Bull. Math. Soc. Sci. Math. Roumanie
Tome 66 (114), No. 1, 2023, 15–29

Graded local cohomology modules with respect to the linked ideals

by

Maryam Jahangiri(1), Azadeh Nadali(2), Khadijeh Sayyari(3)

Abstract

Let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a standard graded ring, M be a finitely generated graded R-module and $R_+ := \oplus_{n\in\mathbb{N}} R_n$ denotes the irrelevant ideal of R. In this paper, considering the new concept of linkage of ideals over a module, we study the graded components $H^i_{\mathfrak{a}}(M)_n$ when $\mathfrak{a}$ is an h-linked ideal over M. More precisely, we show that $H^i_{\mathfrak{a}}(M)$ is tame in each of the following cases:

(i) $i = f^{R_+}_{\mathfrak{a}}(M)$, the first integer i for which $R_+ \nsubseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$;

(ii) $i = \operatorname{cd}(R_+, M)$, the last integer i for which $H^i_{R_+}(M) \neq 0$, and $\mathfrak{a} = \mathfrak{b} + R_+$ where $\mathfrak{b}$ is an h-linked ideal with R_+ over M.

Also, among other things, we describe the components $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is radically h-M-licci with respect to R_+ of length 2.

Key Words: Graded local cohomology modules, linkage of ideals, finiteness dimension, cohomological dimension, relative Cohen-Macaulay.
2010 Mathematics Subject Classification: Primary 13D45, 13A02, 13C40.

1 Introduction

Throughout the paper, $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded Noetherian ring, i.e. R_0 is a commutative Noetherian ring and R is generated, as an R_0-algebra, by finitely many elements of degree one, $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ is the irrelevant ideal of R and $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals of R. Also, M denotes a finitely generated graded R-module.

For $i \in \mathbb{N}_0$, the set of non-negative integers, and $n \in \mathbb{Z}$, the set of integers, let $H^i_{\mathfrak{a}}(M)_n$ denotes the n-th component of graded local cohomology module $H^i_{\mathfrak{a}}(M)$ of M with respect to $\mathfrak{a}$ (our terminology on local cohomology comes from [3]). It is well-known that $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ ([3, 16.1.5]). The asymptotic behavior of the components $H^i_{R_+}(M)_n$ when $n \to -\infty$ has been studied by many authors, too. See for example [1], [2], [5] and [12]. But, we know not much about the graded components $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is an arbitrary homogeneous ideal of R. Although, there are some studies in this topic, see for example [4], [10] and [16].

In a recent paper ([9]), the authors introduce the concept of linkage of ideals over a module, which is a generalization of its classical concept introduced by Peskine and Szpiro ([15]).

Let $\mathfrak{c}$ and $\mathfrak{d}$ be ideals of the commutative Noetherian ring A with $1 \neq 0$ and N be a finitely generated A-module. Assume that $\mathfrak{c}N \neq N \neq \mathfrak{d}N$ and $I \subseteq \mathfrak{c} \cap \mathfrak{d}$ is an ideal generated by an N-regular sequence. Then the ideals $\mathfrak{c}$ and $\mathfrak{d}$ are said to be linked by I over N, denoted by $\mathfrak{c} \sim_{(I;N)} \mathfrak{d}$, if $\mathfrak{c}N = IN :_N \mathfrak{d}$ and $\mathfrak{d}N = IN :_N \mathfrak{c}$. In [7] and [8], the authors studied some cohomological properties of linked ideals.

In this paper, we consider the above concept in the graded case. More precisely, the homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be homogeneously linked (or h-linked) by I over M, denoted by $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, if I is generated by a homogeneous M-regular sequence and $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M. We consider homogeneously linked ideals and study some of their cohomological properties.

This paper is divided into three sections. In Section 2, we study some basic properties of homogeneously linked ideals. We Show by examples that if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, this doesn't imply that $\mathfrak{a} \cap R_0 \sim_{(I \cap R_0; M_n)} \mathfrak{b} \cap R_0$ for all $n \in \mathbb{Z}$ and vice versa. Although, in some cases it does. Due to the importance of irrelevant ideal in a standard graded ring, it is natural to ask whether a homogeneous ideal could be h-linked with R_+. In Section 2, we answer this question in some cases, too.

Section 3, which is the main part of the paper, is devoted to study the graded components of local cohomology modules $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is an h-linked ideal over M. Let $N = \bigoplus_{n\in\mathbb{Z}} N_n$ be a graded R-module, $end(N)$ is defined to be the last integer n for which $N_n \neq 0$. In [10], it is shown that $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \in \mathbb{N}_0$ and all homogeneous ideal $\mathfrak{a} \supseteq R_+$. In Theorem 3.9, we show that if $\mathfrak{a}$ is an h-linked ideal with R_+ over M, then $end(H^i_{\mathfrak{a}}(M)) < \infty$ for any $i \neq \operatorname{grade}(\mathfrak{a}, M)$ and that $end(H^{\operatorname{grade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)) < \infty$ or $H^{\operatorname{grade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$ where, $\operatorname{grade}(\mathfrak{a}, M)$ denotes the length of a maximal M-regular sequence in $\mathfrak{a}$.

$f^{R_+}_{\mathfrak{a}}(M)$ is defined to be the first integer i for which $R_+ \nsubseteq \sqrt{0 : H^i_{\mathfrak{a}}(M)}$. This invariant was studied in [12] as well in [3, §9]. Also, following [1], a graded R-module $N = \bigoplus_{n\in\mathbb{Z}} N_n$ is said to be tame if $\{n \in \mathbb{Z} | N_n = 0 \text{ and } N_{n+1} \neq 0\}$ is a finite set. Tameness of local cohomology modules is one of the most fundamental concepts concerning these modules and attracts lots of interests, see for example [1], [2], [4], [5] and [18]. In [5, 2.2], the authors show that $H^{f^{R_+}_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M)$ is tame whenever $\mathfrak{a} \supseteq R_+$. In Theorem 3.13, we prove it without any restriction on $\mathfrak{a}$. Also, in Theorem 3.14, it is shown that if $(R_0, \mathfrak{m}_0)$ is local, then $H^{\operatorname{cd}(R_+,M)}_{\mathfrak{a}+R_+}(M)$ is tame in the case where $\mathfrak{a}$ is an h-linked ideal with respect to R_+ over M. Here, $\operatorname{cd}(R_+, M)$ is the cohomological dimension of M with respect to R_+, that is the last integer i for which $H^i_{R_+}(M) \neq 0$.

Throughout the paper we keep the notations introduced in the introduction.

2 Homogenously linked ideals over a module

We start by the basic concept of the paper.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by a homogeneous M-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneously* *linked (or h-linked) by* I *over* M*, denoted* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$*, if* $\mathfrak{b}M = IM :_M \mathfrak{a}$ *and* $\mathfrak{a}M = IM :_M \mathfrak{b}$*. The ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are said to be geometrically h-linked by* I *over* M *if, in addition,* $\mathfrak{a}M \cap \mathfrak{b}M = IM$*. Also, we say that the ideal* $\mathfrak{a}$ *is h-linked over* M *if there exist homogeneous ideals* $\mathfrak{b}$ *and* I *of* R *such that* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$*.* $\mathfrak{a}$ *is h-*M*-selflinked by* I *if* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{a}$*.*

Remark 2.2. *Note that, this definition is a special case of linkage of ideals over a module, studied in [9]. Moreover, if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are h-linked by* I *over* M *and* $\operatorname{grade}(\mathfrak{a}, M) = t$ *then the following statements hold.*

(i) If $\mathfrak{a}M \cap \mathfrak{b}M \neq IM$*, then* $IM :_M (\mathfrak{a} + \mathfrak{b}) \neq IM$*. So,* $(\mathfrak{a} + \mathfrak{b}) \subseteq Z(M/IM)$*, the set of zero divisors of* M/IM*, that results* $\operatorname{grade}(\mathfrak{a} + \mathfrak{b}, M) = t$*.*

(ii) If $\mathfrak{a}M \cap \mathfrak{b}M = IM$ *(i.e.* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are geometrically h-linked), then, by [7, 2.9],* $\operatorname{grade}(\mathfrak{a} + \mathfrak{b}, M) = t + 1$*.*

In the next example, we show that there is no bilateral relation between h-linkedness of ideals $\mathfrak{a}$ and $\mathfrak{b}$ by I over M and linkage of $\mathfrak{a} \cap R_0$ and $\mathfrak{b} \cap R_0$ by $I \cap R_0$ over homogeneous components of M.

Example 2.3. *Let* $(R_0, \mathfrak{m}_0)$ *be local with* $\operatorname{depth} R_0 > 0$*,* $\mathfrak{m} = \mathfrak{m}_0 + R_+$ *be the homogeneous maximal ideal of* R *and* $x_1, x_2, \ldots, x_s (s \geq 2)$ *be a homogeneous* R*-regular sequence in* $\mathfrak{m}$*. Assume that* $1 \leq l < s$ *such that deg* $x_i = 0$ *for all* $1 \leq i \leq l$ *and deg* $x_i \geq 1$ *for all* $l < i \leq s$*.*

- *Set* $\mathfrak{a} := (x_1, x_2, \ldots, x_s)$*,* $I := (x_1, \ldots, x_l, x_{l+1}^2, \ldots, x_s)$*,* $\mathfrak{a}_0 := \mathfrak{a} \cap R_0 = (x_1, \ldots, x_l)R_0$ *and* $I_0 := I \cap R_0 = (x_1, \ldots, x_l)R_0$*. So, by [9, 2.2],* $\mathfrak{a}$ *is h-*R*-selflinked by* I*. But, since* $\mathfrak{a}_0 = I_0$*,* $\mathfrak{a}_0$ *is not* R_n*-selflinked by* I_0*, for all* n*.*
- *Again, set* $\mathfrak{a} := (x_1, x_2, \ldots, x_s)$*,* $I := (x_1^2, x_2, \ldots, x_{s-1})$*,* $\mathfrak{a}_0 := \mathfrak{a} \cap R_0 = (x_1, \ldots, x_l)R_0$ *and* $I_0 := I \cap R_0 = (x_1^2, x_2, \ldots, x_l)R_0$*. Then,* $\operatorname{grade}(\mathfrak{a}, R) = s \neq \operatorname{grade}(I, R)$*, so, by [9, 2.6(i)],* $\mathfrak{a}$ *is not h-*R*-selflinked by* I*. But,* $\mathfrak{a}_0$ *is* R_n*-selflinked by* I_0 *for all* n*, using [9, 2.2] and the fact that* $x_1^2, x_2, \ldots, x_l$ *is an* R_n*-regular sequence for all* n*.*

Remark 2.4. *Assume that* $(R_0, \mathfrak{m}_0)$ *is local and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are generated by elements of degree zero. Then, despite the above example,* $\mathfrak{a} \overset{h}{\sim}_{(0;M)} \mathfrak{b}$ *if and only if* $(\mathfrak{a} \cap R_0) \sim_{(0;M_n)} (\mathfrak{b} \cap R_0)$ *for all* $n \in \mathbb{Z}$*.*

In view of the importance of the irrelevant ideal in a standard graded ring, it is natural to study homogeneous ideals which are h-linked with R_+.

If $R = R_0[x_1, \ldots, x_n]$ is a polynomial ring graded in the usual way, then $R_+ = (x_1, \ldots, x_n)$ is h-R-selflinked by $(x_1^2, x_2, \ldots, x_n)$, using [9, 2.2]. In the next example, we find some homogeneous ideals that are h-linked with R_+. It will be used in the next section, too.

Example 2.5.

1. *Let* $R = R_0[x]$ *and* $x \notin Z(M)$, *then* $R_+ \overset{h}{\sim}_{((r_0x^t);M)} (r_0x^{t-1})$ *for all non-unit elements* $r_0 \in R_0 \backslash Z(M)$ *and all* $t \geq 1$.

2. *Let* $R = R_0[x, y]$ *and* r_1x, r_2y *be an* M*-regular sequence where* $r_1, r_2 \in R_0$. *Then, it is straight forward to see that* $R_+ \overset{h}{\sim}_{((r_1x^t, r_2y^{t'});M)} (r_1x^t, r_2y^{t'}, r_1r_2x^{t-1}y^{t'-1})$ *for all* $t, t' \geq 1$.

It's natural to ask whether a homogeneous ideal which is linked could be an h-linked ideal? In the following, we answer it in a special case.

Remark 2.6. *Let* R_0 *be reduced and* R_+ *be a linked ideal by* I *over* R. *As* R_0 *is reduced,* R_+ *is radical. Also, in view of [3, 16.1.2], there exists an ideal, say* I', *generated by a homogeneous* R*-regular sequence of length* t *in* R_+, *where* $t := \operatorname{grade} R_+$ *and* $I' \neq R_+$. *So, using [9, 2.8], [6, Theorem 1] and [19, 1.4],*

$$\begin{aligned}\operatorname{Ass} R/R_+ \subseteq \operatorname{Ass} R/I \cap V(R_+) &= \operatorname{Ass} Hom_R(R/R_+, R/I)\\ &= \operatorname{Ass} Ext^t_R(R/R_+, R) = \operatorname{Ass} R/I' \cap V(R_+),\end{aligned}$$

where $V(R_+)$ *denotes the set of prime ideals of* R *containing* R_+. *This implies that* R_+ *is an h-linked ideal by* I', *by [8, 2.8].*

Definition 2.7. *Following [14, 2.1], a sequence* $x_1, x_2, \ldots, x_t$ *of homogeneous elements of* $\mathfrak{b}$ *is said to be a homogeneous* $\mathfrak{b}$*-filter regular sequence on* M *if* $x_i \notin \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Ass}(\frac{M}{(x_1,\ldots,x_{i-1})M}) \backslash V(\mathfrak{b})$ *and all* $i = 1, \ldots, t$.

Assume that $\mathfrak{a}$ is generated by elements of positive degrees and $\mathfrak{a} \subseteq \mathfrak{b}$. By [10, 1.5], if $\operatorname{Supp}(\frac{M}{\mathfrak{a}M}) \nsubseteq V(\mathfrak{b})$, then all maximal homogeneous $\mathfrak{b}$-filter regular sequences in $\mathfrak{a}$ on M have the same finite length, that is denoted by f-grade$(\mathfrak{b}, \mathfrak{a}, M)$. Also f-grade$(\mathfrak{b}, \mathfrak{a}, M) := \infty$ whenever $\operatorname{Supp}(\frac{M}{\mathfrak{a}M}) \subseteq V(\mathfrak{b})$. Note that $\operatorname{grade}(\mathfrak{a}, M) \leq$ f-grade$(\mathfrak{b}, \mathfrak{a}, M)$.
Moreover, Chu and Gu in [4, 2.4] in the case where $\mathfrak{b} = R_+$, show that if $\operatorname{Supp}(\frac{M}{\mathfrak{a}M}) \nsubseteq V(R_+)$ then

$$\text{f-grade}(R_+, \mathfrak{a}, M) = \max\{i \mid H^j_{\mathfrak{a}}(M)_n = 0, \text{ for all } n \gg 0 \text{ and all } j < i\}.$$

In the following proposition, we consider a polynomial ring and see whether a homogeneous ideal could be h-linked with R_+.

Proposition 2.8. *Let* $(R_0, \mathfrak{m}_0)$ *be a regular local ring containing a field of characteristic zero and* $R = R_0[x_1, \ldots, x_t]$ *be the polynomial ring graded in the usual way, that is* $\deg(x_i) = 1$ *for all* $i = 1, \ldots, t$. *Then* R_+ *can't be h-linked with any ideal* $\mathfrak{a} \supsetneq R_+$. *Moreover, if* $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ *and* $\mathfrak{a} \nsubseteq R_+$, *then* $\mathfrak{a}$ *and* R_+ *are geometrically h-linked by* I *over* R.

Proof. Let $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ and suppose to the contrary that $R_+ \subsetneq \mathfrak{a}$. Since $t := \operatorname{grade}(R_+) \leqslant f - \operatorname{grade}(\mathfrak{a}, R_+, R)$, so $H^t_{\mathfrak{a}}(R)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$, by [10,

1.7]. Thus $H^t_{\mathfrak{a}}(R) = 0$, using [16, 8.1], that is a contradiction, in view of [9, 2.6(i)].
Now, assume that $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ and $\mathfrak{a} \nsubseteq R_+$. By [10, 1.7] and [16, 8.1], $H^i_{\mathfrak{a}+R_+}(R) = 0$ for all $i \leq f - \operatorname{grade}(\mathfrak{a} + R_+, R_+, R)$. So, $f - \operatorname{grade}(\mathfrak{a} + R_+, R_+, R) \lneq \operatorname{grade}(\mathfrak{a} + R_+)$. Thus $\operatorname{grade}(R_+) \lneq \operatorname{grade}(\mathfrak{a} + R_+)$ and, by 2.2(i), $\mathfrak{a} \cap R_+ = I$.

□

In the following proposition, we study the set $\operatorname{Ass}_{R_0}(M/\mathfrak{a}M)$ where $\mathfrak{a}$ is an h-linked ideal over M. It will be used later in the paper, too.

Proposition 2.9. *Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically h-linked by I over M and $\mathfrak{b} \supseteq R_+$. Then*

(i) $\operatorname{Ass}_{R_0}(M/\mathfrak{a}M) = \operatorname{Ass}_{R_0}(M/IM) \bigcap V(\mathfrak{a}_0)$;

(ii) $\operatorname{Ass}_{R_0}(M/\mathfrak{a}M) \bigcap \operatorname{Ass}_{R_0}(M/\mathfrak{b}M) = \varnothing$;

(iii) $\operatorname{Ass}_{R_0}(M/\mathfrak{b}M) \bigcap V(\mathfrak{a}_0) = \varnothing$.

The first case also holds if $\mathfrak{a}$ and $\mathfrak{b}$ are just h-linked and $\operatorname{Ass}_R(M/IM) = \operatorname{minAss}_R(M/IM)$ (e.g. M is a Cohen-Macaulay module).

Proof. (i) By [9, 2.9] and [13, Exercise 6.7], $\operatorname{Ass}_{R_0}(M/\mathfrak{a}M) = \{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \operatorname{Ass}_R(M/IM) \bigcap V(\mathfrak{a})\}$ and that $\operatorname{Ass}_{R_0}(M/IM) = \{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \operatorname{Ass}_R(M/IM)\}$. This implies that $\operatorname{Ass}_{R_0}(M/\mathfrak{a}M) \subseteq \operatorname{Ass}_{R_0}(M/IM) \bigcap V(\mathfrak{a}_0)$.
Now, let $\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(M/IM) \bigcap V(\mathfrak{a}_0)$. Then, there exists $\mathfrak{p} \in \operatorname{Ass}_R(M/IM)$ such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$. $\sqrt{0 : M + I} = \sqrt{0 : M/IM} \subseteq \mathfrak{p}$. Thus, by [7, 2.2] and the assumption, $\sqrt{0 : M + \mathfrak{a} \cap R_+} \subseteq \mathfrak{p}$. So, $\mathfrak{p} \supseteq \mathfrak{a}$ and, again by [9, 2.9], $\mathfrak{p} \in \operatorname{Ass}_R(M/\mathfrak{a}M)$.

(ii) Let $\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(M/\mathfrak{b}M)$ then, by [9, 2.9], there exists $\mathfrak{p} \in \operatorname{Ass}_R(M/IM) \bigcap V(\mathfrak{b})$ such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$ and $\mathfrak{p} \notin \operatorname{Ass}_R(M/\mathfrak{a}M)$. So, $\mathfrak{p} \notin V(\mathfrak{a})$. On the other hand, $\mathfrak{p} \supseteq \mathfrak{b} \supseteq R_+$ thus $\mathfrak{p}_0 \nsupseteq \mathfrak{a}_0$ and by (i), $\mathfrak{p}_0 \notin \operatorname{Ass}_{R_0}(M/\mathfrak{a}M)$.

(iii) Follows from (i) and (ii).

□

If we remove the condition $\mathfrak{b} \supseteq R_+$, then the above proposition does not hold any more, as the following example shows.

Example 2.10. *Let $\mathfrak{b}$ and R_+ be geometrically h-linked by I over R, then* $\operatorname{Ass}_{R_0}(R/R_+) \neq \operatorname{Ass}_{R_0}(R/I)$.

Proof. Since $I = \mathfrak{b} \cap R_+$, so $\mathfrak{b} \nsupseteq R_+$. Assume that $\operatorname{Ass}_{R_0}(R/R_+) = \operatorname{Ass}_{R_0}(R/I)$, then by 2.9(i), $\operatorname{Ass}_{R_0}(R/\mathfrak{b}) \subseteq \operatorname{Ass}_{R_0}(R/R_+)$ that is a contradiction, by 2.9(ii). □

3 Graded components of $H^i_{\mathfrak{a}}(M)$ where $\mathfrak{a}$ is an h-linked ideal

In this section, which is the main part of the paper, we study the graded components of $H^i_{\mathfrak{a}}(M)$ where $\mathfrak{a}$ is h-linked with the irrelevant ideal R_+ over M.

For a graded R-module $N = \bigoplus_{n\in\mathbb{Z}} N_n$, set

$$end(N) := sup\{n \in \mathbb{Z}|\ N_n \neq 0\}.$$

Note that $end(N)$ could be ∞ and that the supremum of the empty set is to be taken as $-\infty$.

The following lemma, which consider a case where $end(H^i_{\mathfrak{a}}(M)) < \infty$, will be used later in the paper, too.

Lemma 3.1. *Let $t \in \mathbb{N}_0$ and assume that $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \neq t$. Then for all $n \gg 0$ and all $i \in \mathbb{N}_0$,*

$$H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n \cong \begin{cases} H^{i-t}_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M))_n, & i \geq t \\ 0 & i \lneq t. \end{cases}$$

Proof. We have the following convergence of spectral sequences, by [17, 11.38],

$$(E_2^{i,j})_n = H^i_{\mathfrak{b}}(H^j_{\mathfrak{a}}(M))_n \overset{i}{\Rightarrow} H^{i+j}_{\mathfrak{a}+\mathfrak{b}}(M)_n.$$

Since $end(H^j_{\mathfrak{a}}(M)) < \infty$ for all $j \neq t$, $H^j_{\mathfrak{a}}(M)$ is R_+-torsion for all $j \neq t$. So, by [3, 2.1.9],

$$H^i_{\mathfrak{b}}(H^j_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{b}+R_+}(H^j_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{b}_0 R}(H^j_{\mathfrak{a}}(M)) \qquad \text{for all } j \neq t \text{ and all } i \geq 0,$$

where $\mathfrak{b}_0 := \mathfrak{b} \cap R_0$. Hence, by [3, 14.1.12] and the assumption, $(E_2^{i,j})_n = H^i_{\mathfrak{b}_0}(H^j_{\mathfrak{a}}(M)_n) = 0$ for all $j \neq t$ and all $n \gg 0$. As a result, $H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n \cong (E_2^{i-t,t})_n$ for all $i \geq t$ and that $H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n = 0$ for all $i \lneq t$, when $n \gg 0$. □

The following corollary, which is immediate by the above lemma, generalizes [10, 1.1].

Corollary 3.2. *Let $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \in \mathbb{N}_0$. Then for any homogeneous ideal $\mathfrak{b} \supseteq \mathfrak{a}$, $end(H^i_{\mathfrak{b}}(M)) < \infty$ for all $i \in \mathbb{N}_0$.*

The following lemma will be used several times in the paper.

Lemma 3.3. *Let $\mathfrak{a}$ be linked by I over M. Then* $\operatorname{Supp} H^t_{\mathfrak{a}}(M) = \operatorname{Supp} M/\mathfrak{a}M$*, where* $t := \operatorname{grade}(\mathfrak{a}, M)$.

Proof. By [9, 2.8], [6, Theorem 1] and [19, 1.4], $\operatorname{Ass} M/\mathfrak{a}M \subseteq \operatorname{Ass} M/IM \cap V(\mathfrak{a}) = \operatorname{Ass} Hom_R(R/\mathfrak{a}, M/IM) = \operatorname{Ass} Ext^t_R(R/\mathfrak{a}, M) = \operatorname{Ass} H^t_{\mathfrak{a}}(M)$.

On the other hand, $\operatorname{Supp} H^t_{\mathfrak{a}}(M) \subseteq \operatorname{Supp} M/\mathfrak{a}M$, which proves the claim. □

In the following, we present some equivalent conditions for $end(H^i_{\mathfrak{a}}(M)) < \infty$, where $\mathfrak{a}$ is an h-linked ideal over M.

Proposition 3.4. *Let $\mathfrak{a}$ be an h-linked ideal by I over M with $\operatorname{grade}(\mathfrak{a}, M) = t$. Then the following statements are equivalent.*

(i) $end(H^i_{\mathfrak{a}}(M)) < \infty$ *for all* $i \in \mathbb{N}_0$,

(ii) $end(H^t_{\mathfrak{a}}(M)) < \infty$,

(iii) $\operatorname{Supp} M/\mathfrak{a}M \subseteq V(R_+)$.

Also, if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$ *and one of the above equivalent conditions holds, then*

$$H^i_{\mathfrak{b}}(M)_n \cong \begin{cases} 0 & i \neq t \\ H^t_I(M)_n & i = t, \end{cases}$$

for all i and all $n \gg 0$.

Proof. "*(ii)* $\Rightarrow$ *(iii)*" Since $end(H^t_{\mathfrak{a}}(M)) < \infty$, $H^t_{\mathfrak{a}}(M)$ is R_+-torsion and $\operatorname{Ass} H^t_{\mathfrak{a}}(M) \subseteq V(R_+)$. So, the result follows from 3.3.
"*(iii)* $\Rightarrow$ *(i)*" Since $\sqrt{\mathfrak{a} + 0 : M} \supseteq \sqrt{R_+}$, using [3, 2.1.9], [3, 16.1.5(ii)] and 3.2, the statement holds.

The last statement follows from [7, 2.2(i)], 3.2 and the following homogeneous Mayer-Vietoris sequence

$$\ldots \longrightarrow H^i_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{b}}(M) \longrightarrow H^i_I(M) \longrightarrow H^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow \ldots .$$

□

Note that if $\mathfrak{b} \subseteq R_+$ and one of the above conditions holds, then $\mathfrak{a}$ can not be geometrically h-linked with $\mathfrak{b}$. Otherwise, by 3.4(iii), $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}+R_+}(M)$ for all i, so $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{grade}(\mathfrak{a}+\mathfrak{b}, M)$, that is a contradiction by 2.2.

Definition 3.5.

- *We say that the ideal I is generated by an M-regular sequence under radical if there exists an M-regular sequence $\underline{x} = x_1, \ldots, x_t$ such that $\sqrt{I + 0 : M} = \sqrt{\underline{x} + 0 : M}$.*
- *$\mathfrak{a}$ and $\mathfrak{b}$ are said to be in an h-M-linkage class of length n if there exist $n \in \mathbb{N}$ and homogeneous ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and $I_1, \ldots, I_n$ such that $\mathfrak{a} = \mathfrak{a}_0 \overset{h}{\sim}_{(I_1;M)} \mathfrak{a}_1 \overset{h}{\sim} \ldots \overset{h}{\sim}_{(I_n;M)} \mathfrak{a}_n = \mathfrak{b}$. If, in addition, $\mathfrak{b}$ is generated by a homogeneous M-regular sequence under radical, then $\mathfrak{a}$ is called radically h-M-licci with $\mathfrak{b}$ of length n.*

Corollary 3.6. *If $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$ and $t := \operatorname{grade}(\mathfrak{a}, M)$, then the following statements are equivalent.*

(i) $\max\{end(H^t_{\mathfrak{a}}(M)), end(H^t_{\mathfrak{b}}(M))\} < \infty$,

(ii) $\operatorname{Supp} M/IM \subseteq V(R_+)$.

In particular, if $\mathfrak{b} = R_+$ *and* $end(H^t_{\mathfrak{a}}(M)) < \infty$*, then* $\mathfrak{a}$ *is radically h-*M*-licci with* R_+ *of length 1.*

Proof. The results follow from [9, 2.6(iii)], 3.4 and the fact that $end(H^i_{R_+}(M)) < \infty$ for all $i \in \mathbb{N}_0$. □

The above corollary shows, if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, then $end(H^i_{\mathfrak{a}}(M))$ or $end(H^i_{\mathfrak{b}}(M))$ is infinite for some $i \in \mathbb{N}_0$ if and only if $\operatorname{Supp} M/IM \nsubseteq V(R_+)$.

Proposition 3.7. *Assume that* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b} \overset{h}{\sim}_{(J;M)} \mathfrak{c}$ *and* $end(H^t_I(M)) < \infty$*, where* $t := \operatorname{grade}(I, M)$*. Then*

$$H^i_{\mathfrak{c}}(M)_n = \begin{cases} 0 & i \neq t \\ H^t_J(M)_n & i = t, \end{cases}$$

for all i *and all* $n \gg 0$*.*

Proof. Since $end(H^t_I(M)) < \infty$, $H^t_I(M)$ is R_+-torsion. So, by [6, Theorem 1], [19, 1.4] and [9, 2.6], $\operatorname{Supp}(M/\mathfrak{b}M) \subseteq V(R_+)$, in other words $R_+ \subseteq \sqrt{\mathfrak{b} + 0 : M}$. Thus $H^i_{\mathfrak{b}}(M)_n \cong H^i_{\mathfrak{b}+R_+}(M)_n = 0$ for all i and all $n \gg 0$, by 3.2. Also, in view of [7, 2.2], we have the following homogeneous Mayer-Vietoris sequence

$$\ldots \longrightarrow H^i_{\mathfrak{b}+\mathfrak{c}}(M) \longrightarrow H^i_{\mathfrak{b}}(M) \oplus H^i_{\mathfrak{c}}(M) \longrightarrow H^i_J(M) \longrightarrow H^{i+1}_{\mathfrak{b}+\mathfrak{c}}(M) \longrightarrow \ldots .$$

This, in conjunction with 3.2 and [9, 2.6], follows the result.

□

Using 3.7, we can describe the components $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is radically h-M-licci with R_+ of length 2, as follows:

Corollary 3.8. *If* $\mathfrak{a}$ *is radically h-*M*-licci with* R_+ *of length 2, i.e.* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b} \overset{h}{\sim}_{(J;M)} R_+$ *and* $\sqrt{R_+ + 0 : M} = \sqrt{J + 0 : M}$*, then*

$$H^i_{\mathfrak{a}}(M)_n = \begin{cases} 0 & i \neq t \\ H^t_I(M)_n & i = t, \end{cases}$$

for all i *and all* $n \gg 0$ *where* $t := \operatorname{grade}(I, M)$*.*

Proof. By hypothesis $H^t_J(M) \cong H^t_{R_+}(M)$. So, using [3, 16.1.5] and 3.7, the statement holds. □

M is called relative Cohen-Macaulay with respect to $\mathfrak{a}$ of degree n if $H^i_{\mathfrak{a}}(M) = 0$ for all $i \neq n$.

Theorem 3.9. *If* $\mathfrak{a} \overset{h}{\sim}_{(I;M)} R_+$ *and* $t := \operatorname{grade}(R_+, M)$*, then* $end(H^i_{\mathfrak{a}}(M)) < \infty$ *for all* $i \neq t$ *and* $end(H^t_{\mathfrak{a}}(M)) < \infty$ *or* $H^t_{\mathfrak{a}}(M)_n \neq 0$ *for all* $n \gg 0$.
In a special case, $H^t_{\mathfrak{a}}(M)_n$ *is a finitely generated* R_0*-module for all* $n \in \mathbb{Z}$.

Proof. The case $i \neq t$ follows from 3.4 and [3, 16.1.5(ii)]. Also, we have $H^t_{\mathfrak{a}}(M)_n \cong H^t_I(M)_n$ for all $n \gg 0$. We consider two cases:

case 1: Let $\operatorname{Supp}(M/IM) \nsubseteq V(R_+)$. By 3.6 and [3, 16.1.5(ii)], $end(H^t_{\mathfrak{a}}(M)) = \infty$. Now, we prove, by induction on t, that $H^t_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.
If $t = 0$, then $\Gamma_{\mathfrak{a}}(M)_n = \Gamma_{\underline{0}}(M)_n = M_n$ for all $n \gg 0$. On the other hand, by [11, Theorem 1], $R_1 M_n = M_{n+1}$ for all $n \gg 0$, thus $\Gamma_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.
Let $t > 0$ and assume, inductively, that the claim holds for $t-1$. Let $I = (x_1, x_2, \ldots, x_t)$ and $deg(x_1) = l$. Now, the homogeneous exact sequence

$$0 \longrightarrow M \xrightarrow{.x_1} M(l) \longrightarrow (M/x_1M)(l) \longrightarrow 0$$

and [9, 2.6(i)] yield the following exact sequence of R_0-modules for all $n \in \mathbb{Z}$,

$$0 \longrightarrow H^{t-1}_{\mathfrak{a}}(M/x_1M)_{n+l} \longrightarrow H^t_{\mathfrak{a}}(M)_n \xrightarrow{.x_1} H^t_{\mathfrak{a}}(M)_{n+l} \longrightarrow \ldots\ .$$

Since $x_1 \in I$, $\mathfrak{a}/(x_1) \overset{h}{\sim}_{(I/(x_1);M/x_1M)} R_+/(x_1)$ and, by the inductive hypothesis, $H^{t-1}_{\frac{\mathfrak{a}}{(x_1)}}(M/x_1M)_n \neq 0$ for all $n \gg 0$. Hence, $H^t_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.

case 2: Now, assume that $\operatorname{Supp}(M/IM) \subseteq V(R_+)$. Then $\sqrt{0 : M + I} = \sqrt{0 : M + R_+}$ and $H^i_I(M) \cong H^i_{R_+}(M)$ for all i. So, by [3, 16.1.5], we have

1. $end(H^t_I(M)) < \infty$,
2. $H^t_I(M)_n$ is a finitely generated R_0-module for all n,
3. M is relative Cohen-Macaulay with respect to R_+ of degree t.

Therefore, in view of 3.4, $end(H^t_{\mathfrak{a}}(M)) < \infty$. Also, using [14, 3.4], there are homogeneous isomorphisms

$$H^i_{\mathfrak{a}}(M) \cong H^{i-t}_{\mathfrak{a}}(H^t_I(M)) \cong H^{i-t}_{\mathfrak{a}+R_+}(H^t_I(M)) \cong H^i_{\mathfrak{a}+R_+}(M) \qquad \text{for all } i \geq t. \quad (3.1)$$

Using [9, 2.6], $t \leq \text{f-grade}(\mathfrak{a}_0 + R_+, R_+, M)$. Therefore, $H^t_{\mathfrak{a}_0+R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$, by [10, 1.7]. As a result, by (3.1), $H^t_{\mathfrak{a}}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$.

□

As wee have seen in the proof of 3.9, if R_+ is h-linked by I over M and $\mathrm{Supp}(M/IM) \subseteq V(R_+)$ then M is relative Cohen-Macaulay with respect to R_+. However, the converse does not hold any more, as the following example shows. Although, it does in some special cases, see Proposition 3.11.

Example 3.10. *Assume that $(R_0, \mathfrak{m}_0)$ is a domain and $\dim R_0 = 2$. Set $R = R_0[x]$. So, there exists a non zero prime ideal $\mathfrak{p}_0$ of R_0 such that $\mathfrak{p}_0 \subsetneq \mathfrak{m}_0$. By 2.5, for any $0 \neq r_0 \in \mathfrak{p}_0$, $(r_0) \overset{h}{\sim}_{((r_0x);R)} (x)$ and $\mathrm{Supp}(R/(r_0x)) = V(r_0) \bigcup V(x) \nsubseteq V(x)$, while R is relative Cohen-Macaulay with respect to (x) of degree 1.*

The following proposition considers a case where the irrelevant ideal can be generated by an M-regular sequence under radical.

Proposition 3.11. *Let $(R_0, \mathfrak{m}_0)$ be local and M be relative Cohen-Macaulay with respect to R_+ of degree t. Then there exists a maximal homogeneous M-regular sequence I in R_+ such that $\mathrm{Supp}(M/IM) \subseteq V(R_+)$. In other words, R_+ can be generated by a homogeneous M-regular sequence under radical.*

Proof. Assume that $t = 0$. By [1, 2.3], $\dim M/\mathfrak{m}_0M = 0$. Therefore, $M/\mathfrak{m}_0M$ is Artinian and $end(M/\mathfrak{m}_0M) < \infty$, using [11, Theorem 1]. Hence, by Nakayama Lemma, $end(M) < \infty$. This implies that M is R_+-torsion and that $\mathrm{Supp}\, M \subseteq V(R_+)$.
Now, let $t > 0$ and assume inductively that the statement holds for $t-1$. As $t > 0$, $R_+ \nsubseteq (\bigcup_{\mathfrak{p} \in MinAss(M/\mathfrak{m}_0M)} \mathfrak{p}) \cup Z(M)$, where $Z(M)$ denotes the set of zero divisors on M. So, by [3, 16.1.2], there exists a homogeneous element

$$x \in R_+ \backslash (\bigcup_{\mathfrak{p} \in MinAss(M/\mathfrak{m}_0M)} \mathfrak{p}) \cup Z(M).$$

Therefore, $\dim \frac{M/xM}{\mathfrak{m}_0(M/xM)} = t - 1 = \mathrm{grade}(R_+, M/xM)$. In other words, using [1, 2.3], M/xM is relative Cohen-Macaulay with respect to R_+ of degree $t-1$ and, by the induction hypothesis, there is a maximal homogeneous M/xM-regular sequence I' in R_+ such that $\mathrm{Supp}\, \frac{M}{(I' + <x>)M} = \mathrm{Supp}\, \frac{M/xM}{I'(M/xM)} \subseteq V(R_+)$. Now, the result follows by induction. □

Definition and Remark 3.12.

(i) Let $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be a graded R-module. Then following [12],

- *N is called finitely graded if $N_n = 0$ for all but finitely many $n \in \mathbb{Z}$;*
- *$g_{\mathfrak{a}}(N) := sup\{k \in \mathbb{N}_0 | H^i_{\mathfrak{a}}(N)$ is finitely graded for all $i < k\}$;*
- *$f^{R_+}_{\mathfrak{a}}(N) := sup\{k \in \mathbb{N}_0 | R_+ \subseteq \sqrt{0 : H^i_{\mathfrak{a}}(N)}$ for all $i < k\}$;*
- *N is called tame, if the set $\{n \in \mathbb{Z} | N_n = 0, N_{n+1} \neq 0\}$ is finite.*

Note that, by [12, 2.3], if N is finitely generated, then $g_{\mathfrak{a}}(N) = f^{R_+}_{\mathfrak{a}}(N)$.

(ii) Let $\mathfrak{a} \overset{h}{\sim}_{(I;M)} R_+$ and $\mathrm{Supp}\, M/IM \nsubseteq V(R_+)$, then using [9, 2.6(i)], 3.9 and (i), we have $f^{R_+}_{\mathfrak{a}}(M) = \mathrm{grade}(R_+, M)$.

In [5, 2.2], the authors studied tameness of $H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{R_+}(M)}(M)$ under the assumption that $\mathfrak{a} \supseteq R_+$. In the following Theorem, we consider this problem without any restriction on $\mathfrak{a}$. Although, the proof is a modification of [5, 2.2], we bring it here for the reader's convenience. It will be used later in the paper, too.

Theorem 3.13. *Let $(R_0, \mathfrak{m}_0)$ be local and $f_{\mathfrak{a}}^{R_+}(M) < \infty$. Then $H_{\mathfrak{a}}^{f_{\mathfrak{a}}^{R_+}(M)}(M)$ is tame.*

Proof. Let y be an indeterminate. Set $R'_0 = R_0[y]_{\mathfrak{m}_0[y]}$, $R' = R \bigotimes_{R_0} R'_0$ and $M' = M \bigotimes_{R_0} R'_0$. R'_0 is a faithfully flat R_0-algebra so, by [3, 16.2.2(iv)], $H_{\mathfrak{a}}^i(M)_n \bigotimes_{R_0} R'_0 \cong H_{\mathfrak{a}R'}^i(M')_n$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. This results $f_{\mathfrak{a}}^{R_+}(M) = f_{\mathfrak{a}R'}^{R'_+}(M)$. Thus, replacing R by R', we can assume that $R_0/\mathfrak{m}_0$ is an infinite filed. Now, we prove the assertion by induction on $f := f_{\mathfrak{a}}^{R_+}(M)$.
Let $f = 0$. By [11, Theorem 1], $\Gamma_{\mathfrak{a}}(M)_n = 0$ for all $n \ll 0$ and $R_1\Gamma_{\mathfrak{a}}(M)_n = \Gamma_{\mathfrak{a}}(M)_{n+1}$ for all $n \gg 0$ that makes $\Gamma_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$. Now, assume that $f \geq 1$ and the result has been proved for $f - 1$.
$\Gamma_{R_+}(M)$ is a finitely graded R-module so, using [12, 2.2], $H_{\mathfrak{a}}^i(\Gamma_{R_+}(M))$ is finitely graded for all $i \in \mathbb{N}_0$. Hence, the exact sequence

$$\ldots \longrightarrow H_{\mathfrak{a}}^i(\Gamma_{R_+}(M))_n \longrightarrow H_{\mathfrak{a}}^i(M)_n \longrightarrow H_{\mathfrak{a}}^i(M/\Gamma_{R_+}(M))_n \longrightarrow H_{\mathfrak{a}}^{i+1}(\Gamma_{R_+}(M))_n \longrightarrow \ldots$$

implies that $H_{\mathfrak{a}}^i(M)_n \cong H_{\mathfrak{a}}^i(M/\Gamma_{R_+}(M))_n$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z} \setminus X$, where X is a finite set. So, replacing M with $M/\Gamma_{R_+}(M)$, we can assume that M is R_+-torsion free and that there exists a homogeneous element $x \in R_1$ which is a non-zero deviser on M, by [3, 16.1.4(ii)]. Now, consider the homogeneous exact sequence

$$0 \longrightarrow M \xrightarrow{x} M(1) \longrightarrow (M/xM)(1) \longrightarrow 0.$$

It yields the following exact sequence of R_0-modules

$$\ldots \longrightarrow H_{\mathfrak{a}}^{i-1}(M)_{n+1} \longrightarrow H_{\mathfrak{a}}^{i-1}(M/xM)_{n+1} \longrightarrow H_{\mathfrak{a}}^i(M)_n \xrightarrow{x} H_{\mathfrak{a}}^i(M)_{n+1} \longrightarrow \ldots\ .$$

Therefore, by 3.12, $f_{\mathfrak{a}}^{R_+}(M/xM) \geq f_{\mathfrak{a}}^{R_+}(M) - 1$.
If $f_{\mathfrak{a}}^{R_+}(M/xM) \gneq f - 1$, by 3.12, $0 \longrightarrow H_{\mathfrak{a}}^f(M)_n \xrightarrow{x} H_{\mathfrak{a}}^f(M)_{n+1}$ is exact for all $n \gg 0$ and all $n \ll 0$ that shows $H_{\mathfrak{a}}^f(M)$ is tame.
Also, if $f_{\mathfrak{a}}^{R_+}(M/xM) = f - 1$, again by 3.12, we have the following exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^{f-1}(M/xM)_{n+1} \longrightarrow H_{\mathfrak{a}}^f(M)_n \xrightarrow{x} H_{\mathfrak{a}}^f(M)_{n+1}$$

for all $n \gg 0$ and all $n \ll 0$. But, by induction, $H_{\mathfrak{a}}^{f-1}(M/xM)$ is tame, which requires $H_{\mathfrak{a}}^f(M)$ is tame.

□

Regards $t := \operatorname{grade}(\mathfrak{a}, M) \leq f_{\mathfrak{a}}^{R_+}(M)$ and 3.12, $H_{\mathfrak{a}}^t(M)$ is tame.

Let N be an R-module. The cohomological dimension of N with respect to $\mathfrak{a}$ is defined to be

$$\mathrm{cd}(\mathfrak{a}, N) := sup\{i \in \mathbb{Z} | H^i_{\mathfrak{a}}(N) \neq 0\}.$$

The following Theorem considers tameness of $H^{\mathrm{cd}(R_+,M)}_{\mathfrak{a}+R_+}(M)$ with some restrictions on M or linkedness of $\mathfrak{a}$ with R_+.

Theorem 3.14. *Let $(R_0, \mathfrak{m}_0)$ be local and $\mathrm{cd}(R_+, M) \neq 0$. Then $f^{R_+}_{\mathfrak{a}+R_+}(M) < \infty$ and $H^{\mathrm{cd}(R_+,M)}_{\mathfrak{a}+R_+}(M)$ is tame in each of the following cases:*

(i) M is relative Cohen-Macaulay with respect to R_+;

(ii) $\mathfrak{a} \overset{h}{\sim}_{(I;M)} R_+$.

Proof. (i) Let $t := \mathrm{cd}(R_+, M) = \mathrm{grade}(R_+, M)$. Since $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all i and all n ([3, 16.1.5]), by [18, 2.6], $\mathfrak{a}_0 H^t_{R_+}(M)_n \neq H^t_{R_+}(M)_n$ for all $n \ll 0$, where $\mathfrak{a}_0 := \mathfrak{a} \cap R_0$. Therefore, using [3, 6.2.7],

$$\text{for all } n \ll 0 \text{ there exists } k_n \in \mathbb{N}_0 \text{ such that } H^{k_n}_{\mathfrak{a}_0}(H^t_{R_+}(M)_n) \neq 0. \tag{3.2}$$

Now, considering the following Grothendieck's spectral sequence ([17, 11.38])

$$E^{i,j}_2 = H^i_{\mathfrak{a}}(H^j_{R_+}(M)) \overset{i}{\Rightarrow} H^{i+j}_{\mathfrak{a}+R_+}(M),$$

we have $E^{i,j}_2 = 0$ for all i and all $j \neq t$. This implies that

$$H^i_{\mathfrak{a}}(H^t_{R_+}(M)) \cong H^{i+t}_{\mathfrak{a}+R_+}(M) \qquad \text{for all } i \in \mathbb{N}_0. \tag{3.3}$$

On the other hand, by [3, 14.1.12] and the fact that $H^t_{R_+}(M)$ is R_+-torsion, we have

$$H^i_{\mathfrak{a}}(H^t_{R_+}(M))_n \cong H^i_{\mathfrak{a}_0+R_+}(H^t_{R_+}(M))_n \cong H^i_{\mathfrak{a}_0 R}(H^t_{R_+}(M))_n \cong H^i_{\mathfrak{a}_0}(H^t_{R_+}(M)_n) \tag{3.4}$$

for all i and all n. So, if $f^{R_+}_{\mathfrak{a}+R_+}(M) = \infty$, then by (3.4), (3.3) and 3.12,

$$H^i_{\mathfrak{a}_0}(H^t_{R_+}(M)_n) \cong H^{i+t}_{\mathfrak{a}+R_+}(M)_n = 0$$

for all i and all $n \ll 0$ that is a contradiction with (3.2). Therefore, $f^{R_+}_{\mathfrak{a}+R_+}(M) < \infty$. In addition, $t = \mathrm{grade}(R_+, M) \leq \mathrm{grade}(\mathfrak{a} + R_+, M) \leq f^{R_+}_{\mathfrak{a}+R_+}(M)$. So, by 3.12 and 3.13, $H^t_{\mathfrak{a}+R_+}(M)$ is tame.

(ii) Set $t := \mathrm{grade}(R_+, M)$. By the homogeneous Mayer-Vietoris sequence and [7, 2.2], we have the homogeneous exact sequence

$$\ldots \to H^{i-1}_I(M) \to H^i_{\mathfrak{a}+R_+}(M) \to H^i_{\mathfrak{a}}(M) \oplus H^i_{R_+}(M) \to H^i_I(M) \to H^{i+1}_{\mathfrak{a}+R_+}(M) \to \ldots .$$

It yields

$$H^i_{\mathfrak{a}+R_+}(M) \cong H^i_{\mathfrak{a}}(M) \oplus H^i_{R_+}(M) \qquad \text{for all } i \gneq t+1 \tag{3.5}$$

and, by [3, 6.2.7], the exact sequence

$$\begin{aligned} 0 \longrightarrow H^t_{\mathfrak{a}+R_+}(M) \longrightarrow H^t_{\mathfrak{a}}(M) \oplus H^t_{R_+}(M) \longrightarrow H^t_I(M) \longrightarrow H^{t+1}_{\mathfrak{a}+R_+}(M) \\ \longrightarrow H^{t+1}_{\mathfrak{a}}(M) \oplus H^{t+1}_{R_+}(M) \longrightarrow 0. \end{aligned} \tag{3.6}$$

Therefore, in view of (3.5), (3.6) and [18, 2.6], $H^{\mathrm{cd}(R_+,M)}_{\mathfrak{a}+R_+}(M)_n \neq 0$ for all $n \ll 0$. So, by [10, 1.1], $H^{\mathrm{cd}(R_+,M)}_{\mathfrak{a}+R_+}(M)$ is tame and, using 3.12, $f^{R_+}_{\mathfrak{a}+R_+}(M) \leq \mathrm{cd}(R_+, M) < \infty$.

□

Remark 3.15.

(i) Here is another situation for the finiteness of $f^{R_+}_{\mathfrak{a}}(M)$. Assume that $\mathfrak{a}$ and R_+ are geometrically h-linked over M. Then, by [9, 2.9(iii)], $\mathrm{Supp}(M/\mathfrak{a}M) \nsubseteq V(R_+)$ *and there exists a homogeneous prime ideal $\mathfrak{p} \in \mathrm{Supp}\, M \cap V(\mathfrak{a}) \backslash V(R_+)$. Hence, $\mathfrak{p} + \mathfrak{a} + R_+ = \mathfrak{p} + R_+ = \mathfrak{p} \cap R_0 + R_+ \neq R$. Therefore, by [3, 9.3.7],*

$$f^{R_+}_{\mathfrak{a}+R_+}(M) \leq \mathrm{depth}\, M_{\mathfrak{p}} + ht(\mathfrak{a} + R_+ + \mathfrak{p})/\mathfrak{p} < \infty.$$

(ii) Assume that $\mathfrak{b} \supseteq \mathfrak{a}$. Then $\mathfrak{b}$ can be represented as $\mathfrak{b} = \mathfrak{a} + (b_1, \ldots, b_s)$ for some homogeneous elements $\mathfrak{b}_1, \ldots, \mathfrak{b}_s \in R$. Using [3, 14.1.11] and induction on s, one can see that $f^{R_+}_{\mathfrak{a}}(M) \leq f^{R_+}_{\mathfrak{b}}(M)$.

The following proposition presents possibilities for $f^{R_+}_{\mathfrak{a}}(M)$ and $f_{R_+}(M)$ in the case where $\mathfrak{a}$ is h-linked with R_+.

Proposition 3.16. *Let $\mathfrak{a} \overset{h}{\sim}_{(I;M)} R_+$, then $f^{R_+}_{\mathfrak{a}}(M), f_{R_+}(M) \in \{\mathrm{grade}(R_+, M), f^{R_+}_{\mathfrak{a}+R_+}(M)\}$.*

Proof. Set $t := \mathrm{grade}(R_+, M)$. By [9, 2.6(i)] and [3, 6.2.7], $t \leq f^{R_+}_{\mathfrak{a}}(M), f_{R_+}(M)$. Also, by 3.15(ii), $f^{R_+}_{\mathfrak{a}}(M), f_{R_+}(M) \leq f^{R_+}_{\mathfrak{a}+R_+}(M)$.
If $f^{R_+}_{\mathfrak{a}+R_+}(M) \leq t+1$, the result follows. So, let $f^{R_+}_{\mathfrak{a}+R_+}(M) \gneq t+1$. By (3.5) and (3.6), for all $i \geq t+1$, $H^i_{\mathfrak{a}}(M)$ and $H^i_{R_+}(M)$ are finitely graded if and only if $H^i_{\mathfrak{a}+R_+}(M)$ is finitely graded and this proves the claim. □

Acknowledgement. *We would like to thank the referee for her/his careful reading of the paper.*

References

[1] M. Brodmann, Asymptotic behaviour of cohomology: Tameness, supports and associated primes, *Commutative Algebra and Algebraic Geometry*, **390**, 31-61 (2005).

[2] M. Brodmann, M. Hellus, Cohomological patterns of coherent sheaves over projective schemes, *Journal of Pure and Applied Algebra*, **172**, 165-182 (2002).

[3] M. P. Brodmann, R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press (2012).

[4] L. Chu, Y. Gu, A problem of local cohomology modules, *Communications in Algebra*, **36**, 1603-1607 (2008).

[5] S. H. Hassanzadeh, M. Jahangiri, H. Zakeri, Asymptotic behaviour and Artinian property of graded local cohomology modules, *Communications in Algebra*, **37**, 4095-4102 (2009).

[6] M. Hellus, On the set of associated primes of a local cohomology module, *Journal of Algebra*, **237**, 406-419 (2001).

[7] M. Jahangiri, K. Sayyari, Cohomological dimension with respect to the linked ideals, *Journal of Algebra and its Applications*, DOI: 10.1142/s0219498821501048.

[8] M. Jahangiri, K. Sayyari, Characterization of some special rings via linkage, *Journal of Algebra and Related Topics*, **8**, 67-81 (2020).

[9] M. Jahangiri, K. Sayyari, Linkage of ideals over a module, *Journal of Algebraic Systems*, **8**, 267-279 (2021).

[10] M. Jahangiri, H. Zakeri, Local cohomology modules with respect to an ideal containing the irrelevant ideal, *Journal of Pure and Applied Algebra*, **213**, 573-581 (2009).

[11] D. Kirby, Artinian modules and Hilbert polynomials, *The Quarterly Journal of Mathematics*, **24**, 47–57 (1973).

[12] T. Marley, Finitely graded local cohomology and the depths of graded algebras, *Proceedings of the American Mathematical Society*, **123**, 3601-3607 (1995).

[13] H. Matsumura, *Commutative ring theory*, Cambridge University Press (1989).

[14] U. Nagel, P. Schenzel, Cohomological annihilators and Castelnuovo-Mumford regularity, *Commutative Algebra*, **159**, 307-328 (1994).

[15] C. Peskine, L. Szpiro, Liaison des variétés algébriques. I, *Inventiones Mathematicae*, **26**, 271-302 (1974).

[16] T. J. Puthenpurakal, Graded components of local cohomology modules, *Collectanea Mathematica*, 1-37 (2021).

[17] J. J. Rotman, *An introduction to homological algebra*, Academic Press (1979).

[18] C. Rotthaus, L. M. Şega, Some properties of graded local cohomology modules, *Journal of Algebra*, **283**, 232-247 (2005).

[19] A. K. Singh, U. Walther, Local cohomology and pure morphisms, *Illinois Journal of Mathematics*, **51**, 287-298 (2007).

Received: 24.05.2021
Revised: 29.08.2021
Accepted: 04.09.2021

(1) Department of Mathematics, Faculty of Mathematical Sciences and Computer,
Kharazmi University, Tehran, Iran
E-mail: `jahangiri@khu.ac.ir, jahangiri.maryam@gmail.com`

(2) Faculty of Mathematical Sciences and Computer,
Kharazmi University, Tehran, Iran
E-mail: `a.nadali1984@gmail.com`

(3) Faculty of Mathematical Sciences and Computer,
Kharazmi University, Tehran, Iran
E-mail: `sayyarikh@gmail.com`