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On the shifted stability of cohomology of configuration spaces

by
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Abstract

We prove that the unordered configuration spaces of $\mathbb{S}^2 \times \mathbb{S}^4$ have shifted stability property with particular range, shift and length.

Key Words: Configuration spaces, shifted stability, cup product, spectral sequence.
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1 Introduction

For any topological space M one considers the k-points *ordered configuration space*

$$F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \, for \, i \neq j\}.$$

The symmetric group S_k acts freely on $F_k(M)$ permuting the coordinates. The quotient of $F_k(M)$ by this action is the k-points *unordered configuration space* denoted by

$$C_k(M) := F_k(M)/S_k.$$

The cohomology of the unordered configuration spaces for various manifolds M was investigated by several authors. V. I Arnold in [1] studied the case $M = \mathbb{R}^2$. D. McDuff in [9] and G. Segal in [11] generalized Arnold's results to open manifolds. The case when M is a connected oriented manifold of finite type (all Betti numbers are finite) was studied by T. Church in [4]. Church's results were further extended by O. Randal-Williams [10] and B. Knudsen [7].

In the paper [2] we introduced and studied various stability properties for the rational cohomology of unordered configuration spaces of connected manifolds of finite type.

In this paper I will focus my attention on the manifold $\mathbb{S}^2 \times \mathbb{S}^4$. All homology and cohomology groups will be considered with coefficients in $\mathbb{Q}$. The *Betti numbers*, the *Poincaré polynomial* and the *total Betti number* of a manifold M of dimension n are defined as usual:

$$\beta_i(M) = dim_{\mathbb{Q}} H^i(M), \quad P_M(t) = \sum_{i=0}^{n} \beta_i(M) t^i, \quad \beta(M) = \sum_{i=0}^{n} \beta_i(M) = P_M(1),$$

The *top Betti number* $\beta_\tau(M)$ is the last non-zero Betti number of M and the index τ is called the *cohomological dimension* and is denoted by $\mathrm{cd}(M) = \tau(M) = \tau$.

Definition 1. *[2]* The *q-truncated Poincaré polynomial* is the sum of the last q terms in the Poincaré polynomial, more precisely

$$P_M^{[q]}(t) = \beta_{\tau-q+1}(M)t^{\tau-q+1} + \ldots + \beta_\tau(M)t^\tau.$$

Definition 2. *[2]* A connected manifold M satisfies the *shifted stability condition* for its unordered configuration spaces $\{C_k(M)\}_{k\geq 1}$, with *range* r, *shift* σ and *length* q ($r, \sigma, q \geq 1$) if and only if the q-truncated Poincaré polynomial is stable after a shift: for any $k \geq r$ we have

$$P^{[q]}_{C_{k+1}(M)}(t) = t^\sigma P^{[q]}_{C_k(M)}(t).$$

In the paper [2] we proved that unordered configuration spaces of $\mathbb{CP}^3$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ satisfy the shifted stability property for particular range, shift and length.

The main result of this paper is

Theorem 1. *The product of spheres, $\mathbb{S}^2 \times \mathbb{S}^4$ has the shifted stability property with range 8, shift 4 and length 7:*

$$P^{[7]}_{C_{k+1}(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) = t^4 P^{[7]}_{C_k(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) \quad \text{for } k \geq 8.$$

More precisely, we have:

$$\begin{aligned}
\mathrm{P}_{C_{k>8}(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) =& \mathrm{P}_{C_{k-1}(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) + t^{2k} + t^{4k} + s(t^{2k+5} + t^{2k+7} + t^{4k-1} + t^{4k+3}) + \\
&+ s^2(t^{2k+12} + t^{4k+2}), \\
\mathrm{P}_{C_8(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) =& 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + 2t^{12} + t^{14} + 2t^{16} + t^{20} + t^{24} + t^{28} + \\
&+ t^{32} + s(2t^{11} + 2t^{13} + 4t^{15} + 3t^{17} + 4t^{19} + 2t^{21} + 3t^{23} + 2t^{27} + 2t^{31} + \\
&+ t^{35}) + s^2(t^{22} + t^{24} + 2t^{26} + t^{28} + t^{30} + t^{34}).
\end{aligned}$$

Corollary 1. *If $k > 8$, then the length $[l]$ (depending on k) for shifted stability of $C_k(\mathbb{S}^2\times\mathbb{S}^4)$ can be increased. More precisely, we have $l(k) = 2k - 9$ for $k > 8$.*

Remark 1. *The Betti numbers of the spaces $C_3(\mathbb{S}^2 \times \mathbb{S}^4)$ and $C_3(\mathbb{CP}^3)$ were computed by Félix-Thomas [6].*

Remark 2. *Even though the Betti numbers of $\mathbb{S}^2 \times \mathbb{S}^4$ and $\mathbb{CP}^3$ are the same, it is well known that their cohomology rings are not isomorphic. This fact causes the difference of the shifts and lengths in the shifted stability property satisfied by the cohomologies of the configuration spaces of the two manifolds.*

General conventions

- For any graded $\mathbb{Q}$-vector space $U^* = \oplus_{i\in\mathbb{Z}} U^i$ I will use the following notations

$$U^{\geq q} = \bigoplus_{i\geq q} U^i, \quad U^{even} = \bigoplus_{i\in\mathbb{Z}} U^{2i}, \quad \tilde{U}^* = \bigoplus_{i\neq 0} U^i,$$

and similarly $U^{\leq q}$ and U^{odd}.

• The symmetric algebra $Sym(U^*)$ of a graded vector space U^* is the tensor product of a polynomial algebra and an exterior algebra:

$$Sym(U^*) = \bigoplus_{k\geq 0} Sym^k(U^*) = Poly(U^{even}) \bigotimes Ext(U^{odd}),$$

where Sym^k is generated by the monomials of length k.

• The n-th suspension of a graded vector space U is the graded vector space $U[n]$ with $U[n]_i = U_{i-n}$, and the element of $U[n]$ corresponding to $a \in U$ is denoted $s^n a$; for example

$$H_*(\mathbb{S}^2 \times \mathbb{S}^4; \mathbb{Q})[n] = \begin{cases} \mathbb{Q}, & \text{if } * \in \{n, n+2, n+4, n+6\} \\ 0, & \text{otherwise.} \end{cases}$$

2 Félix-Thomas model

Y. Félix and J. C. Thomas [6] (see also [5]) studied the cohomology of $C_k(M)$ for closed oriented manifolds of even dimension. Furthermore, Knudsen [7] extended the result of Félix–Thomas for general even dimensional manifolds. In this section, we recall the definition of the differential bigraded algebra $(\Omega^*(*)(V^*, W^*), \partial)$ introduced by Félix-Thomas [6].

Fix a positive number $2m$. Consider two graded vector spaces V^*, W^*

$$V^* = \bigoplus_{i=0}^{2m} V^i, \ W^* = \bigoplus_{j=2m-1}^{4m-1} W^j.$$

By definition, the elements in V^* have length 1 and weight 0 and the elements in W^* have length 2 and weight 1. For each graded piece V^i and W^j choose bases

$$V^i = \mathbb{Q}\langle v_{i,1}, v_{i,2}, \ldots\rangle, \quad W^j = \mathbb{Q}\langle w_{j,1}, w_{j,2}, \ldots\rangle$$

(the degree of an element is marked by the first lower index; x_i^q stands for the product $x_i \wedge x_i \wedge \ldots \wedge x_i$ of q-factors). I will work only with graded vector spaces for which $V^0 = \mathbb{Q}\langle v_0\rangle \cong \mathbb{Q}$. The definition of the bigraded algebra $\Omega^*(k)$ is

$$\Omega^*(*)(V^*, W^*) = \bigoplus_{k\geq 1} \Omega^*(k)(V^*, W^*),$$

$$\Omega^*(k)(V^*, W^*) = \bigoplus_{i\geq 0} \Omega^i(k)(V^*, W^*) = Sym^k(V^* \oplus W^*),$$

where the total degree i is given by the grading of V^* and W^*. The length degree k is the multiplicative extension of length on V^* and W^*.

Let M be a closed orientable manifold of dimension $2m$. The DG-algebra introduced by Y. Félix–J. C. Thomas in [6] is given by two graded vector spaces $V^* = H_*(M; \mathbb{Q})$ and $W^* = H_*(M; \mathbb{Q})[2m-1]$, and a bidegree $(0, 1)$ map ∂, dual to the cup product:

$$\partial|_{V^*} = 0, \quad \partial|_{W^*} : W^* \simeq H_*(M; \mathbb{Q}) \to Sym^2(V^*) \simeq Sym^2(H_*(M; \mathbb{Q})).$$

Then we have identification:

$$H^*(C_k(M)) \cong H^*(\Omega^*(k)(V^*, W^*), \partial).$$

The subspace of $\Omega^*(k)$ containing the elements of weight ω is denoted ${}^{\omega}\Omega^*(k)$ and we have

$$\Omega^*(k)(V^*, W^*) = \bigoplus_{\omega=0}^{\lfloor \frac{k}{2} \rfloor} {}^{\omega}\Omega^*(k), \quad {}^{0}\Omega(k) = Sym^k(V^*),$$

$$\partial : {}^{\omega}\Omega^*(k) \longrightarrow {}^{\omega-1}\Omega^{*+1}(k).$$

3 Proof of theorem 1

In this section, we will compute the cohomology of configuration spaces on the product of a 2-sphere with 4-sphere.

The graded vector spaces of the Félix–Thomas DG-algebra for $M = \mathbb{S}^2 \times \mathbb{S}^4$ are:

$$V^* = H_*(M, \mathbb{Q}) = \oplus_{i=0}^{6} V^i, \;\; V^i = \begin{cases} \mathbb{Q}\langle v_i \rangle, & i = 0, 2, 4, 6 \\ 0, & i \text{ is odd} \end{cases}$$

and

$$W^* = H_*(M, \mathbb{Q})[5] = \oplus_{j=5}^{11} W^j, \;\; W^j = \begin{cases} \mathbb{Q}\langle w_j \rangle, & j = 5, 7, 9, 11 \\ 0, & j = 6, 8, 10. \end{cases}$$

In the above construction, the corresponding two vector spaces are:

$$V^* = \mathbb{Q}\langle v_0, v_2, v_4, v_6 \rangle, \;\; W^* = \mathbb{Q}\langle w_5, w_7, w_9, w_{11} \rangle.$$

The differential is given by:

$$\partial(v_{2i}) = 0, \text{ where } 0 \le i \le 3,$$

$$\partial(w_5, w_7, w_9, w_{11}) = (2v_0v_6 + 2v_2v_4, 2v_2v_6, 2v_4v_6, v_6^2).$$

We define an increasing filtration of subcomplexes $\{F^i\Omega^*(k)(V^*, W^*)\}_{i=0,\dots,6}$:

$$\begin{aligned}
F^0\Omega^*(k) =& V^0 \otimes \Omega^*(k-1)(V^*, W^*), \\
F^1\Omega^*(k) =& F^0\Omega^*(k), \\
F^2\Omega^*(k) =& [\oplus_{i=0}^{2} V^i \otimes \Omega^*(k-1)(V^*, W^*)] + [W^5 \otimes \Omega^*(k-2)(V^*, W^*)], \\
F^3\Omega^*(k) =& F^2\Omega^*(k), \\
F^4\Omega^*(k) =& [\oplus_{i=0}^{4} V^i \otimes \Omega^*(k-1)(V^*, W^*)] + [\oplus_{j=5}^{9} W^j \otimes \Omega^*(k-2)(V^*, W^*)], \\
F^5\Omega^*(k) =& F^4\Omega^*(k), \\
F^6\Omega^*(k) =& [\oplus_{i=0}^{6} V^i \otimes \Omega^*(k-1)(V^*, W^*)] + [\oplus_{j=5}^{11} W^j \otimes \Omega^*(k-2)(V^*, W^*)].
\end{aligned}$$

The differential ∂ respects the filtration. Also, the filtration $\{F^i\}_{i=0,\dots,6}$ and the weight decomposition $\{{}^{\omega}\Omega^*(k)\}_{\omega=0,\dots,\lfloor\frac{k}{2}\rfloor}$ are compatible:

$$F^i\Omega^*(k) = F^i \cap {}^0\Omega^*(k) \oplus F^i \cap {}^1\Omega^*(k) \oplus \ldots \oplus F^i \cap {}^{\lfloor\frac{k}{2}\rfloor}\Omega^*(k) = \bigoplus_{\omega=0}^{\lfloor\frac{k}{2}\rfloor} F^i\,{}^{\omega}\Omega^*(k),$$

hence the spectral sequence $\mathcal{E}_*^{*,*}(k)$ associated with the filtration $\{F^i\Omega^*(k)\}_{i=0,\dots,6}$ is weight-splitted at any page:

$$\mathcal{E}_*^{*,*}(k) = \bigoplus_{\omega=0}^{\lfloor\frac{k}{2}\rfloor} {}^{\omega}\mathcal{E}_*^{*,*}(k),$$

with differential

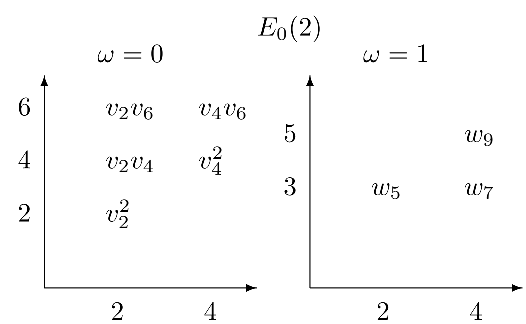
$$\partial_r^{i,j} : {}^{\omega}\mathcal{E}_r^{i,j}(k) \longrightarrow {}^{\omega-1}\mathcal{E}_r^{i-r,j+r+1}(k).$$

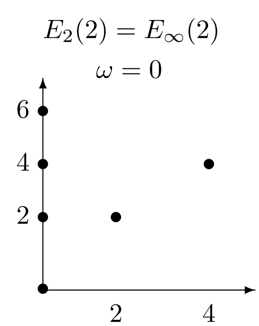
For general properties of spectral sequences obtained from filtered differential graded modules see chapter 2 of [8]. The $\mathcal{E}_*^{*,*}(k)$ is a first quadrant spectral sequence and converges (because the filtration is bounded).

Proof of Theorem 1. The subcomplex generated by v_6^2 and w_{11} is acyclic (the map $\alpha v_6^2 \to \alpha w_{11}$ gives a homotopy $id \simeq 0$). The new spectral sequence ${}^*E_*^{*,*}(k)$ is defined as:

$${}^*E_*^{*,*}(k) = \frac{{}^*\mathcal{E}_*^{*,*}(k)}{\langle w_{11}, v_6^2\rangle}.$$

The sequence of spectral sequences starts with ${}^*E_0^{*,*}(1) = {}^*E_\infty^{*,0}(1) \cong V^*$ and





The results for the spectral sequences ${}^*E_*^{*,*}(k)$, $k = 3, 4, \ldots, 8$, the "weight unstable part," are given in the following tables where I used the notation

$$\triangle_k = P_{C_k(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) - P_{C_{k-1}(\mathbb{S}^2\times\mathbb{S}^4)}(t,s).$$

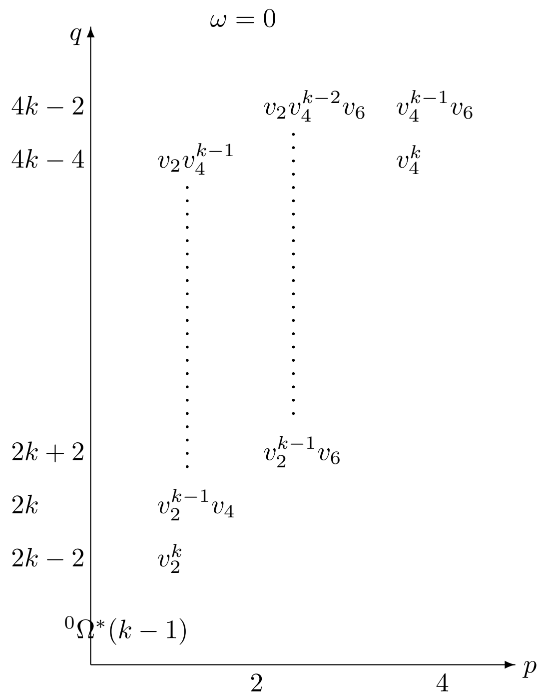
Table 1

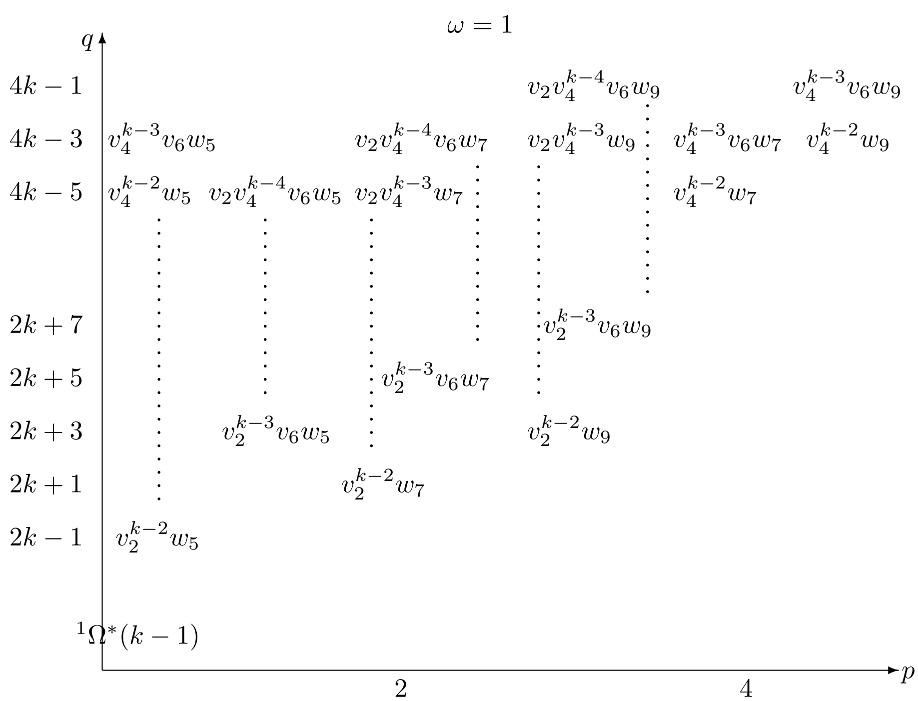
k	non-zero terms ${}^*E_r^{\geq 1,*}(k) = {}^*E_\infty^{\geq 1,*}(k)$
3	${}^0E_1^{2,4} = \langle v_2^3\rangle, {}^0E_1^{4,8} = \langle v_4^3\rangle,\ \ {}^1E_1^{2,9} = \langle \gamma = v_2w_9 - v_6w_5\rangle, {}^1E_1^{4,9} = \langle v_6w_7\rangle,$ ${}^1E_1^{4,7} = \langle v_4w_7\rangle, {}^1E_1^{4,11} = \langle v_6w_9\rangle$
4	${}^0E_1^{2,6} = \langle v_2^4\rangle, {}^0E_1^{4,12} = \langle v_4^4\rangle, {}^1E_1^{2,11} = \langle v_2\gamma\rangle, {}^1E_1^{2,13} = \langle v_2v_6w_7\rangle,$ ${}^1E_1^{2,15} = \langle v_2v_6w_9\rangle, {}^1E_1^{4,11} = \langle v_4^2w_7\rangle, {}^1E_1^{4,15} = \langle v_4v_6w_9\rangle$
5	${}^0E_1^{2,8} = \langle v_2^5\rangle, {}^0E_1^{4,16} = \langle v_4^5\rangle, {}^1E_1^{2,13} = \langle v_2^2\gamma\rangle, {}^1E_1^{2,15} = \langle v_2^2v_6w_7\rangle,$ ${}^1E_1^{4,15} = \langle v_4^3w_7\rangle, {}^1E_1^{4,19} = \langle v_4^2v_6w_9\rangle, {}^2E_1^{4,18} = \langle v_6w_7w_9\rangle$
6	${}^0E_1^{2,10} = \langle v_2^6\rangle, {}^0E_1^{4,20} = \langle v_4^6\rangle,\ \ {}^1E_1^{2,15} = \langle v_2^3\gamma\rangle, {}^1E_1^{2,17} = \langle v_2^3v_6w_7\rangle,$ ${}^1E_1^{4,19} = \langle v_4^4w_7\rangle, {}^1E_1^{4,23} = \langle v_4^3v_6w_9\rangle, {}^2E_1^{2,22} = \langle v_2v_6w_7w_9\rangle, {}^2E_1^{4,22} = \langle v_4v_6w_7w_9\rangle$
7	${}^0E_1^{2,12} = \langle v_2^7\rangle, {}^0E_1^{4,24} = \langle v_4^7\rangle, {}^1E_1^{2,17} = \langle v_2^4\gamma\rangle, {}^1E_1^{2,19} = \langle v_2^4v_6w_7\rangle,$ ${}^1E_1^{4,23} = \langle v_4^5w_7\rangle, {}^1E_1^{4,27} = \langle v_4^4v_6w_9\rangle, {}^2E_1^{2,24} = \langle v_2^2v_6w_7w_9\rangle, {}^2E_1^{4,26} = \langle v_4^2v_6w_7w_9\rangle$
8	${}^0E_1^{2,14} = \langle v_2^8\rangle, {}^0E_1^{4,28} = \langle v_4^8\rangle, {}^1E_1^{2,19} = \langle v_2^5\gamma\rangle, {}^1E_1^{2,21} = \langle v_2^5v_6w_7\rangle,$ ${}^1E_1^{4,27} = \langle v_4^6w_7\rangle, {}^1E_1^{4,31} = \langle v_4^5v_6w_9\rangle, {}^2E_1^{2,26} = \langle v_2^3v_6w_7w_9\rangle, {}^2E_1^{4,30} = \langle v_4^3v_6w_7w_9\rangle$

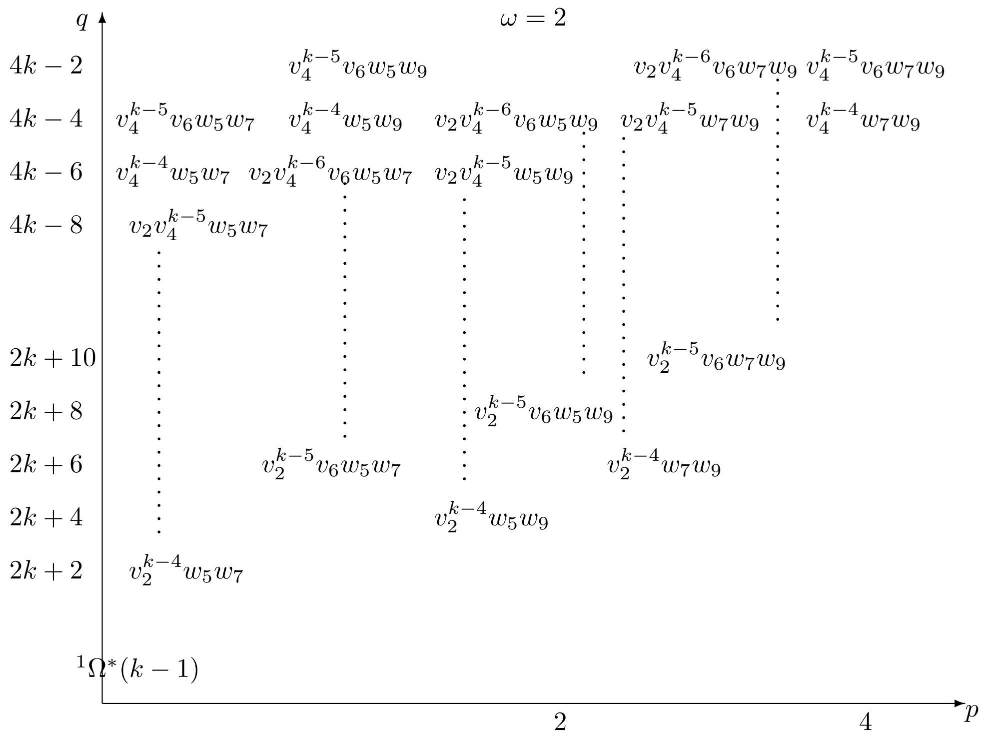
Table 2

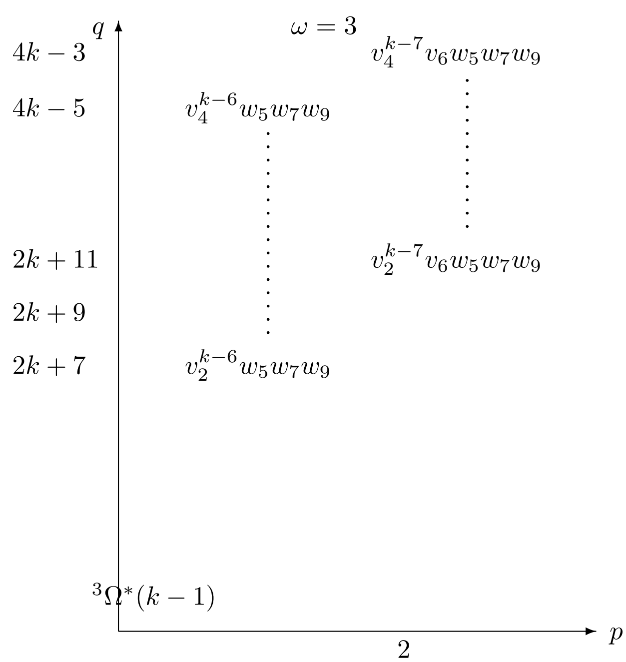
k	$\triangle_k$
3	$t^6 + t^{12} + s(2t^{11} + t^{13} + t^{15})$
4	$t^8 + t^{16} + s(t^{13} + 2t^{15} + t^{17} + t^{19})$
5	$t^{10} + t^{20} + s(t^{15} + t^{17} + t^{19} + t^{23}) + s^2t^{22}$
6	$t^{12} + t^{24} + s(t^{17} + t^{19} + t^{23} + t^{27}) + s^2(t^{24} + t^{26})$
7	$t^{14} + t^{28} + s(t^{19} + t^{21} + t^{27} + t^{31}) + s^2(t^{26} + t^{30})$
8	$t^{16} + t^{32} + s(t^{21} + t^{23} + t^{31} + t^{35}) + s^2(t^{28} + t^{34})$

We have the following picture of the first page of the k-th term ${}^*E_0^{*,*}(k)$:


$\omega = 0$
q
$4k-2$
$4k-4$
$2k+2$
$2k$
$2k-2$
$v_2 v_4^{k-2} v_6$
$v_4^{k-1} v_6$
$v_2 v_4^{k-1}$
v_4^k
$v_2^{k-1} v_6$
$v_2^{k-1} v_4$
v_2^k
${}^0\Omega^*(k-1)$
p
2
4


$\omega = 1$
q
$4k-1$
$4k-3$
$4k-5$
$2k+7$
$2k+5$
$2k+3$
$2k+1$
$2k-1$
$v_2 v_4^{k-4} v_6 w_9$
$v_4^{k-3} v_6 w_9$
$v_4^{k-3} v_6 w_5$
$v_2 v_4^{k-4} v_6 w_7$
$v_2 v_4^{k-3} w_9$
$v_4^{k-3} v_6 w_7$
$v_4^{k-2} w_9$
$v_4^{k-2} w_5$
$v_2 v_4^{k-4} v_6 w_5$
$v_2 v_4^{k-3} w_7$
$v_4^{k-2} w_7$
$v_2^{k-3} v_6 w_9$
$v_2^{k-3} v_6 w_7$
$v_2^{k-3} v_6 w_5$
$v_2^{k-2} w_9$
$v_2^{k-2} w_7$
$v_2^{k-2} w_5$
${}^1\Omega^*(k-1)$
p
2
4


$\omega = 2$
q
p
$4k-2$
$4k-4$
$4k-6$
$4k-8$
$2k+10$
$2k+8$
$2k+6$
$2k+4$
$2k+2$
$^1\Omega^*(k-1)$
2
4
$v_4^{k-5}v_6w_5w_9$
$v_2v_4^{k-6}v_6w_7w_9$
$v_4^{k-5}v_6w_7w_9$
$v_4^{k-5}v_6w_5w_7$
$v_4^{k-4}w_5w_9$
$v_2v_4^{k-6}v_6w_5w_9$
$v_2v_4^{k-5}w_7w_9$
$v_4^{k-4}w_7w_9$
$v_4^{k-4}w_5w_7$
$v_2v_4^{k-6}v_6w_5w_7$
$v_2v_4^{k-5}w_5w_9$
$v_2v_4^{k-5}w_5w_7$
$v_2^{k-5}v_6w_7w_9$
$v_2^{k-5}v_6w_5w_9$
$v_2^{k-5}v_6w_5w_7$
$v_2^{k-4}w_7w_9$
$v_2^{k-4}w_5w_9$
$v_2^{k-4}w_5w_7$


$\omega = 3$
q
p
$4k-3$
$4k-5$
$2k+11$
$2k+9$
$2k+7$
$^3\Omega^*(k-1)$
2
$v_4^{k-7}v_6w_5w_7w_9$
$v_4^{k-6}w_5w_7w_9$
$v_2^{k-7}v_6w_5w_7w_9$
$v_2^{k-6}w_5w_7w_9$

On the column $p = 0$, we get ${}^{\omega}H^*(C_{k-1})$. On the column $p = 4$, we get four cohomology classes v_4^k, $v_4^{k-2}w_7$, $v_4^{k-3}v_6w_9$ and $v_4^{k-5}v_6w_7w_9$:

$$
\begin{gathered}
{}^1E_0^{4,4k-5}(k) = \langle v_4^{k-2}w_7\rangle \to {}^0E_0^{4,4k-4}(k) = \langle v_4^{k}\rangle,\\
{}^2E_0^{4,4k-2}(k) = \langle v_4^{k-5}v_6w_7w_9\rangle \to {}^1E_0^{4,4k-1}(k) = \langle v_4^{k-3}v_6w_9\rangle,\\
{}^2E_0^{4,4k-4}(k) = \langle v_4^{k-4}w_7w_9\rangle \to {}^1E_0^{4,4k-3}(k) = \langle v_4^{k-3}v_6w_7,\ v_4^{k-2}w_9\rangle \to {}^0E_0^{4,4k-2}(k) = \langle v_4^{k-1}v_6\rangle.
\end{gathered}
$$

On the column $p = 2$, we have a four components cochain complex $e(q)$, where q takes values in the interval $[k-2, 2k-1]$:

$$
{}^3E_0^{2,2q-1}(k) \xrightarrow{\ a\ } {}^2E_0^{2,2q}(k) \xrightarrow{\ b\ } {}^1E_0^{2,2q+1}(k) \xrightarrow{\ c\ } {}^0E_0^{2,2q+2}(k).
$$

In the generic case, $q \in [k+6, 2k-4]$, all the four components are non-zero and $e(q)$ is acyclic:

$$
\begin{array}{cccc}
{}^3E_0^{2,2q-1}(k) \xrightarrow{\quad a\quad} & {}^2E_0^{2,2q}(k) \xrightarrow{\quad b\quad} & {}^1E_0^{2,2q+1}(k) \xrightarrow{\quad c\quad} & {}^0E_0^{2,2q+2}(k)\\
\shortparallel & \shortparallel & \shortparallel & \shortparallel\\
v_2^{2k-q-2}v_4^{q-k-4}w_5w_7w_9 & v_2^{2k-q-3}v_4^{q-k-1}w_5w_7 & v_2^{2k-q-3}v_4^{q-k+1}w_5 & v_2^{2k-q-2}v_4^{q-k+2}\\
\oplus & \oplus & \oplus & \oplus\\
v_2^{2k-q-1}v_4^{q-k-6}v_6w_5w_7w_9 & v_2^{2k-q-2}v_4^{q-k-3}v_6w_5w_7 & v_2^{2k-q-2}v_4^{q-k-1}v_6w_5 & v_2^{2k-q-1}v_4^{q-k}v_6\\
 & \oplus & \oplus & \\
 & v_2^{2k-q-2}v_4^{q-k-2}w_5w_9 & v_2^{2k-q-2}v_4^{q-k}w_7 & \\
 & \oplus & \oplus & \\
 & v_2^{2k-q-1}v_4^{q-k-4}v_6w_5w_9 & v_2^{2k-q-1}v_4^{q-k-2}v_6w_7 & \\
 & \oplus & \oplus & \\
 & v_2^{2k-q-1}v_4^{q-k-3}w_7w_9 & v_2^{2k-q-1}v_4^{q-k-1}w_9 & \\
 & \oplus & \oplus & \\
 & v_2^{2k-q}v_4^{q-k-5}v_6w_7w_9 & v_2^{2k-q}v_4^{q-k-3}v_6w_9. &
\end{array}
$$

The matrices of the differential are:

$$
a = \begin{pmatrix} 0 & 0\\ -2 & 0\\ 0 & 0\\ 2 & 0\\ 2 & 0\\ 0 & 2 \end{pmatrix} \quad
b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ -2 & 0 & -2 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & -2 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix} \quad
c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}.
$$

For the last values of q the cochain complex $e(q)$ is still acyclic.

$q = 2k-3$

$$\begin{array}{ccccccc}
{}^3E_0^{2,4k-7}(k) & \xrightarrow{\quad a \quad} & {}^2E_0^{2,4k-6}(k) & \xrightarrow{\quad b \quad} & {}^1E_0^{2,4k-5}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,4k-4}(k) \\
\shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
v_2v_4^{k-7}w_5w_7w_9 & & v_4^{k-4}w_5w_7 & & v_4^{k-2}w_5 & & v_2v_4^{k-1} \\
\bigoplus & & \bigoplus & & \bigoplus & & \bigoplus \\
v_2^2v_4^{k-9}v_6w_5w_7w_9 & & v_2v_4^{k-6}v_6w_5w_7 & & v_2v_4^{k-4}v_6w_5 & & v_2^2v_4^{k-3}v_6 \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2v_4^{k-5}w_5w_9 & & v_2v_4^{k-3}w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^2v_4^{k-7}v_6w_5w_9 & & v_2^2v_4^{k-5}v_6w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^2v_4^{k-6}w_7w_9 & & v_2^2v_4^{k-4}w_9 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^3v_4^{k-8}v_6w_7w_9 & & v_2^3v_4^{k-6}v_6w_9. & &
\end{array}$$

The matrices of the differential are:

$$a = \begin{pmatrix} 0 & 0 \\ -2 & 0 \\ 0 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

$q = 2k-2$

$$\begin{array}{ccccccc}
{}^3E_0^{2,4k-5}(k) & \xrightarrow{\quad a \quad} & {}^2E_0^{2,4k-4}(k) & \xrightarrow{\qquad b \qquad} & {}^1E_0^{2,4k-3}(k) & \xrightarrow{c} & {}^0E_0^{2,4k-2}(k) \\
\shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
v_4^{k-6}w_5w_7w_9 & \begin{pmatrix} -2 & 0 \\ 0 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} & v_4^{k-5}v_6w_5w_7 & \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{pmatrix} & v_4^{k-3}v_6w_5 & \begin{pmatrix} 2 & 0 & 2 & 0 \end{pmatrix} & v_2v_4^{k-2}v_6 \\
\bigoplus & & \bigoplus & & \bigoplus & & \\
v_2v_4^{k-8}v_6w_5w_7w_9 & & v_4^{k-4}w_5w_9 & & v_2v_4^{k-4}v_6w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2v_4^{k-6}v_6w_5w_9 & & v_2v_4^{k-3}w_9 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2v_4^{k-5}w_7w_9 & & v_2^2v_4^{k-5}v_6w_9 & & \\
 & & \bigoplus & & & & \\
 & & v_2^2v_4^{k-7}v_6w_7w_9 & & & &
\end{array}$$

$q = 2k - 1$

$$\begin{array}{ccccc} {}^3E_0^{2,4k-3}(k) & \xrightarrow{\quad a \quad} & {}^2E_0^{2,4k-2}(k) & \xrightarrow{\quad b \quad} & {}^1E_0^{2,4k-1}(k) \\ \| & & \| & & \| \\ v_4^{k-7}v_6w_5w_7w_9 & \begin{pmatrix}0\\2\end{pmatrix} & v_4^{k-5}v_6w_5w_9 & \begin{pmatrix}2 & 0\end{pmatrix} & v_2v_4^{k-4}v_6w_9 \\ & & \bigoplus & & \\ & & v_2v_4^{k-6}v_6w_7w_9 & & \end{array}$$

For the first values of q we obtain non-zero cohomology classes. The value $q = k - 2$ gives the cohomology class v_2^k. The differential of v_2^k is zero. The degrees of all non-zero elements other then v_2^k in column $p > 0$ are bigger than $2k$. So v_2^k is a permanent cocycle and never a coboundary. The next two values of q give the exact sequences:

$q = k - 1$

$$\begin{array}{ccc} {}^1E_0^{2,2k-1}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,2k}(k) \\ \| & & \| \\ v_2^{k-2}w_5 & (1) & v_2^{k-1}v_4 \end{array}$$

$q = k$

$$\begin{array}{ccc} {}^1E_0^{2,2k+1}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,2k+2}(k) \\ \| & & \| \\ v_2^{k-3}v_4w_5 & \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} & v_2^{k-2}v_4^2 \\ \bigoplus & & \bigoplus \\ v_2^{k-2}w_7 & & v_2^{k-1}v_6 \end{array}$$

The value $q = k + 1$ gives the cohomology class $v_2^{k-3}\gamma$.

$q = k + 1$

$$\begin{array}{ccccc} {}^2E_0^{2,2k+2}(k) & \xrightarrow{\quad b \quad} & {}^1E_0^{2,2k+3}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,2k+4}(k) \\ \| & & \| & & \| \\ v_2^{k-4}w_5w_7 & \begin{pmatrix}0\\-2\\2\\0\end{pmatrix} & v_2^{k-4}v_4^2w_5 & \begin{pmatrix}2 & 0 & 0 & 0\\0 & 2 & 2 & 2\end{pmatrix} & v_2^{k-3}v_4^3 \\ & & \bigoplus & & \bigoplus \\ & & v_2^{k-3}v_6w_5 & & v_2^{k-2}v_4v_6 \\ & & \bigoplus & & \\ & & v_2^{k-3}v_4w_7 & & \\ & & \bigoplus & & \\ & & v_2^{k-2}w_9 & & \end{array}$$

The value $q=k+2$ gives the cohomology class $v_2^{k-3}v_6w_7$:

$$q=k+2$$

$$\begin{array}{ccccc}
{}^2E_0^{2,2k+4}(k) & \xrightarrow{\quad b \quad} & {}^1E_0^{2,2k+5}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,2k+6}(k) \\
\shortparallel & & \shortparallel & & \shortparallel \\
\begin{array}{c} v_2^{k-5}v_4w_5w_7 \\ \oplus \\ v_2^{k-4}w_5w_9 \end{array} &
\begin{pmatrix} 0 & 0 \\ -2 & -2 \\ 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix} &
\begin{array}{c} v_2^{k-5}v_4^3w_5 \\ \oplus \\ v_2^{k-4}v_4v_6w_5 \\ \oplus \\ v_2^{k-4}v_4^2w_7 \\ \oplus \\ v_2^{k-3}v_6w_7 \\ \oplus \\ v_2^{k-3}v_4w_9 \end{array} &
\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 \end{pmatrix} &
\begin{array}{c} v_2^{k-4}v_4^4 \\ \oplus \\ v_2^{k-3}v_4^2v_6 \end{array}
\end{array}$$

The cohain complex corresponding to the value $q=k+3$ is acyclic:

$$q=k+3$$

$$\begin{array}{ccccc}
{}^2E_0^{2,2k+6}(k) & \xrightarrow{\qquad b \qquad} & {}^1E_0^{2,2k+7}(k) & \xrightarrow{\quad c \quad} & {}^0E_0^{2,2k+8}(k) \\
\shortparallel & & \shortparallel & & \shortparallel \\
\begin{array}{c} v_2^{k-6}v_4^2w_5w_7 \\ \oplus \\ v_2^{k-5}v_6w_5w_7 \\ \oplus \\ v_2^{k-5}v_4w_5w_9 \\ \oplus \\ v_2^{k-4}w_7w_9 \end{array} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} &
\begin{array}{c} v_2^{k-6}v_4^4w_5 \\ \oplus \\ v_2^{k-5}v_4^2v_6w_5 \\ \oplus \\ v_2^{k-5}v_4^3w_7 \\ \oplus \\ v_2^{k-4}v_4v_6w_7 \\ \oplus \\ v_2^{k-4}v_4^2w_9 \\ \oplus \\ v_2^{k-3}v_6w_9 \end{array} &
\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix} &
\begin{array}{c} v_2^{k-5}v_4^5 \\ \oplus \\ v_2^{k-4}v_4^3v_6 \end{array}
\end{array}$$

The cohain complex corresponding to the value $q=k+4$ is acyclic:

$q = k + 4$

$$\begin{array}{ccccccc}
{}^3E_0^{2,2k+7}(k) & \xrightarrow{a} & {}^2E_0^{2,2k+8}(k) & \xrightarrow{b} & {}^1E_0^{2,2k+9}(k) & \xrightarrow{c} & {}^0E_0^{2,2k+10}(k) \\
\parallel & & \parallel & & \parallel & & \parallel \\
v_2^{k-6}w_5w_7w_9 & & v_2^{k-7}v_4^3w_5w_7 & & v_2^{k-7}v_4^5w_5 & & v_2^{k-6}v_4^6 \\
 & & \bigoplus & & \bigoplus & & \bigoplus \\
 & & v_2^{k-6}v_4v_6w_5w_7 & & v_2^{k-6}v_4^3v_6w_5 & & v_2^{k-5}v_4^4v_6 \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-6}v_4^2w_5w_9 & & v_2^{k-6}v_4^4w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-5}v_6w_5w_9 & & v_2^{k-5}v_4^2v_6w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-5}v_4w_7w_9 & & v_2^{k-5}v_4^3w_9 & & \\
 & & & & \bigoplus & & \\
 & & & & v_2^{k-4}v_4v_6w_9 & &
\end{array}$$

The value $q = k+5$ gives the cohomology class $v_2^{k-5}v_6w_7w_9$. The differential of $v_2^{k-5}v_6w_7w_9$ is zero, so it is cocycle. Also:

$$\partial(v_2^{k-7}v_4w_5w_7w_9) = -2v_2^{k-7}v_4^2v_6w_5w_7 + 2v_2^{k-6}v_4v_6w_5w_9 + 2v_2^{k-6}v_4^2w_7w_9.$$

The element $v_2^{k-5}v_6w_7w_9$ is not a coboundary.

$q = k + 5$

$$\begin{array}{ccccccc}
{}^3E_0^{2,2k+9}(k) & \xrightarrow{a} & {}^2E_0^{2,2k+10}(k) & \xrightarrow{b} & {}^1E_0^{2,2k+11}(k) & \xrightarrow{c} & {}^0E_0^{2,2k+12}(k) \\
\parallel & & \parallel & & \parallel & & \parallel \\
v_2^{k-7}v_4w_5w_7w_9 & & v_2^{k-8}v_4^4w_5w_7 & & v_2^{k-8}v_4^6w_5 & & v_2^{k-7}v_4^7 \\
 & & \bigoplus & & \bigoplus & & \bigoplus \\
 & & v_2^{k-7}v_4^2v_6w_5w_7 & & v_2^{k-7}v_4^4v_6w_5 & & v_2^{k-6}v_4^5v_6 \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-7}v_4^3w_5w_9 & & v_2^{k-7}v_4^5w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-6}v_4v_6w_5w_9 & & v_2^{k-6}v_4^3v_6w_7 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-6}v_4^2w_7w_9 & & v_2^{k-6}v_4^4w_9 & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & v_2^{k-5}v_6w_7w_9 & & v_2^{k-5}v_4^2v_6w_9 & &
\end{array}$$

The matrices of the differential for $q = k + 4$ are:

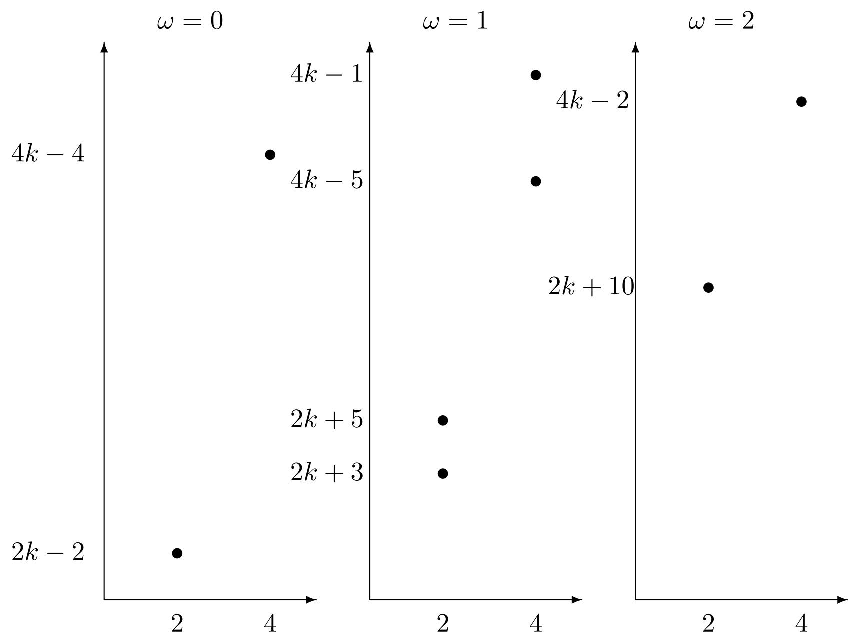
$$a = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 2 \\ 2 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The matrices of the differential for $q = k + 5$ are:

$$a = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The final picture of the spectral sequence for $p > 0$ is:

$$E_1^{\geq 1}(k) = E_\infty^{\geq 1}(k)$$



□

Remark 3. *In section 6 of [2], Berceanu-Yameen introduced several notions of shifted stabilities. In particular (see Definition 3 of [2]), the manifold M satisfies the spectral shifted stability condition with range r and shift σ $(r, \sigma \geq 1)$ if and only if, for any $k \geq r$, any $p \geq 1$ and any $\omega \geq 0$, we have*

$${}^{\omega}E_\infty^{p,q+\sigma}(k+1) = {}^{\omega}E_\infty^{p,q}(k) \text{ and this is non-zero.}$$

In the paper [2] (see Proposition 10) Berceanu-Yameen proved that :

$$\textit{Spectral shifted stability} \Rightarrow \textit{Shifted stability.}$$

The Theorem 1 shows that the converse of this fact is not true (for more details see the final picture in the proof of Theorem 1).

Proof of Corollary 1. Let $k > 8$. From formulas in Theorem 1, we have

$$\Delta_k = [t^{2k} + s(t^{2k+5} + t^{2k+7}) + s^2 t^{2k+12}] + [t^{4k} + s(t^{4k-1} + t^{4k+3}) + s^2 t^{4k+2}].$$

where $\Delta_k = \mathrm{P}_{C_k(\mathbb{S}^2\times\mathbb{S}^4)}(t,s) - \mathrm{P}_{C_{k-1}(\mathbb{S}^2\times\mathbb{S}^4)}(t,s)$. This formula gives that

$$\beta_{4k+3}(C_k(\mathbb{S}^2 \times \mathbb{S}^4)) = 1.$$

Moreover, for $2k + 12 < i < 4k + 3$, we have

$$\beta_i(C_k(\mathbb{S}^2 \times \mathbb{S}^4)) = \begin{cases} 2, & \text{if } i \equiv 3\,(mod\,4) \\ 1, & \text{if } i \equiv 0\,(mod\,4) \text{ or } i \equiv 2\,(mod\,4) \\ 0, & \text{if } i \equiv 1\,(mod\,4) \end{cases}$$

We know that the shift is 4. Therefore the length is an increasing function in k :

$$l(k) = (4k + 3) - (2k + 12) = 2k - 9, \quad \text{for } k > 8.$$

□

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