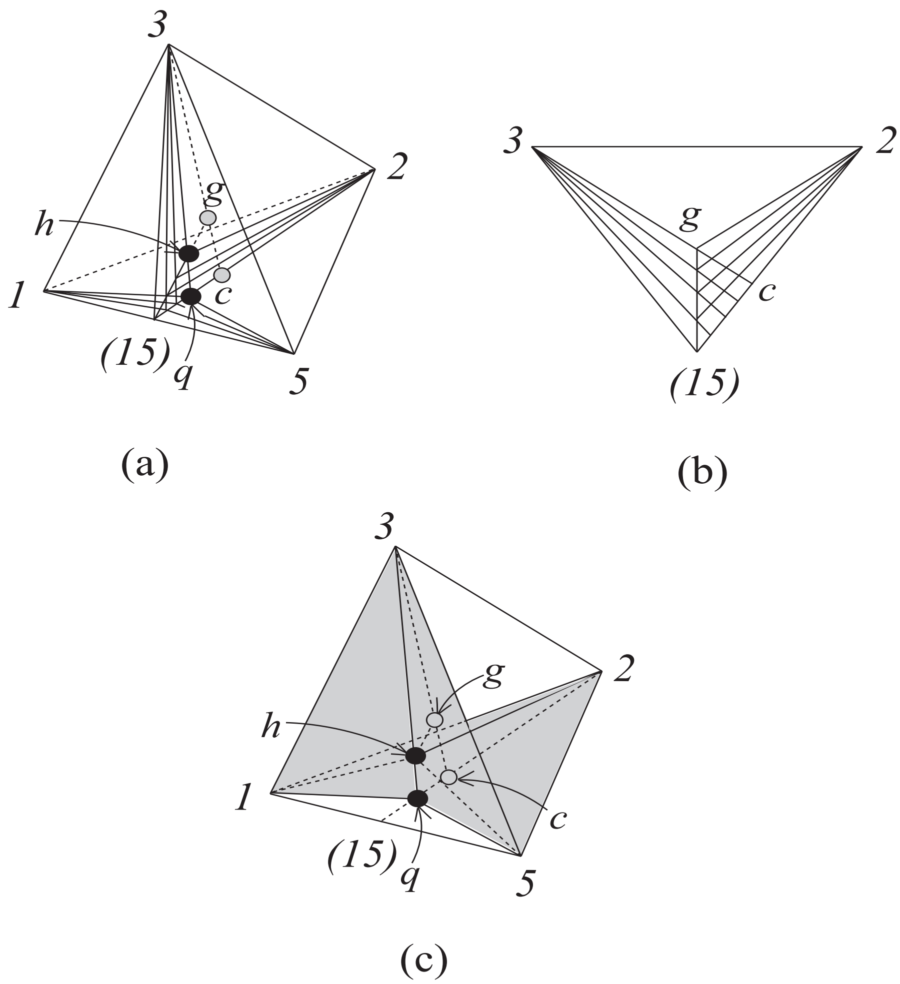

Figure 8: (a) Moving creases in [125] and T_4; (b) Moving creases in T_4 in a plane; (c) Moving creases in Q_4 for some $0 < t < 1$.

However, (15) is moved onto (12). By the priority rule for facets Q_2 and Q_4, if Q_4 follows Q_2 only, which means Q_3 is ignored, T_4 is folded, in a similar way as T_3, onto the union of [23(124)] and [3(12)(123)], as shown in Fig. 7 (e). However, by the priority rule for facets, Q_4 follows both Q_3 and Q_2, the center of gravity (125) is moved onto (123) as mentioned in the motion of faces (see Fig. 3, Fig. 4, and Fig. 7 (d), (e), and (f)).

The point q is in $[(15)(125)]$ for any fixed $t (0 \leq t \leq 1)$, as mentioned in the motion of Q_3. Let h be the point obtained as the intersection of $[(15)g]$ and $[3q]$ where $g = (1235)$. See Fig. 8(a). The set $S(T_4)$ of moving creases of T_4 are the union of two sets, that is,

$$S(T_4) = \{[3q] : q \in [(15)(125)]\} \cup \{[2h] : h \in [(15)g]\}.$$

See Fig. 8 (b). The triangle $T_4 = [(15)23]$ is finally folded onto the union of $[(12)3g]$, $[23g]$, and $[2(123)g]$ (see Fig.7 (f)).

The set $S(Q_4)$ of moving creases in Q_4 consists of four sets of triangles, as shown in Fig. 8 (c), that is,

$$S(Q_4) = \{[q13], [q35]; q \in [(15)(125)]\} \cup \{[12h], [25h]; h \in [(15)g]\}.$$

Therefore, the final folded state of Q_4 is a multilayered triangular pyramid onto [123(1234)] (see Fig. 2). The two triangular pyramids $[123g]$ and $[235g]$ are not folded with any creases, that is, they are rigid, and move onto the triangular pyramid [123(1234)]. The triangular pyramid $[135g]$ is folded onto [13(12)(1234)] with moving creases $\{[13h], [35h] : h \in [(15)g]\}$. The triangular pyramid $[125g]$ is folded onto [12(123)(1234)] with moving creases $\{[qh1], [qh5] : q \in [(15)(125)]\} \cup \{[12h], [25h] : h \in [(15)g]\}$. Therefore, the total volume used for moving creases in Q_4 is one half of its surface volume.

4.3 Volume of moving creases

The total volume used for moving creases in Q is one sixth of its surface volume. Because Q_1 and Q_5 are rigid, Q_2 has no moving creases, and in Q_3 and Q_4 their one third and one half, respectively, are occupied by creasses.

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