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Weak Rickart and dual weak Rickart objects in abelian categories: transfer via functors

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Dedicated to Professors Toma Albu and Constantin Năstăsescu
on the occasion of their 80th birthdays

Abstract

Weak relative Rickart objects generalize relative Rickart objects in abelian categories. We study how such a property is preserved or reflected by fully faithful functors and adjoint pairs of functors. Various consequences are obtained for (co)reflective subcategories, adjoint triples of functors and endomorphism rings of modules. In particular, for a right R-module M with endomorphism ring S, we prove that if M is a weak self-Rickart right R-module, then S is a weak self-Rickart right S-module, while the converse holds provided M is a flat left S-module or M is a k-local-retractable right R-module.

Key Words: Abelian category, (dual) weak Rickart object, Grothendieck category, (graded) module, comodule, endomorphism ring.
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1 Introduction

Weak Rickart objects and their duals in abelian categories were investigated by Crivei and Keskin Tütüncü [8]. These objects are placed in between some important classes of objects as follows. On one hand, they generalize (dual) Rickart objects in abelian categories studied by Crivei, Kör and Olteanu [12, 13] and regular objects in abelian categories studied by Dăscălescu, Năstăsescu, Tudorache and Dăuş [15]. On the other hand, they satisfy some non-singularity conditions, namely: if N is a weak M-Rickart object, then N is M-$\mathcal{K}$-nonsingular, while if N is a dual weak M-Rickart object, then M is N-$\mathcal{T}$-nonsingular [8]. An important feature of (dual) weak Rickart objects is their good behaviour with respect to products and coproducts [8, Theorem 2.7], unlike the (dual) relative Rickart properties. All these categorical notions have module-theoretic roots in (dual) Rickart modules studied by Lee, Rizvi and Roman [25, 26], regular modules in the sense of Zelmanowitz [40], (dual) Baer modules studied by Rizvi and Roman [32, 33] and Keskin Tütüncü and Tribak [23] respectively, or (dual) weak Rickart modules investigated by Keskin Tütüncü, Orhan Ertaş and Tribak [22, 37]. One of the advantages to approach them in abelian categories is that the dual results will automatically hold by the duality principle. Thus we unify the study of dual notions which have been studied separately in the literature. This is our main reason for using categorical tools, our primary interest still being to obtain module-theoretic results.

For further motivation on considering (dual) weak Rickart objects in abelian categories and, in particular, (dual) weak Rickart modules, the reader is referred to [8, 22, 37].

This paper continues the investigation of (dual) weak Rickart objects by analyzing their behaviour under functors between abelian categories as well as their endomorphism rings, the latter in case of (graded) module categories. We present various ways of constructing new weak relative Rickart objects from old by using suitable functors between abelian categories. We will implicitly use the duality principle, which allows us to prove just properties on weak relative Rickart objects. Also, it is enough to consider covariant (additive) functors, the situation of contravariant functors being dual.

In Section 2 we prove that left exact fully faithful functors preserve and reflect the weak relative Rickart property under some mild condition. For applications we consider finitely accessible and exactly definable abelian categories, abelian coreflective and reflective full subcategories of abelian categories, and we obtain results on preservation and reflection of the (dual) weak relative Rickart property in Grothendieck categories, and in particular to finitely accessible Grothendieck categories, module categories, categories $\sigma[M]$ or comodule categories. For instance, if $\mathcal{A}$ is a Grothendieck category and R is the endomorphism ring of a generator U of $\mathcal{A}$, then the functor $\mathrm{Hom}_{\mathcal{A}}(U,-):\mathcal{A}\to \mathrm{Mod}(R)$ from $\mathcal{A}$ to the category $\mathrm{Mod}(R)$ of right R-modules preserves and reflects the weak relative Rickart property. We also derive consequences to adjoint triples of functors, and we illustrate them for (graded) module categories.

In Section 3 we consider covariant functors $L:\mathcal{A}\to\mathcal{B}$ and $R:\mathcal{B}\to\mathcal{A}$ such that (L,R) is an adjoint pair with counit $\varepsilon: LR\to 1_{\mathcal{B}}$. If $M,N\in \mathrm{Stat}(R)=\{B\in\mathcal{B}\mid \varepsilon_B \text{ is an isomorphism}\}$ and L is exact, then we prove that N is weak M-Rickart in $\mathcal{B}$ if and only if $R(N)$ is weak $R(M)$-Rickart in $\mathcal{A}$. We also discuss the case of an adjoint pair of contravariant functors. These results allow us to deduce consequences on preservation and reflection of (dual) weak Rickart properties related to endomorphism rings of modules, modules in categories $\sigma[M]$, graded modules and comodules. For instance, for a right R-module M with endomorphism ring S, if M is a weak self-Rickart right R-module, then S is a weak self-Rickart right S-module, and the converse holds provided M is a flat left S-module or M is a k-local-retractable right R-module.

2 Transfer via fully faithful functors

Let $f:M\to N$ be a morphism in an abelian category $\mathcal{A}$. Then $\ker(f):\mathrm{Ker}(f)\to M$ and $\mathrm{coker}(f):N\to\mathrm{Coker}(f)$ will denote the kernel and the cokernel of f respectively. The morphism f is called a *section* (*retraction*) if it has a left (right) inverse.

We recall from [8] the notions of (dual) weak relative Rickart objects.

Definition 1. Let M and N be objects of an abelian category $\mathcal{A}$. Then N is called:

(1) *weak M-Rickart* if for every nonzero morphism $f:M\to N$, $\ker(f):\mathrm{Ker}(f)\to M$ factors through a section $s:Q\to M$ which is not an isomorphism, i.e., $\ker(f)=si$ for some section $s:Q\to M$ which is not an isomorphism and (mono)morphism $i:\mathrm{Ker}(f)\to Q$. Equivalently, the kernel object $\mathrm{Ker}(f)$ of every nonzero morphism $f:M\to N$ is included in some proper direct summand of M.

(2) *dual weak M-Rickart* if for every nonzero morphism $f : M \to N$, $\mathrm{coker}(f) : N \to \mathrm{Coker}(f)$ factors through a retraction $r : N \to Q$ which is not an isomorphism, i.e., $\mathrm{coker}(f) = dr$ for some retraction $r : N \to Q$ which is not an isomorphism and (epi)morphism $d : Q \to \mathrm{Coker}(f)$. Equivalently, the image object $\mathrm{Im}(f)$ of every nonzero morphism $f : M \to N$ contains some nonzero direct summand of N.

(3) *weak self-Rickart* if N is weak N-Rickart.

(4) *dual weak self-Rickart* if N is dual weak N-Rickart.

Functors between abelian categories need not preserve or reflect the above properties.

Example 1. Consider the ring $R = \mathbb{Z} \oplus \mathbb{Z}_p$ for some prime p. Then the forgetful covariant functor $F : \mathrm{Mod}(R) \to \mathrm{Ab}$ from the category of right R-modules to the category of abelian groups is exact and faithful. Since the endomorphism rings $\mathrm{End}_R(R_R) \cong R$ and $\mathrm{End}_{\mathbb{Z}}(R, +) \cong \left(\begin{smallmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$ are not isomorphic, it follows that F is not full. The right R-module R is weak self-Rickart, being hereditary [8, Theorem 4.10], but the abelian group $F(R) = \mathbb{Z} \oplus \mathbb{Z}_p$ is not weak self-Rickart [22, Example 4.8].

Example 2. For an abelian group G, let $t(G)$ and $d(G)$ be the largest torsion subgroup of G and the largest divisible (i.e., injective) subgroup of G respectively. Let $\mathcal{T}$ be the abelian category of torsion abelian groups. The covariant functor $F : \mathrm{Ab} \to \mathcal{T}$ given by $F(G) = t(d(G))$ is left exact and full, but not faithful [11, Example 4.1]. Consider the abelian group $G = \mathbb{Z} \oplus \mathbb{Z}_p$ for some prime p. Then $F(G) = 0$ is weak self-Rickart, while G is not [22, Example 4.8].

Let us now consider fully faithful covariant (additive) functors. Denote by $\mathrm{Im}(F)$ the essential image of a functor $F : \mathcal{A} \to \mathcal{B}$. As usual, we say that F *preserves the (dual) weak relative Rickart property* if $F(N)$ is (dual) weak $F(M)$-Rickart in $\mathcal{B}$ whenever N is (dual) weak M-Rickart in $\mathcal{A}$. Also, we say that F *reflects the (dual) weak relative Rickart property* if N is (dual) weak M-Rickart in $\mathcal{A}$ whenever $F(N)$ is (dual) weak $F(M)$-Rickart in $\mathcal{B}$.

Theorem 1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful covariant functor between abelian categories.*

(1) *(i) If F is left exact, then F preserves the weak relative Rickart property.*

(ii) If F is left exact and $\mathrm{Im}(F)$ *is closed under direct summands, then F reflects the weak relative Rickart property.*

(2) *(i) If F is right exact, then F preserves the dual weak relative Rickart property.*

(ii) If F is right exact and $\mathrm{Im}(F)$ *is closed under direct summands, then F reflects the dual weak relative Rickart property.*

Proof. (1) (i) Let N be a weak M-Rickart object of $\mathcal{A}$, and let $g : F(M) \to F(N)$ be a nonzero morphism in $\mathcal{B}$. We may write $g = F(f)$ for some nonzero morphism $f : M \to N$ in $\mathcal{A}$, because F is full. Since N is weak M-Rickart, $\ker(f)$ factors through a section $s : Q \to M$ which is not an isomorphism. Since F is left exact, $\mathrm{Ker}(g) = \mathrm{Ker}(F(f)) = F(\ker(f))$ factors through the section $F(s) : F(Q) \to F(M)$, which is not an isomorphism. Hence $F(N)$ is weak $F(M)$-Rickart.

(ii) Let M and N be objects of $\mathcal{A}$ such that $F(N)$ is weak $F(M)$-Rickart, and let $f : M \to N$ be a nonzero morphism in $\mathcal{A}$. Consider the nonzero morphism $F(f) : F(M) \to F(N)$ in

$\mathcal{B}$. By hypothesis, $F(\ker(f)) = \ker(F(f))$ factors through a section $s' : Q' \to F(M)$ which is not an isomorphism. But then $Q' \cong F(Q)$ for some object Q in $\mathcal{A}$. Since F is fully faithful, we may write $s' = F(s)$ for some section $s : Q \to M$ which is not an isomorphism. Then $\ker(f)$ factors through the section $s : Q \to M$ which is not an isomorphism. Hence N is weak M-Rickart. □

Corollary 1. *Equivalences between abelian categories preserve and reflect the (dual) weak relative Rickart properties.*

For a first application, let us recall some terminology on functor categories associated to module categories, following [2, 20] (also, see [9]). Let $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{\mathrm{op}})$ be the categories of right R-modules and left R-modules respectively, and let $(\mathrm{mod}(R), \mathrm{Ab})$ be the category of covariant functors from the category $\mathrm{mod}(R)$ of finitely presented right R-modules to Ab, and by $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$ the category of contravariant functors from $\mathrm{mod}(R)$ to Ab.

Corollary 2. *(1) The functor* $H : \mathrm{Mod}(R) \to ((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$, $H(M) = \mathrm{Hom}_R(-, M)$ *for any right R-module M, preserves and reflects the weak relative Rickart property.*

(2) The functor $T : \mathrm{Mod}(R^{\mathrm{op}}) \to (\mathrm{mod}(R), \mathrm{Ab})$ *given by* $T(M) = - \otimes_R M$ *on left R-modules M preserves and reflects the dual weak relative Rickart property.*

Proof. One may consider flat and absolutely pure objects in the Grothendieck categories $(\mathrm{mod}(R), \mathrm{Ab})$ and $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$ [34]. Note that H is a fully faithful left exact functor, which induces an equivalence between $\mathrm{Mod}(R)$ and the full subcategory of flat objects of $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$, while T is a fully faithful right exact functor, which yields an equivalence between $\mathrm{Mod}(R^{\mathrm{op}})$ and the full subcategory of absolutely pure objects of $(\mathrm{mod}(R), \mathrm{Ab})$. Also, the classes of flat objects of $((\mathrm{mod}(R))^{\mathrm{op}}, \mathrm{Ab})$ and absolutely pure objects of $(\mathrm{mod}(R), \mathrm{Ab})$ are closed under direct summands. Finally, use Theorem 1. □

We illustrate Corollary 2 with one situation picked from several charaterizations of rings in terms of (dual) weak Rickart modules established in [8]. Recall that a ring is called *right semiartinian* if every nonzero right R-module contains a simple submodule, and *right V-ring* if every simple right R-module is injective. Then Corollary 2 together with [8, Corollary 4.4] or [37, Theorem 2.20] yields the following consequence.

Corollary 3. *Let R be a left semiartinian left V-ring. Then $-\otimes_R M$ is a dual weak Rickart object in* $(\mathrm{mod}(R), \mathrm{Ab})$ *for every left R-module M.*

More generally than module categories, we may consider finitely accessible and exactly definable categories, and certain associated categories.

Let $\mathcal{C}$ be an additive category and let $\mathrm{fp}(\mathcal{C})$ be its full subcategory of finitely presented objects. Then $\mathcal{C}$ is called *finitely accessible* if $\mathcal{C}$ has direct limits, $\mathrm{fp}(\mathcal{C})$ is skeletally small, and every object of $\mathcal{C}$ may be written as a direct limit of objects from $\mathrm{fp}(\mathcal{C})$ [29]. Finitely accessible additive categories offer a suitable framework for defining purity [29]. Note that a finitely accessible abelian category is even Grothendieck [29, Theorem 3.15]. Module and comodule categories are finitely accessible Grothendieck categories. Denote by

$(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ the category of all contravariant (additive) functors from $\mathrm{fp}(\mathcal{C})$ to Ab. Note that this functor category is equivalent to the module category over the functor ring of $\mathcal{C}$ (e.g., see [17]). For every finitely accessible additive category $\mathcal{C}$, there is a fully faithful functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ (naturally isomorphic to the inclusion functor) that induces an equivalence between $\mathcal{C}$ and the full subcategory of flat objects of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ [6, Theorem 1.4, Lemma 3.1].

Corollary 4. *Let $\mathcal{C}$ be a finitely accessible Grothendieck category. Then the above functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ preserves and reflects the weak relative Rickart property.*

Proof. The image of the fully faithful left exact functor H is equivalent to the full subcategory of flat objects of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$, which is closed under direct summands. Then use Theorem 1. □

Corollary 5. *Let $\mathcal{C}$ be a finitely accessible Grothendieck category. Assume that every morphism in $\mathcal{C}$ is the composition of a pure epimorphism followed by a monomorphism (e.g., $\mathcal{C}$ is regular in the sense that every short exact sequence is pure). Then every finitely presented object of $\mathcal{C}$ is weak self-Rickart.*

Proof. The equivalence induced by the functor $H : \mathcal{C} \to (\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ between $\mathcal{C}$ and the full subcategory of flat objects of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ restricts to an equivalence between the full subcategories of finitely presented objects of $\mathcal{C}$ and finitely generated projective objects of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ (e.g., see [17, p. 178]). Also, the functor category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is semihereditary [17, Proposition 3.1]. Now if $X \in \mathrm{fp}(\mathcal{C})$, then $H(X)$ is a finitely generated projective object of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$, and so $H(X)$ is a weak self-Rickart object of $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ [8, Theorem 4.10]. Finally, X is weak self-Rickart by Corollary 4. □

Note that the module category $\mathrm{Mod}(R)$ is regular if and only if the ring R is von Neumann regular. It is known that every finitely generated projective right module over a von Neumann regular ring is weak self-Rickart [22, Corollary 2.3]. Now we may generalize this result as follows, as a consequence of Corollary 5.

Corollary 6. *Let R be a von Neumann regular ring. Then every finitely presented right R-module is weak self-Rickart.*

An additive category $\mathcal{C}$ is called *exactly definable* if it is equivalent to the category of exact contravariant additive functors from a skeletally small abelian category to the category Ab [24]. Exactly definable additive categories allow a natural theory of purity [24]. Module and comodule categories are not only finitely accessible, but also exactly definable, Grothendieck categories. Though, there are examples of exactly definable categories that are not finitely accessible (e.g., the category of divisible abelian groups). For every exactly definable additive category $\mathcal{C}$, there is a fully faithful functor $T : \mathcal{C} \to \mathcal{D}(\mathcal{C})$ (naturally isomorphic to the inclusion functor) that induces an equivalence between $\mathcal{C}$ and the full subcategory of absolutely pure objects of some locally coherent Grothendieck category $\mathcal{D}(\mathcal{C})$ [24, Theorem 2.8].

Corollary 7. *Let $\mathcal{C}$ be an exactly definable abelian category. Then the above functor $T : \mathcal{C} \to \mathcal{D}(\mathcal{C})$ preserves and reflects the weak relative Rickart property.*

Proof. The image of the fully faithful left exact functor T is equivalent to the full subcategory of absolutely pure objects of $\mathcal{D}(\mathcal{C})$, which is closed under direct summands. Then use Theorem 1. $\square$

Recall that a finitely accessible or exactly definable additive category is called *pure semisimple* if every pure exact sequence splits. An object M of an abelian category is called *self-Baer* if for every family $(f_i)_{i\in I}$ of endomorphisms of M, $\bigcap_{i\in I} \mathrm{Ker}(f_i)$ is a direct summand of M [12].

Corollary 8. *Let $\mathcal{C}$ be a pure semisimple exactly definable Grothendieck category. Then every finitely presented weak self-Rickart object of $\mathcal{C}$ is self-Baer.*

Proof. A Grothendieck category is exactly definable if and only if it is finitely accessible [30, Theorem 3.6]. Since $\mathcal{C}$ is finitely accessible, the above functor $T : \mathcal{C} \to \mathcal{D}(\mathcal{C})$ preserves finitely presented objects [6, (3.3), Lemma 3]. Since $\mathcal{C}$ is pure semisimple, its associated Grothendieck functor category $\mathcal{D}(\mathcal{C})$ is locally noetherian (e.g., see [8, Proposition 4.7]), hence it has a family of noetherian generators. Then every finitely generated object of $\mathcal{D}(\mathcal{C})$ is noetherian. Now if X is a finitely presented weak self-Rickart object of $\mathcal{C}$, then $T(X)$ is a finitely presented noetherian weak self-Rickart object of $\mathcal{D}(\mathcal{C})$ by the above considerations and Corollary 7. Then $T(X)$ is self-Baer by [8, Theorem 6.4], hence X is self-Baer by [13, Corollary 3.4], bearing in mind that T preserves products [6, (3.3), Lemma 2]. $\square$

We immediately have the following illustration in module categories. Recall that a ring R is called *right pure semisimple* if the category $\mathrm{Mod}(R)$ is pure semisimple.

Corollary 9. *Let R be a right pure semisimple ring. Then every finitely presented weak self-Rickart right R-module is self-Baer.*

Now let $\mathcal{C}$ be a full subcategory of an abelian category $\mathcal{A}$. Then $\mathcal{C}$ is called *reflective* (*coreflective*) if the inclusion functor $i : \mathcal{C} \to \mathcal{A}$ has a left (right) adjoint, in which case i is fully faithful.

Corollary 10. *Let $\mathcal{A}$ be an abelian category. If $\mathcal{C}$ is an abelian reflective (coreflective) full subcategory of $\mathcal{A}$, then the inclusion functor $i : \mathcal{C} \to \mathcal{A}$ preserves and reflects the (dual) weak relative Rickart property.*

Proof. Since $\mathrm{Im}(i)$ is closed under direct summands, one may use Theorem 1. $\square$

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. It is well known that, for every object A of $\mathcal{A}$ we have a short exact sequence $0 \to t(A) \to A \to f(A) \to 0$, where t is the idempotent radical associated to $(\mathcal{T}, \mathcal{F})$. Hence there are two functors $t : \mathcal{A} \to \mathcal{T}$ and $f : \mathcal{A} \to \mathcal{F}$, and moreover, t is a right adjoint to the inclusion i and f is a left adjoint to the inclusion j. Hence $\mathcal{T}$ is a coreflective subcategory of $\mathcal{A}$, while $\mathcal{F}$ is a reflective subcategory of $\mathcal{A}$ (e.g., see [35, Chapter X, Section 1]). Note that $\mathcal{T}$ is an abelian category, but $\mathcal{F}$ need not be an abelian category (e.g., the category of torsionfree abelian groups is not abelian). Now Corollary 10 yields the following consequence.

Corollary 11. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then the inclusion functor $i:\mathcal{T}\to\mathcal{A}$ preserves and reflects the dual weak relative Rickart property.*

Example 3. Consider the torsion pair $(\mathcal{T},\mathcal{F})$ in the category Ab, where $\mathcal{T}$ is the class of torsion abelian groups and $\mathcal{F}$ is the class of torsionfree abelian groups. Let $\mathbb{Z}(p^\infty)$ be the Prüfer abelian group for some prime p. By Corollary 11 and [37, Example 2.6], the torsion abelian group $\mathbb{Z}(p^\infty)$ is dual weak self-Rickart in $\mathcal{T}$, because it is dual weak self-Rickart in Ab. By Corollary 11 and [37, Example 2.14], the torsion abelian group $\mathbb{Z}_{p^3}$ is not dual weak self-Rickart in $\mathcal{T}$, because it is not dual weak self-Rickart in Ab.

Let $\mathcal{C}$ be a Serre subcategory of a locally small abelian category $\mathcal{A}$. Following [28, Chapter 4], one may consider the quotient abelian category $\mathcal{A}/\mathcal{C}$, and the exact *quotient* functor $T:\mathcal{A}\to\mathcal{A}/\mathcal{C}$. Then $\mathcal{C}$ is called *localizing* (*colocalizing*) if T has a right (left) adjoint $S:\mathcal{A}/\mathcal{C}\to\mathcal{A}$. The so-called *section* functor S is fully faithful. If $\mathcal{C}$ is a localizing (colocalizing) subcategory of $\mathcal{A}$, then S induces an equivalence between the quotient category $\mathcal{A}/\mathcal{C}$ and a reflective (coreflective) subcategory of $\mathcal{A}$.

Now we have the following consequence of Corollaries 1 and 10.

Corollary 12. *Let $\mathcal{A}$ be a locally small abelian category. If $\mathcal{C}$ is a localizing (colocalizing) Serre subcategory of $\mathcal{A}$, then the section functor $S:\mathcal{A}/\mathcal{C}\to\mathcal{A}$ preserves and reflects the (dual) weak relative Rickart property.*

For Grothendieck categories we obtain the following result.

Corollary 13. *Let $\mathcal{A}$ be a Grothendieck category and let R be the endomorphism ring of a generator U of $\mathcal{A}$. Then the functor $S=\mathrm{Hom}_{\mathcal{A}}(U,-):\mathcal{A}\to\mathrm{Mod}(R)$ preserves and reflects the weak relative Rickart property.*

Proof. By the Gabriel-Popescu Theorem [35, Chapter X, Theorem 4.1], the functor S is fully faithful and induces an equivalence between $\mathcal{A}$ and the full subcategory of $\mathrm{Mod}(R)$ consisting of τ-torsionfree τ-injective right R-modules for some hereditary torsion theory τ on $\mathrm{Mod}(R)$ (see [35, Definition p. 198, Chapter X, Theorem 4.1]). The latter is a reflective abelian full subcategory of $\mathrm{Mod}(R)$ [35, Chapter IX, Proposition 1.11]. Finally, use Corollaries 1 and 10. □

For a right R-module M, let $\sigma[M_R]$ be the full subcategory of $\mathrm{Mod}(R)$ of M-subgenerated modules, or equivalently, the smallest Grothendieck category containing M [38].

Corollary 14. *Let M be a right R-module. Then the inclusion functor $i:\sigma[M_R]\to\mathrm{Mod}(R)$ preserves and reflects the dual weak relative Rickart property.*

Proof. Use Corollary 10 for the coreflective abelian full subcategory $\sigma[M_R]$ of $\mathrm{Mod}(R)$ (see [38, 45.11]). □

Following [14, Section 2.2], let ${}^C\mathcal{M}$ be the (Grothendieck) category of left C-comodules, where C is a coalgebra over a field k. The category ${}^C\mathcal{M}$ may be identified with the full subcategory of the category $\mathrm{Mod}(C^*)$ of right C^*-modules consisting of rational right C^*-modules, where $C^*=\mathrm{Hom}_k(C,k)$.

Corollary 15. *Let C be a coalgebra over a field. Then the inclusion functor $i : {}^C\mathcal{M} \to \mathrm{Mod}(C^*)$ preserves and reflects the dual weak relative Rickart property.*

Proof. The category ${}^C\mathcal{M}$ is equivalent to the category $\sigma[C_{C^*}]$ (e.g., [14, Corollary 2.5.2]). Then use Corollary 14. □

It is well known that for two algebras A and B and an A-B-bimodule M one can consider the upper triangular matrix algebra $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Following [21], one may dually construct the upper triangular co-matrix coalgebra as follows. Let C and D be coalgebras over a field k, and let M be a C-D-bicomodule. Using Sweedler's convention with the summation symbol omitted, let $c \mapsto c_1 \otimes c_2$ and $d \mapsto d_1 \otimes d_2$ be the comultiplications of C and D respectively, and let $m \mapsto m_{[-1]} \otimes m_{[0]}$ and $m \mapsto m_{(0)} \otimes m_{(1)}$ be the left C-coaction and the right D-coaction on M respectively. The upper triangular co-matrix coalgebra is given by $T = \begin{pmatrix} C & M \\ 0 & D \end{pmatrix} = C \oplus M \oplus D$, where the comultiplication Δ and counit ε are defined by

$$\begin{aligned} \Delta \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} &= \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} m_{[-1]} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_{[0]} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & m_{(0)} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & m_{(1)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}, \\ \varepsilon \begin{pmatrix} c & m \\ 0 & d \end{pmatrix} &= \varepsilon_C(c) + \varepsilon_D(d). \end{aligned}$$

Note that its dual algebra is (isomorphic to) the upper triangular matrix algebra $T^* = \begin{pmatrix} C^* & M^* \\ 0 & D^* \end{pmatrix}$. Now Corollary 15 yields the following consequence.

Corollary 16. *With the above notation, the inclusion functor $i : {}^T\mathcal{M} \to \mathrm{Mod}(T^*)$ preserves and reflects the dual weak relative Rickart property.*

Next let (L, F) and (F, R) be adjoint pairs of covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $L, R : \mathcal{B} \to \mathcal{A}$. Then the triple (L, F, R) is called *adjoint*. Note that F is an exact functor, while L is fully faithful if and only if so is R [16, Lemma 1.3]. In particular, Frobenius functors [4] and recollements of abelian categories [31] induce adjoint triples.

Now we have the following consequence of Theorem 1.

Corollary 17. *Let (L, F, R) be an adjoint triple of covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $L, R : \mathcal{B} \to \mathcal{A}$ between abelian categories.*

(i) If F is fully faithful, then F preserves and reflects the (dual) weak relative Rickart property.

(ii) If L (or R) is fully faithful, then:

(1) R preserves and reflects the weak relative Rickart property.

(2) L preserves and reflects the dual weak relative Rickart property.

Proof. (i) As F is fully faithful, $\mathcal{A}$ is equivalent to $\mathrm{Im}(F)$. Let $i : \mathrm{Im}(F) \to \mathcal{B}$ be the inclusion functor. Since F has a left and right adjoint, so does i, hence $\mathrm{Im}(F)$ is a reflective and coreflective subcategory of $\mathcal{B}$. Now the conclusion follows by Corollaries 1 and 10.

(ii) This follows similarly as for (i). □

Now let $\varphi : R \to S$ be a ring homomorphism. Then we may consider the *extension of scalars* functor $\varphi^* : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ given by $\varphi^*(M) = M \otimes_R S$, the *restriction of scalars* functor $\varphi_* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ given by $\varphi_*(N) = N$, and the functor $\varphi^! : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$ given by $\varphi^!(M) = \mathrm{Hom}_R(S, M)$. Then the triple $(\varphi^*, \varphi_*, \varphi^!)$ is adjoint [35, Chapter IX, p. 105].

Corollary 18. *Let $\varphi : R \to S$ be a ring epimorphism. Then $\varphi_* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ preserves and reflects the (dual) weak relative Rickart property.*

Proof. Note that $\varphi_* : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ is fully faithful [35, Chapter XI, Proposition 1.2], and use Corollary 17. □

Let ${}_CB_A$ be a bimodule and let $R = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ be the associated formal triangular matrix ring [18, Chapter 4, Section A]. Let M be a right A-module, N a right C-module and $f : N \otimes_C B \to M$ a right A-module homomorphism. Then the elements of the abelian group $P = M \oplus N$ will be written in the form $\begin{pmatrix} m & 0 \\ & n \end{pmatrix}$, and one may define a scalar multiplication by

$$\begin{pmatrix} m & 0 \\ & n \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ma+f(n\otimes b) & 0 \\ & nc \end{pmatrix}.$$

Then one has a right R-module $P = \begin{pmatrix} M & 0 \\ & N \end{pmatrix}$ and, moreover, every right R-module has this form.

Now consider the following covariant functors:

$$\begin{aligned}
&J_1 : \mathrm{Mod}(A) \to \mathrm{Mod}(R), \quad J_1(M) = \begin{pmatrix} M & 0 \\ & 0 \end{pmatrix}, \\
&J_{23} : \mathrm{Mod}(C) \to \mathrm{Mod}(R), \quad J_{23}(N) = \begin{pmatrix} N\otimes_C B & 0 \\ & N \end{pmatrix}, \\
&J_3 : \mathrm{Mod}(C) \to \mathrm{Mod}(R), \quad J_3(N) = J_{23}(N)/\begin{pmatrix} N\otimes_C B & 0 \\ & 0 \end{pmatrix}, \\
&P_{12} : \mathrm{Mod}(R) \to \mathrm{Mod}(A), \quad P_{12}(M) = M \left(\begin{smallmatrix} A & 0 \\ B & 0 \end{smallmatrix}\right), \\
&P_1 : \mathrm{Mod}(R) \to \mathrm{Mod}(A), \quad P_1(M) = P_{12}(M)/M \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right), \\
&P_3 : \mathrm{Mod}(R) \to \mathrm{Mod}(C), \quad P_3(M) = M \left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right).
\end{aligned}$$

Corollary 19. *With the above notation:*

(i) The functor J_1 preserves and reflects the (dual) weak relative Rickart property.

(ii) (1) J_3 preserves and reflects the weak relative Rickart property.

(2) J_{23} preserves and reflects the dual weak relative Rickart property.

Proof. Note that (P_1, J_1, P_{12}) and (J_{23}, P_3, J_3) are adjoint triples [18, Chapter 4, Section A, Exercises 19, 20, 22].

(i) The functor J_1 is fully faithful [18, Chapter 4, Section A, Exercise 21], and we may use Corollary 17.

(ii) The functors J_{23} and J_3 are fully faithful [18, Chapter 4, Section A, Exercise 22], and we may use Corollary 17. □

We recall some terminology on graded modules following [27]. Let G be a group with identity element e, and let R be a G-graded ring. For a G-graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$, let $\mathrm{gr}(R)$ be the (Grothendieck) category of G-graded unital right R-modules. For $\sigma \in G$, consider the functor $(-)_\sigma : \mathrm{gr}(R) \to \mathrm{Mod}(R_e)$, which associates to every graded right R-module $M = \bigoplus_{\tau\in G} M_\tau$ the right R_e-module M_σ. The *induced functor* $\mathrm{Ind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ associates to every right R_e-module N the graded right R-module $\mathrm{Ind}(N) = M = N \otimes_{R_e} R$, where the gradation of $M = \bigoplus_{\sigma\in G} M_\sigma$ is given by $M_\sigma = N_\sigma \otimes_{R_e} R$ for every $\sigma \in G$. The *coinduced functor* $\mathrm{Coind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ associates to every right R_e-module N the graded right R-module $\mathrm{Coind}(N) = M^* = \bigoplus_{\sigma\in G} M'_\sigma$, where M'_σ consists of all $f \in \mathrm{Hom}_{R_e}(R, N)$ such that $f(R_{\sigma'}) = 0$ for every $\sigma' \neq \sigma^{-1}$.

Corollary 20. *Let R be a G-graded ring. Then:*

(1) The functor $\mathrm{Coind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ *preserves and reflects the weak relative Rickart property.*

(2) The functor $\mathrm{Ind} : \mathrm{Mod}(R_e) \to \mathrm{gr}(R)$ *preserves and reflects the dual weak relative Rickart property.*

Proof. By [27, Theorem 2.5.5], the triple $(\mathrm{Ind}, (-)_e, \mathrm{Coind})$ is adjoint, and the functors Ind and Coind are fully faithful. Then use Corollary 17. □

3 Transfer for static and adstatic objects

We have seen in Theorem 1 that fully faithful functors behave well with respect to (dual) weak relative Rickart properties. Next we would like to weaken that hypothesis by restricting it to certain objects in the case of an adjoint pair of functors with some extra conditions.

Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor between abelian categories, and let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$. Even when $\mathcal{C}$ is not abelian, we will use the following terminology, viewing the objects of $\mathcal{C}$ as objects of the abelian category $\mathcal{A}$. We say that the restriction F' of F to $\mathcal{C}$ *preserves the (dual) weak relative Rickart property* if for every objects M, N of $\mathcal{C}$, $F(N)$ is (dual) weak $F(M)$-Rickart in $\mathcal{B}$ whenever N is (dual) weak M-Rickart in $\mathcal{A}$. Also, we say that the restriction F' of F to $\mathcal{C}$ *reflects the (dual) weak relative Rickart property* if for every objects of M, N of $\mathcal{C}$, N is (dual) weak M-Rickart in $\mathcal{A}$ whenever $F(N)$ is (dual) weak $F(M)$-Rickart in $\mathcal{B}$.

Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be covariant functors between abelian categories such that (L, R) is an adjoint pair with counit $\varepsilon : LR \to 1_{\mathcal{B}}$ and unit $\eta : 1_{\mathcal{A}} \to RL$. Denote

$$\mathrm{Stat}(R) = \{B \in \mathcal{B} \mid \varepsilon_B \text{ is an isomorphism}\},$$

$$\mathrm{Adst}(R) = \{A \in \mathcal{A} \mid \eta_A \text{ is an isomorphism}\}.$$

The objects of these two classes are called *R-static* and *R-adstatic* respectively [5].

Theorem 2. *Let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between abelian categories.*

(1) The restriction R' of R to $\mathrm{Stat}(R)$ *preserves the weak relative Rickart property, and if L is exact, then R' reflects the weak relative Rickart property.*

(2) The restriction L' of L to $\mathrm{Adst}(R)$ *preserves the dual weak relative Rickart property, and if R is exact, then L' reflects the dual weak relative Rickart property.*

Proof. (1) Denote by $\varepsilon : LR \to 1_{\mathcal{B}}$ the counit and by $\eta : 1_{\mathcal{A}} \to RL$ the unit of the adjunction. Then $\varepsilon_{L(A)}L(\eta_A) = 1_{L(A)}$ and $R(\varepsilon_B)\eta_{R(B)} = 1_{R(B)}$ for every objects A of $\mathcal{A}$ and B of $\mathcal{B}$.

Let $M, N \in \mathrm{Stat}(R)$ be such that N is weak M-Rickart in $\mathcal{B}$. Let $f : R(M) \to R(N)$ be a nonzero morphism in $\mathcal{A}$ with kernel $k : K \to R(M)$. By naturality the following diagram is commutative in $\mathcal{A}$:

$$\begin{array}{ccc} R(M) & \xrightarrow{\quad f \quad} & R(N) \\ {\scriptstyle \eta_{R(M)}}\downarrow & & \downarrow{\scriptstyle \eta_{R(N)}} \\ RLR(M) & \xrightarrow[RL(f)]{} & RLR(N) \end{array}$$

Since $M, N \in \mathrm{Stat}(R)$, $R(\varepsilon_M)$ and $R(\varepsilon_N)$ are isomorphisms, hence so are $\eta_{R(M)}$ and $\eta_{R(N)}$. Consider the morphism $g = \varepsilon_N L(f)\varepsilon_M^{-1} : M \to N$ in $\mathcal{B}$. If $g = 0$, then $L(f) = 0$, and so $\eta_{R(N)}f = RL(f)\eta_{R(M)} = 0$, whence $f = 0$, a contradiction. Hence $g : M \to N$ is a nonzero morphism. But N is weak M-Rickart, hence we may write $\ker(g) = si$ for some monomorphism $i : \mathrm{Ker}(g) \to Q$ and section $s : Q \to M$ which is not an isomorphism. We have:

$$f = \eta_{R(N)}^{-1}RL(f)\eta_{R(M)} = R(\varepsilon_N)RL(f)R(\varepsilon_M^{-1}) = R(\varepsilon_N L(f)\varepsilon_M^{-1}) = R(g).$$

Hence $\ker(f) = R(\ker(g)) = R(s)R(i)$, where $R(s)$ is a section and $R(i)$ is a monomorphism. If $R(s)$ is an isomorphism, then so is $s\varepsilon_Q = \varepsilon_M LR(s)$, whence it follows that s is an isomorphism, a contradiction. Hence $\ker(f)$ factors through the section $R(s) : R(Q) \to R(M)$ which is not an isomorphism. This shows that $R(N)$ is weak $R(M)$-Rickart.

Now let M and N be objects of $\mathcal{B}$ such that $R(N)$ is weak $R(M)$-Rickart in $\mathcal{A}$. Let $f : M \to N$ be a nonzero morphism in $\mathcal{B}$ with kernel $k : K \to M$. Consider the morphism $R(f) : R(M) \to R(N)$ in $\mathcal{A}$. By naturality the following diagram is commutative in $\mathcal{B}$:

$$\begin{array}{ccc} LR(M) & \xrightarrow{LR(f)} & R(N) \\ {\scriptstyle \varepsilon_M}\downarrow & & \downarrow{\scriptstyle \varepsilon_N} \\ M & \xrightarrow[f]{\quad\quad} & N \end{array}$$

in which ε_M and ε_N are isomorphisms. If $R(f) = 0$, then $f\varepsilon_M = \varepsilon_N LR(f) = 0$, whence $f = 0$, a contradiction. Hence $R(f) : R(M) \to R(N)$ is a nonzero morphism. But $R(N)$

is weak $R(M)$-Rickart, hence we may write $\ker(R(f)) = si$ for some monomorphism $i : \mathrm{Ker}(R(f)) \to Q$ and section $s : Q \to R(M)$ which is not an isomorphism. Since L is exact, it follows that

$$\ker(f) = \ker(LR(f)) = L(\ker R(f)) = L(s)L(i),$$

where $L(i)$ is a monomorphism and $L(s)$ is a section. If $L(s)$ is an isomorphism, then so is $RL(s)$. Consider a morphism $r : R(M) \to Q$ such that $rs = 1_Q$. Since $RL(r)$ is an isomorphism, so is $\eta_Q r = RL(r)\eta_{R(M)}$. It follows that r is an isomorphism, hence so is s, a contradiction. Hence $\ker(f)$ factors through the section $\varepsilon_M L(s) : L(Q) \to M$ which is not an isomorphism. Therefore, N is weak M-Rickart. □

Corollary 21. *Let (L, R) be an adjoint pair of covariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between abelian categories.*

(1) If R is fully faithful, then R preserves the weak relative Rickart property, and if L is exact, then R reflects the weak relative Rickart property.

(2) If L is fully faithful, then L preserves the dual weak relative Rickart property, and if R is exact, then L reflects the dual weak relative Rickart property.

Example 4. Consider the abelian group $\mathbb{Q}/\mathbb{Z}$ and denote $S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$. Consider the functors $T = - \otimes_S \mathbb{Q}/\mathbb{Z} : \mathrm{Mod}(S) \to \mathrm{Mod}(\mathbb{Z})$ and $H = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, -) : \mathrm{Mod}(\mathbb{Z}) \to \mathrm{Mod}(S)$. Then (T, H) is an adjoint pair, $\mathrm{Stat}(H)$ consists of the divisible torsion abelian groups (see [1, Corollary 4.3] and [39, 3.6]), and $\mathrm{Adst}(H)$ consists of images of divisible torsion abelian groups via H. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, H is exact. By Theorem 2, if $M, N \in \mathrm{Adst}(H)$, then N is dual weak M-Rickart if and only if $T(N)$ is dual weak $T(M)$-Rickart.

We also give the contravariant version of Theorem 2, because it will be useful later on. Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be contravariant functors between abelian categories such that (L, R) is a *right adjoint pair* [38, 45.2] with counit $\varepsilon : 1_{\mathcal{B}} \to LR$ and unit $\eta : 1_{\mathcal{A}} \to RL$. Note that in this case both L and R are left exact. Denote

$$\mathrm{Refl}(R) = \{B \in \mathcal{B} \mid \varepsilon_B \text{ is an isomorphism}\},$$

$$\mathrm{Refl}(L) = \{A \in \mathcal{A} \mid \eta_A \text{ is an isomorphism}\}.$$

The objects of these two classes are called *R-reflexive* and *L-reflexive* respectively [3]. Similarly, if (L, R) a *left adjoint pair* of contravariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between abelian categories [38, 45.2], then both L and R are right exact, and we have some similar classes of reflexive objects.

Theorem 3. *Let (L, R) be a pair of contravariant functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ between abelian categories.*

(1) If (L, R) is left adjoint, then the restriction R' of R to $\mathrm{Refl}(R)$ preserves the weak relative Rickart property, and if L is exact, then R' reflects the weak relative Rickart property.

(2) If (L,R) is right adjoint, then the restriction L' of L to $\mathrm{Refl}(L)$ *preserves the dual weak relative Rickart property, and if R is exact, then L' reflects the dual weak relative Rickart property.*

Next we consider endomorphism rings of (graded) modules and comodules, and analyze how the (dual) weak relative Rickart properties transfer to them. First, we consider the module categories $\mathrm{Mod}(R)$ and $\sigma[M_R]$ for some right R-module M. For a right R-module P, we denote by $\mathrm{PAdd}(P)$ the class of right R-modules which are pure epimorphic images of direct sums of copies of P [7].

Corollary 22. *Let P be a finitely presented right R-module with endomorphism ring S.*

(1) Let $M,N \in \mathrm{PAdd}(P)$. If N is a weak M-Rickart right R-module, then $\mathrm{Hom}_R(P,N)$ *is a weak* $\mathrm{Hom}_R(P,M)$*-Rickart right S-module. The converse holds if P is a flat left S-module.*

(2) Let M,N be flat right S-modules. If N is a dual weak M-Rickart right S-module, then $N \otimes_S P$ is a dual weak $N \otimes_S P$-Rickart right R-module. The converse holds if P is a projective right R-module.

Proof. Consider the functors $T = -\otimes_S P : \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ and $H = \mathrm{Hom}_R(P,-) : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$. Then (T,H) is an adjoint pair of covariant functors, $\mathrm{PAdd}(P)$ is included in $\mathrm{Stat}(H)$ and the class of flat right S-modules is included in $\mathrm{Adst}(H)$ [19, Lemma 2.4]. If P is a flat left S-module, then T is exact. Also, if P is a projective right R-module, then H is exact. Now use Theorem 2. □

Corollary 23. *Let M be a right R-module with endomorphism ring S.*

(1) If M is a weak self-Rickart right R-module, then S is a weak self-Rickart right S-module. The converse holds if M is a flat left S-module.

(2) If M is a dual weak self-Rickart right R-module, then S is a dual weak self-Rickart left S-module. The converse holds if M is a projective right R-module.

(3) If M is a dual weak self-Rickart right R-module, then S is a weak self-Rickart left S-module. The converse holds if M is an injective left S-module.

Proof. (1) Consider the adjoint pair (T,H) of covariant functors

$$T = -\otimes_S M : \mathrm{Mod}(S) \to \mathrm{Mod}(R), \quad H = \mathrm{Hom}_R(M,-) : \mathrm{Mod}(R) \to \mathrm{Mod}(S).$$

Since $TH(M) \cong M$, we have $M \in \mathrm{Stat}(H)$. If M is a flat left S-module, then T is exact. Then use Theorem 2 (1).

(2) Consider again the above adjoint pair (T,H). Since $HT(S) \cong S$, we have $S \in \mathrm{Adst}(H)$. If M is a projective right R-module, then H is exact. Then use Theorem 2 (2).

(3) Consider the right adjoint pair (H_1,H_2) of contravariant functors

$$H_1 = \mathrm{Hom}_R(-,M) : \mathrm{Mod}(R) \to \mathrm{Mod}(S^{\mathrm{op}}), \quad H_2 = \mathrm{Hom}_S(-,M) : \mathrm{Mod}(S^{\mathrm{op}}) \to \mathrm{Mod}(R).$$

Since $H_2H_1(M) \cong M$, we have $M \in \mathrm{Refl}(H_1)$. If M is an injective left S-module, then H_2 is exact. Then use Theorem 3 (2). □

The notion of k-local-retractability is useful when deducing the weak self-Rickart property of a module provided its endomorphism ring has the same property. A right R-module M is called *k-local-retractable* if for every endomorphism $f : M \to M$ with kernel $k : K \to M$ and for every $x \in K$, we have $x \in \mathrm{Im}(hk)$ for some homomorphism $h : M \to K$ [25, Definition 3.6]. Dually, a right R-module M is called *c-local-coretractable* if for every endomorphism $f : M \to M$ with cokernel $c : M \to C$ and for every $z \in C$, we have $z \in \mathrm{Im}(ch)$ for some homomorphism $h : C \to M$ [10, Definition 4.5].

Now we also have the following result in case of module categories. The first part is known [22, Proposition 3.10], but we give a more categorical proof for it, which is easily dualizable.

Corollary 24. *Let M be a right R-module with endomorphism ring S.*

(1) If M is k-local-retractable and S is a weak self-Rickart right S-module, then M is a weak self-Rickart right R-module.

(2) If M is c-local-coretractable and S is a dual weak self-Rickart left S-module, then M is a dual weak self-Rickart right R-module.

Proof. (1) Consider the adjoint pair (T, H) from the proof of Corollary 23. Assume that M is k-local-retractable and S is a weak self-Rickart right S-module. Let $f : M \to M$ be a nonzero R-homomorphism with kernel $k : K \to M$. Then $H(f) : H(M) \to H(M)$ is a nonzero S-homomorphism. Since $H(M) = S$ is a weak self-Rickart right S-module, $H(k) = \ker(H(f)) = s'i'$ for some monomorphism $i' : H(K) \to Q'$ and section $s' : Q' \to H(M)$ which is not an S-isomorphism. Then there exists a homomorphism $r' : H(M) \to Q'$ such that $r's' = 1_{Q'}$, and $s = \varepsilon_M T(s') : T(Q') \to M$ is a section. Since $M \in \mathrm{Stat}(H)$ and $H(\varepsilon_M)\eta_{H(M)} = 1_{H(M)}$, $\eta_{H(M)}$ is an S-isomorphism. By naturality we may construct the following commutative diagram:

$$\begin{array}{ccccc} H(M) & \xrightarrow{r'} & Q' & \xrightarrow{s'} & H(M) \\ \downarrow{\scriptstyle \eta_{H(M)}} & & \downarrow{\scriptstyle \eta_{Q'}} & & \downarrow{\scriptstyle \eta_{H(M)}} \\ HTH(M) & \xrightarrow[HT(r')]{} & HT(Q') & \xrightarrow[HT(s')]{} & HTH(M) \end{array}$$

Then $\eta_{Q'}r' = HT(r')\eta_{H(M)}$ is a retraction and $HT(s')\eta_{Q'} = \eta_{H(M)}s'$ is a monomorphism, which imply that $\eta_{Q'} : Q' \to HT(Q')$ is an S-isomorphism. If $s : T(Q') \to M$ is an R-isomorphism, then $H(s)\eta_{Q'} = H(\varepsilon_M)HT(s')\eta_{Q'} = H(\varepsilon_M)\eta_{H(M)}s' = s'$, is an S-isomorphism, a contradiction. Hence the section $s : T(Q') \to M$ is not an R-isomorphism.

Let $i = T(r')\varepsilon_M^{-1}k : K \to T(Q')$. We claim that $k = si$. To this end, let $x \in K$. Since M is k-local-retractable, we have $x \in \mathrm{Im}(hk)$ for some R-homomorphism $h : M \to K$. Hence $x = hk(y)$ for some $y \in K$. By naturality the following diagram is commutative:

$$\begin{array}{ccc} TH(M) & \xrightarrow{TH(kh)} & TH(M) \\ \downarrow{\scriptstyle \varepsilon_M} & & \downarrow{\scriptstyle \varepsilon_M} \\ M & \xrightarrow[kh]{} & M \end{array}$$

It follows that:

$$\begin{aligned} si(x) &= \varepsilon_M T(s')T(r')\varepsilon_M^{-1}k(x) = \varepsilon_M T(s')T(r')\varepsilon_M^{-1}khk(y) \\ &= \varepsilon_M T(s')T(r')\varepsilon_M^{-1}kh\varepsilon_M\varepsilon_M^{-1}k(y) = \varepsilon_M T(s')T(r')\varepsilon_M^{-1}\varepsilon_M TH(kh)\varepsilon_M^{-1}k(y) \\ &= \varepsilon_M T(s')T(r')TH(kh)\varepsilon_M^{-1}k(y) = \varepsilon_M T(s')T(r')TH(k)TH(h)\varepsilon_M^{-1}k(y) \\ &= \varepsilon_M T(s')T(r')T(s')T(i')TH(h)\varepsilon_M^{-1}k(y) = \varepsilon_M T(s')T(i')TH(h)\varepsilon_M^{-1}k(y) \\ &= \varepsilon_M TH(k)TH(h)\varepsilon_M^{-1}k(y) = \varepsilon_M TH(kh)\varepsilon_M^{-1}k(y) \\ &= kh\varepsilon_M\varepsilon_M^{-1}k(y) = khk(y) = k(x). \end{aligned}$$

Hence $k = si$, which shows that M is a weak self-Rickart right R-module. □

Corollary 25. *Let M be a right R-module with endomorphism ring S.*

(1) If M is a weak self-Rickart module in $\sigma[M_R]$, then S is a weak self-Rickart right S-module. The converse holds if M is a generator in $\sigma[M_R]$.

(2) If M is a dual weak self-Rickart module in $\sigma[M_R]$, then S is a dual weak self-Rickart left S-module. The converse holds if M is a quasi-projective right R-module.

Proof. Consider the adjoint pair (T, H) [38, 45.8] of covariant functors

$$T = -\otimes_S M : \mathrm{Mod}(S) \to \sigma[M_R], \quad H = \mathrm{Hom}_R(M, -) : \sigma[M_R] \to \mathrm{Mod}(S).$$

(1) Since $TH(M) \cong M$, we have $M \in \mathrm{Stat}(H)$. If M is a generator in $\sigma[M_R]$, then M is a flat left S-module [38, 15.9], and so T is exact. Then use Theorem 2 (1).

(2) Since $HT(S) \cong S$, we have $S \in \mathrm{Adst}(H)$. If M is a quasi-projective right R-module, then M is projective in $\sigma[M_R]$, and so H is exact. Then use Theorem 2 (2). □

Following [27], for $M, N \in \mathrm{gr}(R)$ there is the G-graded abelian group $\mathrm{HOM}_R(M, N)$, having the following σ-th homogeneous component:

$$\mathrm{HOM}_R(M, N)_\sigma = \{f \in \mathrm{Hom}_R(M, N) \mid f(M_\lambda) \subseteq N_{\lambda\sigma} \text{ for all } \lambda \in G\}.$$

For $M = N$, one obtains the G-graded ring$S = \mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$ (where the multiplication is the map composition) and M is a graded (S, R)-bimodule, that is, $S_\tau \cdot M_\sigma \cdot R_\lambda \subseteq M_{\tau\sigma\lambda}$ for every $\tau, \sigma, \lambda \in G$. For a graded right S-module N, there is the right R-module $N \otimes_S M$, graded by

$$(N \otimes_S M)_\tau = \left\{ \sum_{\sigma\lambda=\tau} n_\sigma \otimes m_\lambda \mid n_\sigma \in N_\sigma, m_\lambda \in M_\lambda \right\}.$$

Corollary 26. *Let M be a graded right R-module with $S = \mathrm{END}_R(M)$.*

(1) If M is a weak self-Rickart graded right R-module, then S is a weak self-Rickart graded right S-module. The converse holds if M is a flat graded left S-module.

(2) If M is a dual weak self-Rickart graded right R-module, then S is a dual weak self-Rickart graded left S-module. The converse holds if M is a projective graded right R-module.

(3) If M is a dual weak self-Rickart graded right R-module, then S is a weak self-Rickart graded left S-module. The converse holds if M is an injective graded left S-module.

Proof. (1) Consider the adjoint pair (T, H) covariant functors

$$T = - \otimes_S M : \mathrm{gr}(S) \to \mathrm{gr}(R), \quad H = \mathrm{HOM}_R(M, -) : \mathrm{gr}(R) \to \mathrm{gr}(S).$$

Since $TH(M) \cong M$, we have $M \in \mathrm{Stat}(H)$. If M is a flat graded left S-module, then T is exact. Then use Theorem 2 (1).

(2) Consider again the above adjoint pair (T, H). Since $HT(S) \cong S$, we have $S \in \mathrm{Adst}(H)$. If M is a projective graded right R-module, then H is exact. Then use Theorem 2 (2).

(3) Consider the right adjoint pair (H_1, H_2) of contravariant functors

$$H_1 = \mathrm{HOM}_R(-, M) : \mathrm{gr}(R) \to \mathrm{gr}(S^{\mathrm{op}}), \quad H_2 = \mathrm{HOM}_S(-, M) : \mathrm{gr}(S^{\mathrm{op}}) \to \mathrm{gr}(R).$$

Since $H_2H_1(M) \cong M$, we have $M \in \mathrm{Refl}(H_1)$. If M is an injective graded left S-module, then H_2 is exact. Then use Theorem 3 (2). □

A coalgebra C over a field is called *left (right) quasi-co-Frobenius* if one has an embedding of left (right) C^*-modules from C to a free left (right) C^*-module [14, Definition 3.3.1].

Corollary 27. *Let C be a coalgebra over a field with $S = \mathrm{End}_{C^*}(C)$.*

(1) If C is a weak self-Rickart left C-comodule, then S is a weak self-Rickart right S-module. The converse holds if C is a left quasi-co-Frobenius coalgebra.

(2) If C is a dual weak self-Rickart left C-comodule, then S is a dual weak self-Rickart left S-module. The converse holds if C is a left and right quasi-co-Frobenius coalgebra.

Proof. Recall that the category ${}^C\mathcal{M}$ is equivalent to the category $\sigma[C_{C^*}]$ (e.g., [14, Corollary 2.5.2]). If C is a left quasi-co-Frobenius coalgebra, then C is a generator in ${}^C\mathcal{M}$ [14, Corollary 3.3.10]. If C is a left and right quasi-co-Frobenius coalgebra, then C is a projective generator in ${}^C\mathcal{M}$ [14, Corollary 3.3.11]. Then use Corollary 25. □

Let C be a coalgebra over a field. A left C-comodule Q is said to be *quasi-finite* if for every finite dimensional left C-comodule M, $\mathrm{Hom}_C(M, Q)$ is also finite dimensional [36, Definition 1.1]. For a C-D-bicomodule Q such that Q is quasi-finite in $\mathcal{M}^D$, the *cotensor functor* $-\Box_C Q : \mathcal{M}^C \to \mathcal{M}^D$ has a left adjoint $h_D(Q, -) : \mathcal{M}^D \to \mathcal{M}^C$, called the *cohom functor* (see [14, p. 87] and [36, Proposition 1.10]).

Corollary 28. *Let D be a coalgebra over a field, let Q be a quasi-finite injective right D-comodule, and let $C = \mathrm{h}_D(Q, Q)$. If Q is a dual weak self-Rickart right D-comodule, then C is a dual weak self-Rickart right C-comodule. The converse holds if Q is an injective right D-comodule.*

Proof. Consider the adjoint pair (L, R) of covariant functors $L = h_D(Q, -) : \mathcal{M}^D \to \mathcal{M}^C$ and $R = -\Box_C Q : \mathcal{M}^C \to \mathcal{M}^D$. Since $RL(Q) \cong Q$, we have $Q \in \mathrm{Adst}(R)$. If Q is an injective right D-comodule, then R is exact [14, Theorem 2.4.17]. Finally, use Theorem 2 (2). □

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