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Algebraic methods for Bloch-Iserles Hamiltonian systems

by

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To the memory of Doru Ştefănescu

Abstract

We survey some results on Bloch-Iserles Hamiltonian systems and we present solutions in terms of Theta-functions in the cases n=3 and n=4. This paper is based on the talks of the authors at two Conferences: "Transient Transcendence in Transylvania", Braşov, ROMANIA, May 13-17, 2019, and "5-th Conference of the Mathematical Society of the Republic of Moldova", Chişinău, September 28 - October 1, 2019.

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1 Introduction

For a given skew symmetric real $n \times n$ matrix N, the bracket $[X, Y]_N = XNY - YNX$ defines a Lie algebra structure on the space $Sym(n, N)$ of real symmetric $n \times n$ matrices.

This type of bracket has appeared before in [14] (Trofimov - Fomenko) in the study of the integrability of the Euler equations via the argument shift method and in the construction of the sectional operators as well as in [11] (Morosi - Pizzocchero) where it was used to produce the second Hamiltonian structure of the n-dimensional free rigid body equations.

For any $f, g \in C^\infty(Sym(n, N))$, the corresponding Lie-Poisson bracket is hence given by

$$\{f, g\}_N(X) := -\mathrm{trace}\left(X \left[\nabla f(X), \nabla g(X)\right]_N\right),$$

where ∇f is the gradient of f on $Sym(n, N)$ relative to the inner product $\langle\langle Y_1, Y_2\rangle\rangle := \mathrm{trace}(Y_1 Y_2)$, for any symmetric matrices Y_1, Y_2.

The inner product $\langle\langle \cdot, \cdot\rangle\rangle$ is used to identify $Sym(n, N)$ with its dual.

The geometry, integrability, and linearizability of the flow of the Bloch-Iserles Hamiltonian system [4] (Bloch - Iserles)

$$\dot{X} = [X^2, N] = [X, XN + NX] \tag{1.1}$$

on $Sym(n, N)$ were investigated in [5] (Bloch - Brînzănescu - Iserles - Marsden - Raţiu), under the hypothesis that N has distinct eigenvalues.

Theorem 1. *([5]) (Bloch - Brînzănescu - Iserles - Marsden - Raţiu) If N has distinct eigenvalues, this system is complete integrable on a generic symplectic leaf of the Lie-Poisson structure of $Sym(n, N)$.*

The Hamiltonian function is $h(X) = \mathrm{trace}(X^2/2)$ and (1.1) is equivalent to Hamilton's equations in Poisson bracket form $\dot{f} = \{f, h\}_N$ for any $f \in C^\infty(Sym(n,N))$.

We continue the study of this Hamiltonian system by establishing its algebraic complete integrability, if N has distinct eigenvalues. In order to do that, we shall complexify the problem. $Sym^{\mathbb{C}}(n,N)$ is the complexification of $Sym(n,N)$, i.e., the space of complex symmetric matrices. From now on all Hamiltonian systems are complexified.

2 Algebraic integrability: case $\mathfrak{gl}_n(\mathbb{C})$

For the convenience we shall present shortly the known case $\mathfrak{gl}_n(\mathbb{C})$ (see [3] (Beauville) and [9] (Gavrilov)).

We shall use the following definitions (compare to [2] (Adler - van Moerbeke), [1], [8] (Adams - Harnad - Hurtubise), [10] (Li - Tomei)):

Definition 1. *Let $h : \mathcal{M} \to \mathcal{V}$ be a (complex) completely integrable system, where the Poisson manifold $\mathcal{M}$ is a non-singular affine variety, $\mathcal{V}$ is an affine space, and $h = (h_1, ..., h_s)$ is given by regular algebraic functions. We say that the system $h : \mathcal{M} \to \mathcal{V}$ is*

- (i) *an* algebraically completely integrable system (a.c.i. system) *if each generic fiber of h is a Zariski open subset of an Abelian variety, on which the Hamiltonian vector fields generated by h_i are translation invariant*
- (ii) *a* generalized algebraically completely integrable system (generalized a.c.i. system) *if each generic fiber of h is a Zariski open subset of a commutative algebraic group, on which the Hamiltonian vector fields generated by h_i are translation invariant.*

The system (1.1) is equivalent to the following Lax pair system with parameter (see [5] (Bloch - Brînzănescu - Iserles - Marsden - Raţiu))

$$\frac{d}{dt}(X + \lambda N) = \left[X + \lambda N, NX + XN + \lambda N^2\right]. \tag{2.1}$$

We follow below [9] (Gavrilov) and [3] (Beauville) (see also, [2] (Adler - van Moerbeke), [12] (Reyman - Semenov-Tian-Shansky), [8] (Deift - Li - Tomei)).

Let $X(\lambda) := \lambda N + X$, where $X \in \mathfrak{gl}_n(\mathbb{C})$ is an arbitrary matrix and $Q(\lambda, z) := \det(zI_n - X(\lambda))$ its characteristic polynomial.

Denote by $M^N := \lambda N + \mathfrak{gl}_n(\mathbb{C}) = \{X(\lambda) = \lambda N + X \mid X \in \mathfrak{gl}_n(\mathbb{C})\}$ the affine space of all complex matrix polynomials of degree one whose leading coefficient is the constant real skew-symmetric matrix N.

To stress the dependence on N and Q (both fixed), we denote the associated isospectral variety by M_Q^N, i.e.,

$$M_Q^N := \{X(\lambda) \in M^N \mid \det(zI_n - X(\lambda)) = Q(\lambda, z)\}.$$

The plane algebraic curve (called a *spectral curve*), associated to each $X(\lambda)$, namely,

$$\Gamma_{X(\lambda)} := \{(\lambda, z) \in \mathbb{C} \times \mathbb{C} \mid \det(zI_n - X(\lambda)) = 0\},$$

is preserved by the flow of (2.1).

The coefficients of the characteristic polynomial $Q(\lambda, z)$ of $X(\lambda)$ are polynomials which are constants of motion for the dynamics defined by (2.1). Similarly, for each $X(\lambda)$ the isospectral variety is preserved by the flow of (2.1).

Notice that the spectral curve and the isospectral variety depend on the values of the constants of motion only (i.e., on the vector $\mathbf{c} = (q_{kl})$, where q_{kl} is the coefficient of $\lambda^k z^l$ in $Q(\lambda, z)$). Sometimes, one writes $\Gamma_{\mathbf{c}}$ instead of $\Gamma_{X(\lambda)}$.

Notice that the spectral curve $\Gamma_{\mathbf{c}}$ is non-singular for generic values of $\mathbf{c}$. Let $\overline{\Gamma}_{\mathbf{c}}$ be the compactification in the projective plane $\mathbb{P}^2_{\mathbb{C}}$ of $\Gamma_{\mathbf{c}}$. For generic values of $\mathbf{c}$ the projective curve $\overline{\Gamma}_{\mathbf{c}}$ is also non-singular. This is the case that we will consider.

The subgroup $G := \mathbb{P}GL_n(\mathbb{C}; N)$ of the projective group $\mathbb{P}GL_n(\mathbb{C})$ formed by matrices which commute with N is a symmetry group of the system (2.1).

Moreover, since $X + \lambda N$ commutes with its square, (2.1) can be written as the Lax equation with parameter

$$\frac{d}{dt}(X + \lambda N) = \left[X^2/\lambda, X + \lambda N\right]. \tag{2.2}$$

This Lax equation (2.2) for an arbitrary matrix $X \in \mathfrak{gl}_n(\mathbb{C})$ was studied, for example, in [3] (Beauville), [9] (Gavrilov).

Let $\mathcal{V}$ be the affine space of polynomials

$$Q(\lambda, z) = z^n + s_1(\lambda)z^{n-1} + ... + s_n(\lambda),$$

where $s_i(\lambda)$ are polynomials in λ of degree $\deg s_i \leq i$ for all $i = 1, ..., n$.

Consider the map

$$h : M^N \to \mathcal{V},$$

which sends a matrix of M^N to its characteristic polynomial (the components of the map h are the coefficients of the characteristic polynomial). Since G acts freely and properly (by conjugation) on the affine space M^N, it follows that the quotient M^N/G is a smooth variety.

Moreover, we have the commutative diagram

$$\begin{tikzcd} M^N \arrow[rr, "h"] \arrow[dr, "q"'] & & \mathcal{V} \\ & M^N/G \arrow[ur, "\tilde{h}"'] & \end{tikzcd}$$

where $q : M^N \to M^N/G$ is the quotient map and $\tilde{h} : M^N/G \to \mathcal{V}$ is the map induced by h (the action by conjugation preserves the characteristic polynomials).

Since G acts freely and properly on the isospectral variety M^N_Q, it follows that M^N_Q can be considered as the total space of a holomorphic principal fiber bundle with base space M^N_Q/G, structural group G, and natural projection map

$$M^N_Q \longrightarrow M^N_Q/G.$$

Note that M^N_Q is a fiber of h and M^N_Q/G is a fiber of $\tilde{h}$.

Generically, the spectral curve $\Gamma_{\mathbf{c}}$ (where $\mathbf{c}$ is the vector of the coefficients of the polynomial Q) is smooth. Then, the manifold M_Q^N/G is bi-holomorphic to a Zariski open subset of the usual Jacobian $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$; see [3] (Beauville).

By Theorem 2.1 in [9] (Gavrilov), the isospectral variety M_Q^N is smooth and bi-holomorphic to a Zariski open subset of the generalized Jacobian variety $\mathrm{Jac}(\Gamma'_{\mathbf{c}})$, where $\Gamma'_{\mathbf{c}}$ is the singular curve obtained from $\overline{\Gamma}_{\mathbf{c}}$ by identifying its points at infinity $\{P_1, \ldots, P_n\}$ with a single point ∞ (for details, see [13] (Serre)).

The generalized Jacobian $\mathrm{Jac}(\Gamma'_{\mathbf{c}})$ is a non-compact commutative algebraic group given by a non-trivial extension of the usual Jacobian $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$ by the algebraic group $G = \mathbb{P}GL_n(\mathbb{C}; N) \cong (\mathbb{C}^*)^{n-1}$, namely

$$0 \longrightarrow G \longrightarrow \mathrm{Jac}(\Gamma'_{\mathbf{c}}) \xrightarrow{\Phi} \mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}}) \longrightarrow 0. \tag{2.3}$$

The generalized Jacobian $\mathrm{Jac}(\Gamma'_{\mathbf{c}})$ can also be considered as total space of a holomorphic principal fiber bundle with base space $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$ and structure group G and has dimension $g + n - 1$, where g is the genus of $\overline{\Gamma}_{\mathbf{c}}$.

Beauville [3] and Gavrilov [9] proved that the Hamiltonian vector fields generated by the components of $\tilde{h}$, respectively of h, are translation invariant. We thus have the following results:

Theorem 2. *([3]) (Beauville) For N having distinct eigenvalues, the system $\tilde{h} : M^N/G \to \mathcal{V}$ is an algebraically completely integrable system.*

Theorem 3. *([9]) (Gavrilov) For N having distinct eigenvalues, the system $h : M^N \to \mathcal{V}$ is a generalized algebraically completely integrable system.*

3 Algebraic integrability: case $Sym^{\mathbb{C}}(n, N)$

Let us now consider the case of the Lie subalgebra

$$(Sym^{\mathbb{C}}(n, N), [\,,]_N) \subset (\mathfrak{gl}_n(\mathbb{C}), [\,,]_N),$$

where $\mathfrak{gl}_n(\mathbb{C})$ is endowed with the bracket $[\,,]_N$ (see [5] (Bloch - B - Iserles - Marsden - Ratiu)). We shall follow [6]:

Let $M^{N,Sym} := \lambda N + Sym^{\mathbb{C}}(n, N) = \left\{X(\lambda) := \lambda N + X \mid X \in Sym^{\mathbb{C}}(n, N)\right\}$ be the affine space of all complex matrix polynomials of degree one whose leading coefficient is the constant real skew-symmetric matrix N.

We denote the associated isospectral variety by $M_Q^{N,Sym}$, i.e.,

$$M_Q^{N,Sym} := \{X(\lambda) \in M^{N,Sym} \mid \det(zI_n - X(\lambda)) = Q(\lambda, z)\},$$

where $Q(\lambda, z)$ is the characteristic polynomial of $X(\lambda)$. We consider the generic case when the completion of the plane curve given by $Q = 0$ is non-singular.

We note that the affine subspace $M^{N,Sym} = \lambda N + Sym^{\mathbb{C}}(n, N)$ is invariant by the automorphism (involution)

$$\tau : \lambda N + \mathfrak{gl}_n(\mathbb{C}) \to \lambda N + \mathfrak{gl}_n(\mathbb{C}),$$

given by $\tau(X(\lambda)) = X^{\mathsf{T}}(-\lambda)$.

Indeed, we have:

$$\tau(X(\lambda)) = -\lambda N^{\mathsf{T}} + X^{\mathsf{T}} = \lambda N + X = X(\lambda),$$

since $N^{\mathsf{T}} = -N$ and $X^{\mathsf{T}} = X$.

Let N be a given real skew symmetric matrix. For N invertible with distinct eigenvalues $(n := 2p)$, choose an orthonormal basis of $\mathbb{R}^{2p}$ in which N is written as

$$\left[\begin{array}{cc} 0 & V \\ -V & 0 \end{array}\right], \tag{3.1}$$

where V is a real diagonal matrix whose entries are $v_1, ..., v_p$.

For N having distinct eigenvalues and nullity one, i.e., $n := 2p+1$ and rank $N = 2p$, choose an orthonormal basis of $\mathbb{R}^{2p+1}$ in which N is written as

$$\left[\begin{array}{ccc} 0 & V & 0 \\ -V & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \tag{3.2}$$

where V is a real diagonal matrix whose entries are $v_1, ..., v_p$.

Now, we compute explicitly the subgroup $G := \mathbb{P}GL_n(\mathbb{C}; N)$ of the projective group $\mathbb{P}GL_n(\mathbb{C})$ formed by matrices which commute with N. In Lemma 1 below, we use the form of N given in (3.1).

Lemma 1. *Let $n = 2p$ and N invertible. Then, the group $GL_n(\mathbb{C}; N)$ of matrices which commute with N is commutative and given by complex matrices P of the form*

$$P = \left[\begin{array}{cc} A & B \\ -B & A \end{array}\right],$$

where $A = \operatorname{diag}(a_1, ..., a_p)$, and $B = \operatorname{diag}(b_1, ..., b_p)$ are diagonal matrices and $\det(P) \neq 0$. Finally, $G := \mathbb{P}GL_n(\mathbb{C}; N) = GL_n(\mathbb{C}; N)/\mathbb{C}^{\times}$, also a commutative group.

Similarly, using the form of N given in (3.2), we get the following result.

Lemma 2. *Let $n = 2p+1$, $\operatorname{rank} N = 2p$, and N has nullity one. Then, the group $GL_n(\mathbb{C}; N)$ of the matrices which commute with N is commutative and given by complex matrices P_1 of the form*

$$P_1 = \left[\begin{array}{ccc} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & \alpha \end{array}\right],$$

where $A = \operatorname{diag}(a_1, ..., a_p)$, and $B = \operatorname{diag}(b_1, ..., b_p)$ are diagonal matrices and $\det(P_1) \neq 0$. Finally, $G := \mathbb{P}GL_n(\mathbb{C}; N) = GL_n(\mathbb{C}; N)/\mathbb{C}^{\times}$, also a commutative group.

The affine subspace $M^{N,Sym}$ is not invariant by the action (by conjugation) of the commutative group G. We shall construct a quotient of the group G, which will act by conjugation on this affine subspace.

Let G_1 be the subgroup of the group G generated by the matrices of the form

$\mathrm{diag}(d_1, ..., d_p, d_1, ..., d_p)$, where $d_k \in \mathbb{C}^\times$, $k = 1, ..., p$, if $n = 2p$ and of the form $\mathrm{diag}(d_1, ..., d_p, d_1, ..., d_p, \alpha)$, where $d_k \in \mathbb{C}^\times$, $k = 1, ..., p$, and $\alpha \in \mathbb{C}^\times$, if $n = 2p + 1$. Denote the quotient G/G_1 by G_0 and observe that this quotient group G_0 is isomorphic to the subgroup of orthogonal matrices ($T^\mathsf{T} T = I_n$) of the group $GL_n(\mathbb{C}; N)$.

Since G_0 acts freely and properly on the isospectral variety $M_Q^{N,Sym}$, it follows that $M_Q^{N,Sym}$ is the total space of a holomorphic principal bundle with base space $M_Q^{N,Sym}/G_0$, structural group G_0, and natural projection map

$$\phi : M_Q^{N,Sym} \longrightarrow M_Q^{N,Sym}/G_0.$$

Recall that the bi-holomorphic map $l : M_Q^N/G \to U$, where U is a Zariski open subset of the usual Jacobian $\mathrm{Jac}(\overline{\Gamma}_\mathbf{c})$ (see [3] (Beauville) and [9] (Gavrilov)) is given by the eigenvector map.

More precisely, let $f(\lambda, z)$ be a normalized eigenvector of the matrix $X(\lambda)$, where (λ, z) is a point on the spectral curve $\overline{\Gamma}_\mathbf{c}$, with eigenvalue z. Then, it defines a line subbundle of the trivial vector bundle $\overline{\Gamma}_\mathbf{c} \times \mathbb{C}^n$, by taking at each point (λ, z) of the spectral curve, the line along the normalized eigenvector $f(\lambda, z)$. Denote its dual by L; then $l(X(\lambda)) = L$ (see [9, page 495] (Gavrilov)).

By Theorem 2.1 in [9] (Gavrilov), the isospectral variety M_Q^N is smooth and bi-holomorphic to a Zariski open subset of the generalized Jacobian variety $\mathrm{Jac}(\Gamma'_\mathbf{c})$, where $\Gamma'_\mathbf{c}$ is the singular curve obtained from $\overline{\Gamma}_\mathbf{c}$ by identifying its points at infinity $\{P_1, \ldots, P_n\}$ with a single point ∞.

The bi-holomorphic map $l' : M_Q^N \to U'$, where U' is a Zariski open subset of the generalized Jacobian variety, is given by a similar eigenvector map (see [9] (Gavrilov)).

Since $(X + \lambda N)^\mathsf{T} = X - \lambda N$, we have $Q(-\lambda, z) = Q(\lambda, z)$, hence there is an involution

$$\tau : \overline{\Gamma}_\mathbf{c} \to \overline{\Gamma}_\mathbf{c}$$

of the spectral curve defined by $\tau(\lambda, z) = (-\lambda, z)$.

The quotient smooth curve $C_1 := \overline{\Gamma}_\mathbf{c}/\tau$ has a double covering $\overline{\Gamma}_\mathbf{c} \to C_1$ and associated to this double covering is the Prym variety $\mathrm{Prym}(\overline{\Gamma}_\mathbf{c}/C_1)$, with the property that $\mathrm{Jac}(\overline{\Gamma}_\mathbf{c})$ is isogenous to (see [5])

$$\mathrm{Jac}(C_1) \times \mathrm{Prym}(\overline{\Gamma}_\mathbf{c}/C_1).$$

We have the following key result (Lemma 3.3 in [6].

Lemma 3. *The natural map* $j : M_Q^{N,Sym}/G_0 \to M_Q^N/G$ *induced by the inclusion* $M_Q^{N,Sym} \hookrightarrow M_Q^N$ *on the quotients is injective. The map* $l \circ j : M_Q^{N,Sym}/G_0 \to \mathrm{Prym}(\overline{\Gamma}_\mathbf{c}/C_1)$ *is injective and maps bi-holomorphically* $M_Q^{N,Sym}/G_0$ *onto an open set of Prym*$(\overline{\Gamma}_\mathbf{c}/C_1)$*.*

As in [12, Section 8] (Reyman - Semenov-Tian-Shannsky), we write for the equivalent Lax equations with parameter (2.2) and (2.3):

$$M_+ := NX + XN + \lambda N^2, \qquad M_- := -X^2/\lambda,$$

and we get

$$M := M_+ - M_- = (X + \lambda N)^2/\lambda.$$

Since $[X+\lambda N, M]=0$, it follows that the eigenvectors of $X+\lambda N$ are the also eigenvectors of M (see [12, Theorem 8.3, page 177] (Reyman - Semenov-Tian-Shannsky)).

Now, since the involution τ acts on M by

$$\tau(M)=-M,$$

it follows for any eigenvalue μ of M, we have $\mu(\tau(P))=-\mu(P)$ for any point P on the spectral curve $\Gamma_{\mathbf{c}}$ (see [12, page 181] (Reyman - Semenov-Tian-Shannsky)).

By [12, Proposition 9.3] (Reyman - Semenov-Tian-Shannsky), it follows that the line bundle, which is the image of $X+\lambda N$ by the map l, belongs to the anti-invariant part of the Jacobian $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$, i.e., it belongs to the Prym variety $\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1)$.

The previous lemma immediately implies the following result, see [6]:

Theorem 4. *(Brînzănescu - Raţiu; 2015) For N having distinct eigenvalues, the system $\tilde{h}_0 : M^{N,Sym}/G_0 \to \mathcal{V}_{Sym}$ is an algebraically completely integrable system.*

Here, $\mathcal{V}_{Sym}$ consists of polynomials in $\mathcal{V}$ which have only even degrees in λ.

The exact sequence (2.3) gives the following exact sequence of commutative groups:

$$0 \longrightarrow G \xrightarrow{i} \Phi^{-1}(\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1)) \longrightarrow \mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1) \longrightarrow 0. \tag{3.3}$$

Let $\beta : G \to G_0$ be the quotient map and

$$\eta : \mathrm{Ext}^1(\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1), G) \longrightarrow \mathrm{Ext}^1(\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1), G_0)$$

the natural map of extensions induced by β.

Now we know that the extension (3.3) gives us the extension

$$\eta(\Phi^{-1}(\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1)))$$

of commutative groups

$$0 \longrightarrow G_0 \longrightarrow E \longrightarrow \mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1) \longrightarrow 0, \tag{3.4}$$

where

$$E := G_0 \oplus \Phi^{-1}(\mathrm{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1))/K,$$

and

$$K := \{(-\beta(g), i(g)) \mid g \in G\}.$$

Applying the result of Gavrilov ([9]) and a similar computation of dimensions, we get the following result.

Theorem 5. *(Brînzănescu - Raţiu; 2015) For N having distinct eigenvalues, the system $h : M^{N,Sym} \to \mathcal{V}_{Sym}$ is a generalized algebraically completely integrable system.*

Here, the generic fiber $M_Q^{N,Sym}$ is bi-holomorphic to a Zariski open set of the commutative algebraic group E.

Recall that the system (1.1) is equivalent to the following Lax pair system with parameter (see [5] (Bloch - Brînzănescu - Iserles - Marsden - Raţiu))

$$\frac{d}{dt}(X+\lambda N)=\left[X+\lambda N, NX+XN+\lambda N^2\right]. \tag{3.5}$$

Recall that we wrote for the equivalent Lax equations with parameter (2.1) and (2.2):

$$M_+ := NX+XN+\lambda N^2, \qquad M_- := -X^2/\lambda,$$

and we get

$$M := M_+ - M_- = (X+\lambda N)^2/\lambda.$$

It is well known that a Lax system with parameter has the solution of the following form (see [12]):

$$X(\lambda,t)=g_+(\lambda,t)X(\lambda,0)g_+(\lambda,t)^{-1},$$

(the same formula for g_-), where $g_+(\lambda,t)$ and $g_-(\lambda,t)$ are matrix-valued functions which solve the Riemann factorisation problem

$$e^{tM(\lambda)}=g_+(\lambda,t)^{-1}g_-(\lambda,t),$$

where $g_+(\lambda,t)$, respectively $g_-(\lambda,t)$ are analytic in $\mathbb{C}P_1-\{\infty\}$, respectively in $\mathbb{C}P_1-\{0\}$ with the normalisation condition $g_-(\infty,t)=Id$.

Explicit expressions for $g_+(\lambda,t)$, $g_-(\lambda,t)$ and for the solutions of the Lax system may be obtain in terms of Baker-Akhiezer functions.

A *Baker-Akhiezer function* for $X(\lambda)$ is its eigenvector

$$\Psi=(\psi_1,...,\psi_n)^T$$

parametized by the spectral curve Γ.

Explicit formulae for the Baker-Akhiezer functions can be obtained in terms of Theta-functions of the Riemann surface associated to the spectral curve.

Of course, general formulae are not easy to describe. We made computations for the particular cases $n=3$ and $n=4$ (see [7]).

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