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Some q-congruences related to double sums

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Abstract

In this paper, we provide two new q-congruences associated with double basic hypergeometric sums. A related conjecture on q-congruences modulo the cube and fourth powers of a cyclotomic polynomial is also proposed.

Key Words: q-congruences, supercongruences, cyclotomic polynomial, basic hypergeometric series.
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1 Introduction

In 1997, Van Hamme [21] conjectured the following nice p-adic analogue:

$$\sum_{k=0}^{(p-1)/3}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6}\equiv -p\Gamma_p\left(\frac{1}{3}\right)^9 \pmod{p^4}, \tag{1.1}$$

where p is an odd prime such that $p\equiv 1 \pmod 6$, $\Gamma_p(x)$ is the p-adic Gamma function and $(x)_n=\Gamma(x+n)/\Gamma(x)$ is the shifted-factorial for any nonnegative integer n and complex number x. In 2016, Long and Ramakrishna [15, Theorem 2] showed that (1.1) can be generalized to the modulus p^6 case. Later Guo and Schlosser [8, Theorem 2.3] established a partial q-analogue of the generalization of (1.1). Quite recently, together with the creative microscoping method developed by Guo and Zudilin [9] and the Chinese remainder theorem for coprime polynomials, Wei [25] further extended this partial q-analogue as follows: let n be a positive integer, then for $n\equiv 1 \pmod 3$, modulo $[n]\Phi_n(q)^4$,

$$\begin{aligned}&\sum_{k=0}^{(n-1)/3}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k}\\ &\quad\equiv[n]\frac{(q^2;q^3)_{(n-1)/3}^3}{(q^3;q^3)_{(n-1)/3}^3}\left\{1+[n]^2(2-q^n)\sum_{j=1}^{(n-1)/3}\left(\frac{q^{3j-1}}{[3j-1]^2}-\frac{q^{3j}}{[3j]^2}\right)\right\},\end{aligned} \tag{1.2}$$

and for $n\equiv 2 \pmod 3$, modulo $[n]\Phi_n(q)^5$,

$$\begin{aligned}&\sum_{k=0}^{(2n-1)/3}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k}\\ &\quad\equiv[2n]\frac{(q^2;q^3)_{(2n-1)/3}^3}{(q^3;q^3)_{(2n-1)/3}^3}\left\{1+[2n]^2(2-q^{2n})\sum_{j=1}^{(2n-1)/3}\left(\frac{q^{3j-1}}{[3j-1]^2}-\frac{q^{3j}}{[3j]^2}\right)\right\}.\end{aligned} \tag{1.3}$$

Here $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*; $[n]=[n]_q=1+q+\cdots+q^{n-1}$ denotes the *q-integer* and $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q:

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta_n^k),$$

where ζ_n is an n-th primitive root of unity. During the past few years, congruences and q-analogues have attracted broad attentions of many authors [6, 10, 11, 12, 13, 16, 17, 18, 19, 23, 24, 26]. Especially, in 2015, Swisher [20] proved the following congruence involving double sums: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 8^k}\sum_{j=1}^{k}\left(\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\right)\equiv 0 \pmod{p}, \tag{1.4}$$

which was originally conjectured by Long [14]. Later, Gu and Guo [3] gave a q-analogue of (1.4): for any positive odd integer n, modulo $\Phi_n(q)$,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3}\sum_{j=1}^{k}\left(\frac{q^{2j-1}}{[2j-1]^2}-\frac{q^{4j}}{[4j]^2}\right)\equiv 0. \tag{1.5}$$

Shortly afterwards, Guo and Lian [7], Wang and Yu [22] as well as Fang and Guo [1] carried on this topic and presented several similar results.

Motivated by the above work, in this paper, we shall provide two new q-supercongruences on double basic hypergeometric sums, which are analogous to (1.5).

Theorem 1. *Let $n>1$ be an odd integer. Then, modulo $\Phi_n(q^2)^2$,*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\sum_{j=1}^{k}\left(\frac{q^{4j-6}}{[2j-3]_{q^2}^2}-\frac{q^{4j}}{[2j]_{q^2}^2}\right)\equiv 0. \tag{1.6}$$

Setting $n=p$ be a positive odd prime and then taking $q\to 1$ in (1.6), we obtain

$$\sum_{k=0}^{(p+1)/2}(4k-1)^3\frac{(-\frac{1}{2})_k^4}{k!^4}\sum_{j=1}^{k}\left(\frac{1}{(2j-3)^2}-\frac{1}{4j^2}\right)\equiv 0 \pmod{p^2}.$$

Theorem 2. *Let n be a positive integer such that $n\equiv t \pmod 3$ with $t\in\{1,2\}$. Then, modulo $\Phi_n(q)^2$,*

$$\begin{aligned}\sum_{k=0}^{(tn-1)/3}&[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k}\sum_{j=1}^{k}\left(\frac{q^{3j-2}}{[3j-2]^2}-\frac{q^{3j}}{[3j]^2}\right)\\ &\equiv[tn]\frac{(q^2;q^3)_{(tn-1)/3}^3}{(q^3;q^3)_{(tn-1)/3}^3}\sum_{j=1}^{(tn-1)/3}\left(\frac{q^{3j-1}}{[3j-1]^2}-\frac{q^{3j}}{[3j]^2}\right).\end{aligned} \tag{1.7}$$

When $n=p$ is a positive prime with $p\equiv t \pmod 3$ and $q\to 1$ in (1.7), we arrive at

$$\sum_{k=0}^{(tp-1)/3}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6}\sum_{j=1}^{k}\left(\frac{1}{(3j-2)^2}-\frac{1}{9j^2}\right)$$
$$\equiv tp\frac{(\frac{2}{3})^3_{(tp-1)/3}}{(1)^3_{(tp-1)/3}}\sum_{j=1}^{(tp-1)/3}\left(\frac{1}{(3j-1)^2}-\frac{1}{9j^2}\right)\pmod{p^2}.$$

Conjecture 1. *The q-congruence* (1.7) *holds modulo $\Phi_n(q)^3$ when $t=1$ and holds modulo $\Phi_n(q)^4$ when $t=2$.*

2 Proof of Theorem 1

We first require the following assistant results. Note that the first one was in fact given in the proof of Guo [4, Eq. (5.5)] and the second one due to Guo [5, Eq. (1.11)].

Lemma 1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^2(q^{-2+2n};q^4)_k(q^{-2-2n};q^4)_k}{(q^4;q^4)_k^2(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k}q^{4k}=0. \tag{2.1}$$

Proof. Recall Watson's ${}_8\phi_7$ transformation formula [2, Appendix (III.18)]:

$${}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]$$
$$=\frac{(aq;q)_n(aq/de;q)_n}{(aq/d;q)_n(aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right],$$

where the basic hypergeometric series ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix}a_1,\ a_2,\ \ldots,\ a_{r+1}\\ b_1,\ b_2,\ \ldots,\ b_r\end{matrix};q,\ z\right]=\sum_{k=0}^{\infty}\frac{(a_1;q)_k(a_2;q)_k\cdots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k}z^k.$$

The expression on the left-hand side of (2.1) is equal to

$$-q^{-4}{}_8\phi_7\left[\begin{matrix}q^{-2}, & q^3, & -q^3, & q^3, & q^3, & q^{-2}, & q^{-2+2n}, & q^{-2-2n}\\ & q^{-1}, & -q^{-1}, & q^{-1}, & q^{-1}, & q^4, & q^{4-2n}, & q^{4+2n}\end{matrix};q^4,\ q^4\right]$$
$$=-q^{-4}\frac{(q^2;q^4)_{(n+1)/2}(q^{6-2n};q^4)_{(n+1)/2}}{(q^4;q^4)_{(n+1)/2}(q^{4-2n};q^4)_{(n+1)/2}}\,{}_4\phi_3\left[\begin{matrix}q^{-4},\ q^{-2},\ q^{-2+2n},\ q^{-2-2n}\\ q^{-1},\ q^{-1},\ q^{-4}\end{matrix};q^4,\ q^4\right].$$

The proof of (2.1) is completed because of $(q^{6-2n};q^4)_{(n+1)/2}$ in the numerator. ∎

Lemma 2. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv 0 \pmod{[n]_{q^2}\Phi_n(q^2)^3}. \tag{2.2}$$

Proof of Theorem 1. By (2.1), we get

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}-0\\
&=\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k}\left(\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}-\frac{(q^{-2+2n};q^4)_k(q^{-2-2n};q^4)_k}{(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k}\right)\\
&=\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^2}{(q^4;q^4)_k^2}q^{4k}\\
&\quad\times\left(\frac{(q^{-2};q^4)_k^2(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k-(q^4;q^4)_k^2(q^{-2+2n};q^4)_k(q^{-2-2n};q^4)_k}{(q^4;q^4)_k^2(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k}\right).
\end{aligned}\tag{2.3}$$

Observing that

$$(1-q^{a+n+dj})(1-q^{a-n+dj})=(1-q^{a+dj})^2-(1-q^n)^2q^{a+dj-n},$$

and $q^n\equiv 1 \pmod{\Phi_n(q)}$, we obtain that

$$\begin{aligned}
(q^{2-n};q^2)_k(q^{2+n};q^2)_k&=\prod_{j=1}^{k}(1-q^{-n+2j})(1-q^{n+2j})\\
&=\prod_{j=1}^{k}((1-q^{2j})^2-(1-q^n)^2q^{2j-n})\\
&\equiv(q^2;q^2)_k^2-(q^2;q^2)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{2j})^2}q^{2j-n} \pmod{\Phi_n(q)^4},
\end{aligned}$$

since the remaining terms are multiples of $(1-q^n)^4$. Similarly, there holds

$$\begin{aligned}
&(q^{-1-n};q^2)_k(q^{-1+n};q^2)_k\\
&\quad\equiv(q^{-1};q^2)_k^2-(q^{-1};q^2)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{2j-3})^2}q^{2j-n-3} \pmod{\Phi_n(q)^4}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(q^{-2};q^4)_k^2(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k-(q^4;q^4)_k^2(q^{-2+2n};q^4)_k(q^{-2-2n};q^4)_k\\
&\quad\equiv(q^{-2};q^4)_k^2(q^4;q^4)_k^2[n]_{q^2}^2\sum_{j=1}^{k}\left(\frac{q^{4j-2n-6}}{[2j-3]_{q^2}^2}-\frac{q^{4j-2n}}{[2j]_{q^2}^2}\right) \pmod{\Phi_n(q^2)^4}.
\end{aligned}\tag{2.4}$$

Inserting (2.2) and (2.4) into (2.3), we conclude

$$\sum_{k=0}^{(n+1)/2} \frac{[4k-1]_{q^2}[4k-1]^2(q^{-2};q^4)_k^4 q^{4k}}{(q^4;q^4)_k^2(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k} \sum_{j=1}^{k} \left(\frac{q^{4j-2n-6}}{[2j-3]_{q^2}^2} - \frac{q^{4j-2n}}{[2j]_{q^2}^2} \right) \equiv 0 \pmod{\Phi_n(q^2)^2},$$

which is equal to (1.6) since $(q^{4-2n};q^4)_k(q^{4+2n};q^4)_k \equiv (q^4;q^4)_k^2 \pmod{\Phi_n(q^2)^2}$. □

3 Proof of Theorem 2

We first give the following lemma (or see the proof of [25, Proposition 4.1]).

Lemma 3. *Let n be an odd integer. Then, for $n \equiv t \pmod 3$ with $t = \{1,2\}$,*

$$\sum_{k=0}^{(tn-1)/3} [6k+1] \frac{(q;q^3)_k^4(q^{1-tn};q^3)_k(q^{1+tn};q^3)_k}{(q^3;q^3)_k^4(q^{3+tn};q^3)_k(q^{3-tn};q^3)_k} q^{3k} = [tn] \frac{(q^2;q^3)_{(tn-1)/3}^3}{(q^3;q^3)_{(tn-1)/3}^3}. \tag{3.1}$$

Proof. By means of Jackson's ${}_8\phi_7$ transformation [2, Appendix (II.22)]:

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ;q,\, q\right] \\ &= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/cd;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n}, \end{aligned}$$

where $a^2q = bcdeq^{-n}$, we have

$${}_8\phi_7\left[\begin{matrix} q, & q^{\frac72}, & -q^{\frac72}, & q, & q, & q, & q^{1+tn}, & q^{1-tn} \\ & q^{\frac12}, & -q^{\frac12}, & q^3, & q^3, & q^3, & q^{3-tn}, & q^{3+tn} \end{matrix} ;q^3,\, q^3\right] = [tn] \frac{(q^2;q^3)_{(tn-1)/3}^3}{(q^3;q^3)_{(tn-1)/3}^3},$$

where $t \in \{1,2\}$, which leads us to (3.1) immediately. □

Proof of Theorem 2. We first consider $n \equiv 1 \pmod 3$ case. By (3.1), we arrive at

$$\begin{aligned} &\sum_{k=0}^{(n-1)/3} [6k+1] \frac{(q;q^3)_k^6}{(q^3;q^3)_k^6} q^{3k} - [n] \frac{(q^2;q^3)_{(n-1)/3}^3}{(q^3;q^3)_{(n-1)/3}^3} \\ &\quad= \sum_{k=0}^{(n-1)/3} [6k+1] \frac{(q;q^3)_k^4}{(q^3;q^3)_k^4} q^{3k} \left(\frac{(q;q^3)_k^2}{(q^3;q^3)_k^2} - \frac{(q^{1-n};q^3)_k(q^{1+n};q^3)_k}{(q^{3+n};q^3)_k(q^{3-n};q^3)_k} \right) \\ &\quad= \sum_{k=0}^{(n-1)/3} [6k+1] \frac{(q;q^3)_k^4}{(q^3;q^3)_k^4} q^{3k} \\ &\qquad \times \left(\frac{(q;q^3)_k^2(q^{3+n};q^3)_k(q^{3-n};q^3)_k - (q^3;q^3)_k^2(q^{1-n};q^3)_k(q^{1+n};q^3)_k}{(q^3;q^3)_k^2(q^{3+n};q^3)_k(q^{3-n};q^3)_k} \right). \end{aligned} \tag{3.2}$$

Analogous to (2.4), we deduce

$$\begin{aligned}(q;q^3)_k^2(q^{3+n};q^3)_k(q^{3-n};q^3)_k-(q^3;q^3)_k^2(q^{1-n};q^3)_k(q^{1+n};q^3)_k\\ \equiv(q;q^3)_k^2(q^3;q^3)_k^2[n]^2\sum_{j=1}^{k}\left(\frac{q^{3j-n-2}}{[3j-2]^2}-\frac{q^{3j-n}}{[3j]^2}\right)\pmod{\Phi_n(q)^4}.\end{aligned}\tag{3.3}$$

Substituting (1.2) and (3.3) into (3.2), we obtain

$$\begin{aligned}\sum_{k=0}^{(n-1)/3}\frac{[6k+1](q;q^3)_k^6q^{3k}}{(q^3;q^3)_k^4(q^{3-n};q^3)_k(q^{3+n};q^3)_k}\sum_{j=1}^{k}\left(\frac{q^{3j-n-2}}{[3j-2]^2}-\frac{q^{3j-n}}{[3j]^2}\right)\\ \equiv[n]\frac{(q^2;q^3)_{(n-1)/3}^3}{(q^3;q^3)_{(n-1)/3}^3}(2-q^n)\sum_{j=1}^{(n-1)/3}\left(\frac{q^{3j-1}}{[3j-1]^2}-\frac{q^{3j}}{[3j]^2}\right)\pmod{\Phi_n(q)^2}.\end{aligned}$$

which is equivalent to the $t=1$ case of (1.7).

Using the similar argument as above, for $n\equiv 2 \pmod 3$, we replace n by $2n$ in (3.2) and (3.3) respectively and then substitute the responding results into (1.3). This finishes the proof of the case $t=2$ of (1.7). □

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