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Some remarks on the category $\sigma[L]$ of all L-subgenerated lattices

by
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Dedicated to the memory of Professor Doru Ştefănescu (1952-2021)

Abstract

In this paper we present some remarks on the category $\sigma[L]$ of all subgenerated lattices by a modular complete lattice L, defined in [Albu, Dăscălescu, Iosif, Lattices subgenerated by a lattice, and applications (I), in preparation] "somehow" related with the concept of product in L of two lattices, introduced and studied in [Albu, Pérez, Ríos, Prime, irreducible, and completely irreducible lattice preradicals on modular complete lattices, J. Algebra Appl. **21** (2022), 2250097 [33 pages] DOI: 10.1142/S0219498822500979].

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Introduction

The concept of a lattice subgenerated by a modular complete lattice L has been introduced and investigated in [5]. In this paper we relate this concept with the one of product of two lattices, introduced and studied in [4].

In Section 0 we list some definitions and results about lattices, especially from [2].

Section 1 is devoted to the concepts of *trace* and *generators* defined and investigated in [1], [5], [7]. We present a new definition of trace related with the concept of product in L of two lattices, introduced and studied in [4].

In Section 2 we discuss several properties of the category $\sigma[L]$ introduced and investigated in [5], but not covered there, as those of *self-generator* and *fully invariance* in lattices.

Applications of our latticial results to Grothendieck categories and module categories equipped with a hereditary torsion theory will be given in a subsequent paper.

0 Preliminaries

All posets and lattices considered in this paper are assumed to be *bounded,* i.e., to have a least element denoted by 0 and a last element denoted by 1, and L will always denote such a lattice. If the lattices L and L' are isomorphic, we denote this by $L \simeq L'$. We denote by $\mathcal{L}$ (respectively, $\mathcal{M}$, $\mathcal{M}_c$) the class of all bounded (respectively, bounded modular, bounded modular complete) lattices.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{ \ x \in L \mid a \leqslant x \leqslant b \ \}.$$

For basic notation and terminology on lattices the reader is referred to [2], [9], [10], and/or [11], but especially to [2].

Recall from [6] the following concept. A mapping $f : L \longrightarrow L'$ between a lattice L with least element 0 and greatest element 1 and a lattice L' with least element $0'$ and greatest element $1'$ is called a *linear morphism* if there exist $k \in L$, called a *kernel* of f, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \vee k), \forall\, x \in L.$
- f induces a lattice isomorphism

$$\bar{f} : 1/k \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

If $f : L \longrightarrow L'$ is a linear morphism of lattices, then, by [6, Proposition 1.3], f is an increasing mapping, commutes with arbitrary joins (i.e., $f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist), preserves intervals (i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$), and its kernel k is uniquely determined.

As in [6], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$ ($\mathcal{L}$ for "*L*inear" and $\mathcal{M}$ for "*M*odular") if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'. A major property of this category is that the subobjects of an object $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ for any $a \in L$ (see [6, Proposition 2.2(5)]).

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$.

The latticial counterpart of the concept of a fully invariant submodule of a module is that of a fully invariant element introduced in [8] as follows. Let $L \in \mathcal{M}$. An element $a \in L$ is said to be *fully invariant,* abbreviated *FI,* if $f(a) \leqslant a$ for any $f \in \mathrm{End}_{\mathcal{LM}}(L) := \mathrm{Hom}_{\mathcal{LM}}(L, L)$, and the set of all fully invariant elements of L will be denoted by $FI(L)$.

1 Trace and Generators

The aim of this section is to present some results on trace and generators. Thus, we define an apparently different concept of *trace* than the ones in [1] and [7], "somehow" related with the concept of product in L of two lattices, introduced and studied in [4].

Definition 1.1. ([1, Definition 3.1], [5, Definition 1.2]). *A poset L is said to be* generated *by a poset G, or G*-generated, *if for any $a \neq 1$ in L there exist $c \in L$ and $g \in G$ such that $c \nleq a$ and $c/0 \simeq 1/g$.*

One denotes by $\mathrm{Gen}\,(G)$ *the class of all modular complete lattices generated by G.* □

The next concept is a particular case of [7, Definition 3.1] for the *trace* $\mathrm{Tr}\,(\mathcal{X}, L)$ of a nonempty class $\mathcal{X}$ of lattices in a complete lattice L.

Definition 1.2. *For any poset G and any modular complete lattice L we set*

$$\mathrm{Tr}\,(G, L) := \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}\,(G)\},$$

and call it the trace *of G in L.* □

Lemma 1.3. *The following assertions are equivalent for $L, L' \in \mathcal{M}_c$.*

(1) *L is L'-generated.*

(2) $L = \mathrm{Tr}\,(L', L)/0$.

Proof. (1) $\Longrightarrow$ (2) As L is L'-generated, then $L = 1/0 \in \mathrm{Gen}\,(L')$ by Definition 1.1. Now, by Definition 1.2, we have

$$\mathrm{Tr}\,(L', L) = \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}\,(L')\} = 1,$$

so $\mathrm{Tr}\,(L', L)/0 = 1/0 = L$.

(2) $\Longrightarrow$ (1) As $1/0 = L = \mathrm{Tr}\,(L', L)/0$, then $1 = \mathrm{Tr}\,(L', L) = \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}\,(L')\}$. Thus $1 = \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}\,(L')\}$. So $L = \mathrm{Tr}\,(L', L)/0 \in \mathrm{Gen}\,(L')$ by [5, Proposition 1.3], i.e., L is L'-generated. □

Notice that by [5, Proposition 1.3] $\mathrm{Tr}\,(L', L)/0 \in \mathrm{Gen}\,(L')$. Hence $\mathrm{Tr}\,(L', L)/0$ is L'-generated. So by 1.3, $\mathrm{Tr}\,(L', \mathrm{Tr}\,(L', L)/0)/0 = \mathrm{Tr}\,(L', L)/0$ for $L, L' \in \mathcal{M}_c$.

Next, we present another definition of $\mathrm{Tr}\,(-,-)$ for two lattices L, L' in $\mathcal{M}_c$. As usually, we denote by 0 (respectively, $0'$) the least element of L (respectively, L'), and by 1 (respectively, $1'$) the greatest element of L (respectively, L').

For any $L, K \in \mathcal{M}_c$ and $N \in \mathrm{Sub}\,(L)$, where $\mathrm{Sub}\,(L)$ is the collection of all sublattices in $\mathcal{LM}$ of L, we have denoted in [4]

$$N_L K := \alpha^L_N(K),$$

and called it the *product* of N and K in L. If $N = n/0$, we showed there that

$$\alpha^L_N = \bigvee\nolimits_{x \in N} \alpha^L_x = \alpha^L_n.$$

This implies that

$$\alpha^L_N(K) = \alpha^L_n(K) = \big(\bigvee\{f(n) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, K)\}\big)/\,0$$

for all $K \in \mathcal{M}_c$.

In particular, for $N = L = 1/0$ and $K = L' = 1'/0'$, we have

$$\alpha_L^L(L') = \alpha_1^L(L') = \big(\bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}\big)/\,0.$$

Definition 1.4. *For lattices L, L' in $\mathcal{M}_c$, the sublattice*

$$\mathrm{Tr}^*(L', L) = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L)\})/\,0$$

of L is called the trace *of L' in L.* □

As one can see, $\mathrm{Tr}^*(L', L)$ is a sublattice of L, while $\mathrm{Tr}(L', L)$ is an element of L. They are related by Proposition 1.7 below.

Proposition 1.5. *For any two lattices $L', L \in \mathcal{M}_c$, one has $\mathrm{Tr}^*(L', L) = L'_{L'}L$.*

Proof. By Definition 1.4

$$\mathrm{Tr}^*(L', L) = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L)\})/\,0.$$

By [3, Definition 2.9] we have

$$\alpha_{1'}^{L'}(L) = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L)\})/\,0,$$

so $\mathrm{Tr}^*(L', L) = \alpha_{1'}^{L'}(L)$. Now, $\alpha_{1'}^{L'}(L) = \alpha_{L'}^{L'}(L)$ by [4, Notation 2.3], which implies that $\mathrm{Tr}^*(L', L) = \alpha_{L'}^{L'}(L)$. Moreover, by [4, Notation 2.4], we have

$$\alpha_{L'}^{L'}(L) = L'_{L'}L,$$

and so $\mathrm{Tr}^*(L', L) = L'_{L'}L$. □

Definition 1.6. *Let $L \in \mathcal{M}_c$. A lattice $L' \in \mathcal{M}_c$ is said to be* generated* *by L or L-* generated* *if $\mathrm{Tr}^*(L', L) = L$.*

One denotes by $\mathrm{Gen}^*(G)$ the class of all modular complete lattices generated* *by G.* □

Observe that if T is another lattice such that T is L-generated*, then $\mathrm{Tr}^*(T, L) = L$.

Notice also that if L is L'-generated (in the sense of Definition 1.1), then by Lemma 1.3 we have $L = \mathrm{Tr}(L', L)/0$. Thus

$$L \text{ is } L'\text{-generated} \Longleftrightarrow L = \mathrm{Tr}(L', L)/0.$$

Proposition 1.7. $\mathrm{Tr}(L', L)/0 = \mathrm{Tr}^*(L', L)$ *for any lattices $L', L \in \mathcal{M}_c$.*

Proof. Let $t/0 = T = \mathrm{Tr}^*(L', L)$. By Lemma 1.3, T is L'-generated. Thus $T \in \mathrm{Gen}(L')$. As $\mathrm{Tr}(L', L) = \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}(L')\}$, we have $t \leqslant \bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}(L')\}$, so

$$T = t/0 \subseteq (\bigvee\{\, a \in L \mid a/0 \in \mathrm{Gen}(L')\})/0 = \mathrm{Tr}(L', L)/0.$$

Hence $\mathrm{Tr}^*(L', L) = T \subseteq \mathrm{Tr}(L', L)/0$.

If we set $h := \mathrm{Tr}(L', L)$, then $\mathrm{Tr}(L', L)/0 = h/0$. As $t/0 = T \subseteq \mathrm{Tr}(L', L)/0 = h/0$, we have $t \leqslant h$. We are going to prove that $t = h$. Suppose that $t < h$. Then, as we remarked above, we have

$$t/0 = T = \mathrm{Tr}^*(L', L) = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L)\})/\,0 = L'_{L'}L.$$

Thus $t = \bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L)\}$. Because $t < h$, we deduce that

$$\bigvee\{f(1') \mid f \in \operatorname{Hom}_{\mathcal{LM}}(L', L)\} < h$$

By [5, Proposition 1.3], $\operatorname{Tr}(L', L)/0 = h/0$ is L'-generated. As $t < h$, there exist $u \in L'$ and $c \in h/0$ such that $c \nleqslant t$ and $1'/u \simeq c/0$. Since $L', L \in \mathcal{M}_c$, then there exists a linear isomorphism of lattices $1'/u \simeq c/0$, say $\varphi : 1'/u \longrightarrow c/0$, and then $\varphi(1') = c$.

Consider the linear morphism $i \circ \varphi \circ \pi : L' = 1'/0' \longrightarrow L = 1/0$, where π is the surjective linear morphism $\pi : L' = 1'/0' \longrightarrow 1'/u$ and i is the canonical inclusion linear morphism $i : c/0 \hookrightarrow L$. Thus

$$(i \circ \varphi \circ \pi)(1') = (i \circ \varphi)(\pi(1')) = (i \circ \varphi)(1') = i(\varphi(1') = i(c) = c.$$

Since $i \circ \varphi \circ \pi \in \operatorname{Hom}_{\mathcal{LM}}(L', L)$, we have

$$c = (i \circ \varphi \circ \pi)(1') \leqslant \bigvee\{f(1') \mid f \in \operatorname{Hom}_{\mathcal{LM}}(L', L)\} = t,$$

and hence $c \leqslant t$, which is a contradiction. Thus $t = h$, and so

$$\operatorname{Tr}^*(L', L) = T = t/0 = h/0 = \operatorname{Tr}(L', L)/0,$$

as desired. □

Observe that by Proposition 1.7, we have

$$\operatorname{Tr}^*(L', L) = \operatorname{Tr}(L', L)/0 = L.$$

Thus

$$L \text{ is } L'\text{-generated} \iff \operatorname{Tr}^*(L', L) = L.$$

If we use now Definition 1.6, then $\operatorname{Tr}^*(L', L) = L$ implies that L' is L- *generated**.

Thus

$$L' \text{ is } L\text{- generated}^* \iff \operatorname{Tr}^*(L', L) = L.$$

$$\text{As one can see, } L \text{ is } L'\text{-generated} \iff \operatorname{Tr}^*(L', L) = L \iff L' \text{ is } L\text{- generated}^*.$$

We are now going to show that the equality $\operatorname{Gen}(L) = \operatorname{Gen}^*(L)$ is not true in general. To do that, consider the ring $\mathbb{Z}_4$ of rational integers modulo 4 and the category $\mathbb{Z}_4$-Mod.

We claim that $\operatorname{Gen}(\mathbb{Z}_4) \neq \operatorname{Gen}^*(\mathbb{Z}_4)$. First, we show that $2\mathbb{Z}_4 \in \operatorname{Gen}(\mathbb{Z}_4)$. By Proposition 1.8, we have

$$\operatorname{Tr}(\mathbb{Z}_4, 2\mathbb{Z}_4)/0 = \operatorname{Tr}^*(\mathbb{Z}_4, 2\mathbb{Z}_4),$$

$$\operatorname{Tr}^*(\mathbb{Z}_4, 2\mathbb{Z}_4) = \mathbb{Z}_4\, {}_{\mathbb{Z}_4}2\mathbb{Z}_4 = \textstyle\sum_{f \in \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, 2\mathbb{Z}_4)} f(\mathbb{Z}_4),$$

$$\textstyle\sum_{f \in \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, 2\mathbb{Z}_4)} f(\mathbb{Z}_4) = 2\mathbb{Z}_4,$$

so

$$\operatorname{Tr}(\mathbb{Z}_4, 2\mathbb{Z}_4)/0 = \operatorname{Tr}^*(\mathbb{Z}_4, 2\mathbb{Z}_4) = 2\mathbb{Z}_4.$$

Thus $\mathrm{Tr}\,(\mathbb{Z}_4, 2\mathbb{Z}_4)/0 = 2\mathbb{Z}_4$. By Lemma 1.3, $2\mathbb{Z}_4$ is $\mathbb{Z}_4$-generated and so $2\mathbb{Z}_4 \in \mathrm{Gen}\,(\mathbb{Z}_4)$.

We are going to show that $2\mathbb{Z}_4 \notin \mathrm{Gen}^*(\mathbb{Z}_4)$. Suppose not, i.e., $2\mathbb{Z}_4 \in \mathrm{Gen}^*(\mathbb{Z}_4)$. By Definition 1.6, we have

$$\mathrm{Tr}^*(2\mathbb{Z}_4, \mathbb{Z}_4) = \mathbb{Z}_4,$$

and by Proposition 1.5,

$$\mathrm{Tr}^*(2\mathbb{Z}_4, \mathbb{Z}_4) = 2\mathbb{Z}_4\ {}_{2\mathbb{Z}_4}2\mathbb{Z}_4 = \sum_{f \in \mathrm{Hom}_{\mathbb{Z}_4}(2\mathbb{Z}_4, \mathbb{Z}_4)} f(2\mathbb{Z}_4).$$

As $2\mathbb{Z}_4$ is a simple ideal of $\mathbb{Z}_4$, we have

$$\sum_{f \in \mathrm{Hom}_{\mathbb{Z}_4}(2\mathbb{Z}_4, \mathbb{Z}_4)} f(2\mathbb{Z}_4) = 2\mathbb{Z}_4.$$

Thus

$$\mathrm{Tr}^*(2\mathbb{Z}_4, \mathbb{Z}_4) = 2\mathbb{Z}_4,$$

and so

$$\mathbb{Z}_4 = \mathrm{Tr}^*(2\mathbb{Z}_4, \mathbb{Z}_4) = 2\mathbb{Z}_4,$$

which is a contradiction. Therefore $2\mathbb{Z}_4 \notin \mathrm{Gen}^*(\mathbb{Z}_4)$. This proves that $\mathrm{Gen}\,(\mathbb{Z}_4) \neq \mathrm{Gen}^*(\mathbb{Z}_4)$, as claimed.

Proposition 1.8. *Let L, L' in $\mathcal{M}_c$. If for every non-zero linear morphism $f : L' \longrightarrow Y$ there exists a linear morphism $g : L \longrightarrow L'$ with $f \circ g \neq 0$, then L' is L-generated.*

Proof. By Lemma 1.3, L' is L-generated $\Longleftrightarrow$ $L' = \mathrm{Tr}\,(L, L')/0'$. Now, $\mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^*(L, L')$ by Proposition 1.7, and $\mathrm{Tr}^*(L, L') = L_L L'$ as we observed just a line above Definition 1.5. Thus L' is L-generated $\Longleftrightarrow$ $L' = L_L L'$. Consequently, it is enough to prove that $L' = L_L L'$.

Set $N := L_L L' = N = n/0'$. Then $n/0' = N \subseteq L' = 1'/0'$, so $n \leqslant 1'$. If $n = 1'$, then $N = n/0' = 1'/0' = L'$, i.e., $L_L L' = N = L'$.

Now, assume that $n < 1'$, and consider the surjective linear epimorphism

$$p : L' = 1'/0' \longrightarrow 1'/n,\ p(a) = a \vee n,\ a \in L'.$$

Since $n < 1'$, p is a non-zero linear morphism. By hypothesis there exists a linear morphism $g : L \longrightarrow L'$ such that $p \circ g \neq 0$. As $p \circ g : L \longrightarrow 1'/n$, there exists $l \in L$ with $(p \circ g)(l) > n$. But $(p \circ g)(l) = p(g(l)) = g(l) \vee n$. Thus $g(l) \vee n > n$, and then $g(l) \leqslant g(1)$ since g is an increasing mapping.

Remember that

$$n/0' = N = L_L L' = \bigvee(\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\})/0',$$

so $n = \bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}$. Hence $n \geqslant f(1)$ for all $f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')$. As $g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')$, we have $n \geqslant g(1)$. Since $g(1) \geqslant g(l)$, we deduce that $n \geqslant g(l)$. So $g(l) \vee n = n$. On the other hand, we have seen above that $g(l) \vee n > n$, which is a contradiction. Consequently, necessarily $n = 1'$, and then $L_L L' = N = L'$. This finishes the proof. □

We present below a new definition, that is a reformulation of [12, Proposition 13.5(2)] showing that if U and L' are two right R-modules, then

$$L' \text{ is } U\text{-generated if and only if } \mathrm{Tr}\,(U, L') = L' \ .$$

Definition 1.9. *A lattice* $L' \in \mathcal{M}_c$ *is said to be* generated** *by* L *or* L generated** *for a lattice* $L \in \mathcal{M}_c$ *if for every non-zero linear morphism* $f : L' \longrightarrow Y$ *there exists a linear morphism* $g : L \longrightarrow L'$ *with* $f \circ g \neq 0$. □

One denotes by $\mathrm{Gen}^{**}(G)$ the class of all modular complete lattices *generated*** by G.

Proposition 1.10. *With the notation above,* L' *is* L-generated *if and only if it is* L-generated**.

Proof. Suppose L' is L-generated**. Then, by Proposition 1.8 we deduce that L' is L-generated.

Conversely, assume that L' is L-generated. Then, $L' = \mathrm{Tr}\,(L, L')/0'$ by Lemma 1.3, so

$$L' = \mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^*(L, L').$$

Then $L_L L' = \mathrm{Tr}^*(L, L')$ by Proposition 1.5. It follows that

$$L' = \mathrm{Tr}^*(L, L') = L_L L'.$$

Now, let $f : L' \longrightarrow Y = 1''/0''$ be a non-zero linear morphism. Then

$$L_L L' = \mathrm{Tr}^*(L, L') = (\bigvee\{g(1) \mid g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\})/\,0',$$

and so

$$1'/0' = L' = L_L L' = (\bigvee\{g(1) \mid g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\})/\,0'.$$

We deduce that $1' = \bigvee\{g(1) \mid g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}$, and because $f : L' \longrightarrow Y = 1''/0''$ is a non-zero linear morphism, we have

$$0'' \neq f(1') = f(\bigvee\{g(1) \mid g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}) = \bigvee\{f(g(1)) \mid g \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\},$$

so there exists a linear morphism $g : L \longrightarrow L'$ with $f(g(1)) \neq 0''$. Then $f \circ g$ is a non-zero linear morphism. This proves that L' is L- *generated***, as desired. □

Lemma 1.11. *Let* $1/0 = L, L' \in \mathcal{M}_c$. *If* $T =\ \mathrm{Tr}^*(L', L)$, *then* T *es* L'*-generated.*

Proof. As above, $T =\ \mathrm{Tr}^*(L', L) = L'_{L'} L$, so, by product of lattices we have

$$t/0 = T = L'_{L'}L = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0,$$

hence $t = \bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\}$.

Let $a \in T, a \neq t$. Then $a < t = \bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\}$, so there exists $f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\}$ with $f(1') \nleqslant a$. As $f : 1'/0' = L' \longrightarrow L = 1/0$ is a linear morphism, there exists $k \in 1'/0'$ such that k is the kernel of f. If $c := f(1')$, then f induces a lattice isomorphism

$$\bar{f} : 1'/k \xrightarrow{\sim} c/0,\ \bar{f}(x) = f(x),\ \forall\, x \in 1'/k, \text{ and so}$$

$c/0 \simeq 1'/k$, i.e., T is L'-generated by Definition 1.1, as desired. □

Proposition 1.12. *The lattice* $\mathrm{Tr}\,(L',L)/0$ *is* L'*-generated.*

Proof. By Proposition 1.7, $\mathrm{Tr}\,(L',L)/0 = \mathrm{Tr}^*(L',L)$, and $\mathrm{Tr}^*(L',L)$ is L'-generated by Lemma 1.11, so $\mathrm{Tr}\,(L',L)/0$ is L'-generated. □

Corollary 1.13. *The following assertions hold for* $L, L' \in \mathcal{M}_c$.

(1) $\mathrm{Tr}^*(L',L)$ *is the largest sublattice of* L *generated by* L'.

(2) $\mathrm{Tr}^*(L',L)$ *is fully invariant in* L.

Proof. (1) Since $\mathrm{Tr}\,(L',L) = \bigvee\{\, a \in L \,|\, a/0 \in \mathrm{Gen}\,(L')\}$, by [5, Proposition 1.3] we have $\mathrm{Tr}\,(L',L)/0 \in \mathrm{Gen}\,(L')$. Hence $\mathrm{Tr}\,(L',L)/0$ is L'-generated. Now, $\mathrm{Tr}\,(L',L)/0 = \mathrm{Tr}^*(L',L)$ by Proposition 1.7, so $\mathrm{Tr}^*(L',L)$ is L'-generated.

Let $T = t/0$ be a sublattice of L such that T is L'-generated, i.e., $T = t/0 \in \mathrm{Gen}\,(L')$. Then

$$t \leqslant \bigvee\{\, a \in L \,|\, a/0 \in \mathrm{Gen}\,(L')\} = \mathrm{Tr}\,(L',L).$$

Therefore $T = t/0 \subseteq \mathrm{Tr}\,(L',L)/0 = \mathrm{Tr}^*(L',L)$. Thus $\mathrm{Tr}^*(L',L)$ is the largest sublattice of L generated by L'.

(2) Let $g : L \longrightarrow L$ be a linear morphism. Then

$$g(\mathrm{Tr}^*(L',L)) = g(\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0).$$

As g commutes with arbitrary joins, we have

$$g(\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0) = \bigvee(\{(g \circ f)(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0.$$

Because $g \circ f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)$ we have

$$\bigvee(\{(g \circ f)(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0 \subseteq \bigvee(\{h(1') \mid h \in \mathrm{Hom}_{\mathcal{LM}}(L',L)\})/0,$$

and then $g(\mathrm{Tr}^*(L',L)) \subseteq \mathrm{Tr}^*(L',L))$, so $\mathrm{Tr}^*(L',L)$ is fully invariant in L, as desired. □

Proposition 1.14. *Let* $L, L' \in \mathcal{M}_c$ *with* $L' \in \mathrm{Gen}^{**}(L)$. *Then* $\mathrm{Gen}^{**}(L') \subseteq \mathrm{Gen}^{**}(L)$.

Proof. Let $1''/0'' = L'' \in \mathrm{Gen}^{**}(L')$, so L'' is L'-generated by Proposition 1.10, and then $L'' = \mathrm{Tr}\,(L', L'')/0''$ by Lemma 1.3. Further, $\mathrm{Tr}\,(L', L'')/0' = \mathrm{Tr}^{*}(L', L'')$, and by Proposition 1.5 we have $\mathrm{Tr}^{*}(L', L'') = L'_{L'}L''$. Thus

$$L'' = \mathrm{Tr}\,(L', L'')/0'' = \mathrm{Tr}^{*}(L', L'') = L'_{L'}L''.$$

So $L'' = L'_{L'}L''$. Moreover, by Definition 1.4, we have

$$L'_{L'}L'' = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L'')\})/\,0''.$$

Notice that if $f \in \mathrm{Hom}_{\mathcal{LM}}(L', L'')$ then $f(L') = f(1'/0') = f(1')/f(0') = f(1')/0''$, so

$$\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} f(L') = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} (f(1')/0'').$$

By Notation [4, 2.4], we have

$$\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} (f(1')/0'') = (\bigvee\{f(1') \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L', L'')\})/\,0'' = L'_{L'}L''.$$

Thus $\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} f(L') = L'_{L'}L''$.

Since $L' \in \mathrm{Gen}^{**}(L)$, then by Proposition 1.10 we deduce that L' is L-generated. So, by Lemma 1.3, $L' = \mathrm{Tr}\,(L, L')/0'$, and by Proposition 1.7 $\mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^{*}(L, L')$. Now, by Proposition 1.5 we have $\mathrm{Tr}^{*}(L, L') = L_L L'$. Thus

$$L' = \mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^{*}(L, L') = L_L L'.$$

On the other hand, by Definition 1.4 we have $L' = L_L L' = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} h(L)$.

If $f \in \mathrm{Hom}_{\mathcal{LM}}(L', L'')$, then

$$f(L') = f(L_L L') = f(\bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} h(L)) = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} f(h(L)),$$

and so $f(L') = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} (f \circ h)(L)$. Since $f \circ h \in \mathrm{Hom}_{\mathcal{LM}}(L, L'')$, it follows that

$$f(L') = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} (f \circ h)(L) \subseteq \bigvee_{g\in \mathrm{Hom}_{\mathcal{LM}}(L,L'')} g(L).$$

Moreover, by Definition 1.4 we have $\bigvee_{g\in \mathrm{Hom}_{\mathcal{LM}}(L,L'')} g(L) = L_L L''$. Thus $f(L') \subseteq L_L L''$ for all $f \in \mathrm{Hom}_{\mathcal{LM}}(L', L'')$. We deduce that

$$\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} f(L') \subseteq L_L L''.$$

Since $L'' = L'_{L'}L'' = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L',L'')} f(L')$, then

$$L'' = L'_{L'}L'' \subseteq L_L L'' \subseteq L''.$$

Thus $L'' = L_L L''$. By Proposition 1.5, we have $\mathrm{Tr}^{*}(L, L'') = L_L L'' = L''$. So, by Lemma 1.11, we deduce that L'' is L-generated, and by Proposition 1.10 it follows that L'' is L-generated**. Therefore $L'' \in \mathrm{Gen}^{**}(L)$, as desired. □

Proposition 1.15. *Let* $L, L' \in \mathcal{M}_c$. *If* $L' \in \mathrm{Gen}^{**}(L)$, *then* $L'/Y \in \mathrm{Gen}^{**}(L)$ *for all* Y *in* $\mathrm{Sub}\,(L')$.

Proof. Let $Y = y/0' \in \mathrm{Sub}\,(L')$. Then $L_L(L'/Y) \subseteq L'/Y$ by the definition of product of lattices. Consider the linear epimorphism

$$p : L' \longrightarrow L'/Y,\ p(x) = x \vee y,\ \forall\, x \in L'.$$

Then $p(L') = L'/Y$. Since L' is *L- generated***, then L' is L-generated by Proposition 1.10, so $L' = \mathrm{Tr}\,(L, L')/0'$ by Lemma 1.3. It follows that $\mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^{\,*}(L, L')$ by Proposition 1.7. Now, by Proposition 1.5 $\mathrm{Tr}^{\,*}(L, L') = L_L L'$, we have

$$L' = \mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^{\,*}(L, L') = L_L L'.$$

On the other hand, $L_L L' = \bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} f(L)$ by Definition 1.4, so

$$L' = \bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} f(L).$$

Thus

$$p(L') = p(L_L L') = p(\bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} f(L)) = \bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} (p \circ f)(L).$$

Clearly $p \circ f \in \mathrm{Hom}_{\,\mathcal{LM}}(L, L'/Y)$ for all $f \in \mathrm{Hom}_{\,\mathcal{LM}}(L, L')$, so

$$(p \circ f)(L) \subseteq \bigvee_{g \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L'/Y)} g(L) \text{ for all } f \in \mathrm{Hom}_{\,\mathcal{LM}}(L, L').$$

Thus

$$\bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} (p \circ f)(L) \subseteq \bigvee_{g \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L'/Y)} g(L).$$

Observe that $\bigvee_{g \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L'/Y)} g(L) = L_L(L'/Y)$.

$$L'/Y = p(L') = \bigvee_{f \in \mathrm{Hom}_{\,\mathcal{LM}}(L,L')} (p \circ f)(L) \subseteq L_L(L'/Y).$$

So $L_L(L'/Y) = L'/Y$. Apply now Proposition 1.5, Proposition 1.7, and Proposition 1.10 to deduce that L'/Y is L-generated**, i.e., $L'/Y \in \mathrm{Gen}^{\,**}(L)$, which finishes the proof. □

Lemma 1.16. *Let $L \in \mathcal{M}_c$, and let $(X_i)_{i \in I}$ be a family in* Sub (L). *If $Y \in$* Sub (L) *and $X_i \subseteq Y$ for all $i \in I$, then $\bigvee_{i \in I} X_i \subseteq Y$.*

Proof. Let $X_i = x_i/0$ for each $i \in I$, and let $Y = y/0$. As $X_i \subseteq Y$, we have $x_i/0 \subseteq y/0$, so $x_i \leqslant y$ for all $i \in I$. Thus $\bigvee x_i \leqslant y$, and then $(\bigvee x_i)/0 \leqslant y/0 = Y$. By [4, Section 2], we have $\bigvee_{i \in I} X_i = (\bigvee x_i)/0$. Consequently $\bigvee_{i \in I} X_i \subseteq Y$, as desired. □

Proposition 1.17. *Let $L, L' \in \mathcal{M}_c$. If $(X_i)_{i \in I}$ is a family in* Sub (L') *and $X_i \in \mathrm{Gen}^{\,**}(L)$ for all $i \in I$, then $\bigvee_{i \in I} X_i \in \mathrm{Gen}^{\,**}(L)$.*

Proof. Set $X := \bigvee_{i \in I} X_i$. Then $L_L X = L_L(\bigvee_{i \in I} X_i) \subseteq \bigvee_{i \in I} X_i$. As $X_i \subseteq X$, then by [4, Proposition 2.7], $L_L X_i \subseteq L_L X$. By Proposition 1.10, X_i is L-generated, so by Lemma 1.3, Proposition 1.7, and Proposition 1.5 we deduce that $X_i = L_L X_i$ for all $i \in I$. Thus $X_i = L_L X_i \subseteq L_L X$ for all $i \in I$. Now $\bigvee_{i \in I} X_i \subseteq L_L X$ by Lemma 1.16, and then $X = \bigvee_{i \in I} X_i = L_L X$. Proposition 1.7, Lemma 1.3, and Proposition 1.10 imply that X is *L- generated***, i.e., $\bigvee_{i \in I} X_i = X \in \mathrm{Gen}^{\,**}(L)$, as desired. □

Lemma 1.18. *Let $L, L' \in \mathcal{M}_c$. Then L' is L-generated** $\Longleftrightarrow L_L L' = L'$.*

Proof. Observe that

$$L' \text{ is } L\text{-generated}^{**} \iff L' \text{ is } L\text{-generated}$$

by Proposition 1.10, and

$$L' \text{ is } L\text{-generated} \iff L' = \mathrm{Tr}\,(L, L')/0'$$

by Lemma 1.3, so

$$L' \text{ is } L\text{-generated}^{**} \iff L' = \mathrm{Tr}\,(L, L')/0'.$$

On the other hand, $L' = \mathrm{Tr}\,(L, L')/0' = \mathrm{Tr}^*(L, L')$ by Proposition 1.7, and $\mathrm{Tr}^*(L, L') = L_L L'$ by Proposition 1.5. Thus

$$L' \text{ is } L\text{-generated}^{**} \iff L_L L' = L',$$

and we are done. □

Proposition 1.19. *Let $L, X \in \mathcal{M}_c$. Then*

$$\bigvee\{\, Y \in \mathrm{Sub}\,(X) \mid Y \in \mathrm{Gen}^{**}(L)\} = \mathrm{Tr}^*(L, X).$$

Proof. By Proposition 1.5, $\mathrm{Tr}^*(L, X) = L_L X$. Let $Y \in \mathrm{Sub}\,(X)$. If $Y \in \mathrm{Gen}^{**}(L)$, then Y is L-generated**, and $L_L Y = Y$ by Lemma 1.18. As $Y \subseteq X$, then by [4, Proposition 2.7] we have $L_L Y \subseteq L_L X$. Thus $L_L Y \subseteq L_L X = \mathrm{Tr}^*(L, X)$, so $Y = L_L Y \subseteq \mathrm{Tr}^*(L, X)$ for all $Y \in \mathrm{Sub}\,(X)$. Since $Y \in \mathrm{Gen}^{**}(L)$, then by Lemma 1.16,

$$\bigvee\{\, Y \in \mathrm{Sub}\,(X) \mid Y \in \mathrm{Gen}^{**}(L)\} \subseteq \mathrm{Tr}^*(L, X).$$

Observe that $\mathrm{Tr}^*(L, X)$ is L-generated by Lemma 1.11, and $\mathrm{Tr}^*(L, X)$ is L-generated** by Proposition 1.10, i.e., $\mathrm{Tr}^*(L, X) \in \mathrm{Gen}^{**}(L)$. Since $\mathrm{Tr}^*(L, X) = L_L X \subseteq X$, then $\mathrm{Tr}^*(L, X) \in \mathrm{Sub}\,(X)$. Hence

$$\mathrm{Tr}^*(L, X) \subseteq \bigvee\{\, Y \in \mathrm{Sub}\,(X) \mid Y \in \mathrm{Gen}^{**}(L)\}.$$

Thus $\bigvee\{\, Y \in \mathrm{Sub}\,(X) \mid Y \in \mathrm{Gen}^{**}(L)\} = \mathrm{Tr}^*(L, X)$, and we are done. □

Lemma 1.20. *The following statements hold for $L, L', L'' \in \mathcal{M}_c$ and $1'/0' = L' \in \mathrm{Gen}^{**}(L)$.*

(1) *If $L' \simeq L''$ in $\mathcal{LM}$, then $L'' \in \mathrm{Gen}^{**}(L)$.*

(2) *If $a' \in L'$ then $1'/a' \in \mathrm{Gen}^{**}(L)$.*

Proof. (1) We are going to prove that $L'' = L_L L''$. By definition of the product of lattices we have $L_L L'' \subseteq L''$. By hypothesis, there exists a linear isomorphism $g : L' \xrightarrow{\sim} L''$, so $g(L') = L''$.

As $L' \in \mathrm{Gen}^{**}(L)$, then $L_L L' = L'$ by Lemma 1.18, and

$$L' = L_L L' = \big(\bigvee\{\, f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}\big)/\,0' = \bigvee\nolimits_{f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')} f(L)$$

by [4, Notation 2.4].

Thus

$$L'' = g(L') = g(\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} f(L)) = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')}(g\circ f)(L),$$

where $g\circ f : L \longrightarrow L''$, so $g\circ f \in \mathrm{Hom}_{\mathcal{LM}}(L, L'')$.

Since $L_L L'' = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L'')} h(L)$, we have

$$L'' = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')}(g\circ f)(L) \subseteq \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,L'')} h(L) = L_L L''.$$

Therefore $L'' \subseteq L_L L''$, so $L'' = L_L L''$. Now, by Lemma 1.18 it follows that L'' is L-generated**, and then $L'' \in \mathrm{Gen}^{**}(L)$.

(2) We are now going to prove that $1'/a' = L_L(1'/a')$. By the definition of product of two lattices we have $L_L(1'/a') \subseteq 1'/a'$.

Consider the surjective linear morphism

$$p : L' \longrightarrow 1'/a',\ p(x) = x \vee a',\ \forall\, x \in L'.$$

Then, as $L' \in \mathrm{Gen}^{**}(L)$, we have $L_L L' = L'$ by Lemma 1.18. On the other hand

$$L' = L_L L' = \big(\bigvee\{ f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}\big)/\,0' = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} f(L)$$

by [4, Notation 2.4].

As p is a linear epimorphism we have $p(L') = 1'/a'$, so

$$1'/a' = p(L') = p(\bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} f(L)) = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')}(p\circ f)(L),$$

where $p\circ f : L \longrightarrow 1'/a'$, so $p\circ f \in \mathrm{Hom}_{\mathcal{LM}}(L, (1'/a'))$.

Since $L_L(1'/a') = \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,(1'/a'))} h(L)$, we deduce that

$$1'/a' = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')}(p\circ f)(L) \subseteq \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,(1'/a'))} h(L) = L_L(1'/a').$$

Therefore $1'/a' \subseteq L_L(1'/a')$, so $1'/a' = L_L(1'/a')$. By Lemma 1.18 it follows that $1'/a'$ is L-generated**. Thus $1'/a' \in \mathrm{Gen}^{**}(L)$, as desired. □

Proposition 1.21. *The following assertions are equivalent for $L, L' \in \mathcal{M}_c$.*

(1) $L' \in \mathrm{Gen}^{**}(L)$.

(2) *There exists a family $(X_i)_{i\in I}$ in* $\mathrm{Sub}\,(L')$*, such that $L' = \bigvee_{i\in I} X_i$ and X_i is a linear homomorphic image of L.*

Proof. (1) $\Longrightarrow$ (2). Assume that $L' \in \mathrm{Gen}^{**}(L)$. Then $L' = L_L L'$ by Lemma 1.18, and by [4, Notation 2.4] we have

$$L' = L_L L' = \big(\bigvee\{ f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')\}\big)/\,0'.$$

Therefore

$$L' = \bigvee_{f\in \mathrm{Hom}_{\mathcal{LM}}(L,L')} f(L).$$

For each $f \in \mathrm{Hom}_{\mathcal{LM}}(L, L')$ we set $X_f := f(L)$ and $I := \mathrm{Hom}_{\mathcal{LM}}(L, L')$. Thus X_f is a linear homomorphic image of L for all $f \in I$, and so

$$L' = \bigvee_{f \in I} X_f,$$

which proves (2).

(2) $\Longrightarrow$ (1) Suppose that $L' = \bigvee_{i \in I} X_i$ and X_i is a linear homomorphic image of L for every $i \in I$, i.e., there exist linear epimorphisms $f_i : L \longrightarrow X_i$ for every $i \in I$. If k_i is the kernel of f_i, then f_i induces a lattice isomorphism

$$\bar{f}_i : 1/k_i \xrightarrow{\sim} f(1)/f(k_i),\ \bar{f}_i(x) = f_i(x),\ \forall\, x \in 1/k_i.$$

Since f_i is linear epimorphism, then $f_i(L) = X_i$. Thus $X_i = f_i(1)/f_i(k_i)$. Hence we deduce that $f_i(L) = X_i \simeq 1/k_i$. As $L = L_L L$, then Lemma 1.18 and Proposition 1.10 imply that L is L-*generated***. As $k_i \in L$, then $1/k_i$ is L-*generated*** by Lemma 1.20, and so X_i is L-*generated*** for all $i \in I$ by Lemma 1.20.

Since $X_i \subseteq L'$ for all $i \in I$, then by Proposition 1.17 $L' = \bigvee_{i \in I} X_i$ is L-*generated***, i.e., $L' \in \mathrm{Gen}^{**}(L)$, and we are done. □

Proposition 1.22. $\mathrm{Gen}\,(L) = \mathrm{Gen}^{**}(L)$ *for any* $L \in \mathcal{M}_c$.

Proof. The result follows immediately from Proposition 1.10. □

2 The category $\sigma[L]$

Definition 2.1. *For any lattice* $L \in \mathcal{M}_c$ *we denote by* $\sigma[L]$ *the full subcategory of* $\mathcal{M}_c$ *that contains all lattices* $L' \in \mathcal{M}_c$ *such that* L' *is linearly isomorphic to a sublattice of an* L-generated *lattice* (*or equivalently,* L-generated** *by Proposition* 1.10). □

Lemma 2.2. *Let* $L, L' \in \mathcal{M}_c$. *If* $h : L \longrightarrow L'$ *is a linear isomorphism and* K *is a sublattice of* L, *then* L/K *and* $L'/h(K)$ *are* (*linearly*) *isomorphic.*

Proof. Let $K = k/0$. As h is a linear morphism, then h preserves intervals by [7, Lemma 0.6], so $h(1/k) = h(1)/h(k)$, $h(1) = 1'$, and $h(1/k) = 1'/h(k)$.

Since $L/K = 1/k$, $L'/h(K) = 1'/h(k)$, and h is a linear isomorphism, then L/K and $L'/h(K)$ are (linearly) isomorphic, as desired. □

Proposition 2.3. *The category* $\sigma[L]$ *is closed under sublattices and quotient linear epimorphisms for any* $L \in \mathcal{M}_c$.

Proof. First, we are going to prove that $\sigma[L]$ is closed under sublattices. Let $L' \in \sigma[L]$ and $L'' \subseteq L'$. Then, by the definition of $\sigma[L]$, we have $L' \simeq T'$, where T' is a sublattice of an L-generated lattice T. Then L'' is isomorphic to a sublattice of T'. Since T' is a sublattice of T, we deduce that L'' is isomorphic to a sublattice of T, which implies that $L'' \in \sigma[L]$, so $\sigma[L]$ is closed under sublattices.

We are now going to show that $\sigma[L]$ is closed under quotient linear epimorphisms. Let $L' \in \sigma[L]$ and $L'' \subseteq L'$. As $L' \in \sigma[L]$, then $L' \simeq T'$ by the definition of $\sigma[L]$, where T' is a sublattice of an L-generated lattice T, so, L'' is isomorphic to sublattice T'' of T', and $L'/L'' \simeq T'/T''$ by Lemma 2.2, As T'' is sublattice of T' and T' is a sublattice of T, we deduce that T'' is sublattice of T, and so $T'/T'' \subseteq T/T''$.

On the other hand, $\mathrm{Gen}^{**}(L) = \mathrm{Gen}(L)$ by Proposition 1.22, so $T \in \mathrm{Gen}(L)$, and then $T \in \mathrm{Gen}^{**}(L)$. Now, $T/T'' \in \mathrm{Gen}^{**}(L)$ by Proposition 1.15. Thus $T/T'' \in \mathrm{Gen}(L)$, so T/T'' is L-*generated.* Since $T'/T'' \subseteq T/T''$ and $L'/L'' \simeq T'/T''$, it follows that $L'/L'' \in \sigma[L]$, as desired. ∎

Lemma 2.4. *If* $\mathrm{Gen}[L] \subseteq \mathrm{Gen}[H]$ *for* $L, H \in \mathcal{M}_c$*, then* $\sigma[L] \subseteq \sigma[H]$*.*

Proof. Let $L' \in \sigma[L]$ and $L'' \subseteq L'$. As $L' \in \sigma[L]$, then by the definition of $\sigma[L]$, we have $L' \simeq T'$, where T' is a sublattice of an L-generated lattice T. Thus $T \in \mathrm{Gen}[L]$. By hypothesis, we have $\mathrm{Gen}[L] \subseteq \mathrm{Gen}[H]$, so $T \in \mathrm{Gen}[H]$, hence T is H-generated. Since $L' \simeq T' \subseteq T$, we have $L' \in \sigma[H]$, which shows that $\sigma[L] \subseteq \sigma[H]$. ∎

Proposition 2.5. *Let* $L, L' \in \mathcal{M}_c$*. If* L' *is* L*-generated* (*or equivalently,* L-generated**), *then* $\sigma[L'] \subseteq \sigma[L]$*.*

Proof. As L' is L-*generated*, $L' \in \mathrm{Gen}[L]$. Then $L' \in \mathrm{Gen}[L]^{**}$ by Proposition 1.22. Now, $\mathrm{Gen}[L']^{**} \subseteq \mathrm{Gen}[L]^{**}$ by Proposition 1.14. Further, by Proposition 1.22, we have $\mathrm{Gen}[L'] \subseteq \mathrm{Gen}[L]$, and $\sigma[L'] \subseteq \sigma[L]$ by Lemma 2.4. ∎

Corollary 2.6. *Let* $L \in \mathcal{M}_c$ *and* $T = t/0 \in \mathrm{Sub}(L)$*. Then* $\sigma[1/t] \subseteq \sigma[L]$*.*

Proof. As $t/0 = T \subseteq L$, then $t \in L$. By Lemma 1.20, we have $1/t \in \mathrm{Gen}[L]^{**}$, so, by Proposition 1.22, we deduce that $1/t \in \mathrm{Gen}[L]$. Now, by Proposition 2.5, we have $\sigma[1/t] \subseteq \sigma[L]$. ∎

Proposition 2.7. *Let* $L, T \in \mathcal{M}_c$ *and let* $(T_i)_{i\in I}$ *be a family in* $\mathrm{Sub}(T)$*. If* $T_i \in \mathrm{Gen}[L]$ *for all* $i \in I$*, then* $\bigvee_{i\in I} T_i \in \sigma[L]$*.*

Proof. Set $T' := \bigvee_{i\in I} T_i$. We claim that $L_L T' = T'$. Indeed, by the definition of product of lattices, we have $L_L T' \subseteq T'$. Since $T_i \in \mathrm{Gen}[L]$ for all $i \in I$, then by Lemma 1.18, we deduce that $L_L T_i = T_i$ for all $i \in I$. As $T_i \subseteq T'$ for all $i \in I$, then, by [4, Proposition 2.7], we have

$$T_i = L_L T_i \subseteq L_L T' \subseteq T' \text{ for all } i \in I.$$

Thus $T_i \subseteq T'$ for all $i \in I$, so we deduce that

$$T' = \bigvee_{i\in I} T_i \subseteq L_L T' \subseteq T'.$$

Therefore $L_L T' = T'$, as claimed.

By Lemma 1.18, it follows that T' is L-generated, and so $\bigvee_{i\in I} T_i = T' \in \mathrm{Gen}\,(L)$. Now, $\mathrm{Gen}\,(L) \subseteq \sigma[L]$ by the definition of $\sigma[L]$. Consequently $T' = \bigvee_{i\in I} T_i \in \sigma[L]$, and we are done. □

Definition 2.8. *A lattice $L \in \mathcal{M}_c$ is said to be a* self-generator *if it generates all its sublattices, and a lattice $L' \in \sigma[L]$ is called a* generator *in $\sigma[L]$ if L' generates all lattices in $\sigma[L]$.* □

Proposition 2.9. *The following assertions are equivalent for $L \in \mathcal{M}_c$ and $X \in \mathrm{Sub}\,(L)$.*

(1) *L is a self-generator.*

(2) *The mapping $X \longmapsto \mathrm{Hom}_{\mathcal{LM}}(L, X)$ from the set $\mathrm{Sub}\,(L)$ of all sublattices of L to the set of all subsets of $\mathrm{End}_{\mathcal{LM}}(L)$ is injective.*

Proof. 1) $\Longrightarrow$ 2) Let $X, Y \in \mathrm{Sub}\,(L)$ be such that $\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, Y)$. Since L is a self-generator, then X and Y are L-generated. By Proposition 1.10, we deduce that X and Y are L-generated**, and by Lemma 1.18, it follows that $L_L X = X$ and $L_L Y = Y$.

On the other hand, we have

$$X = L_L X = \big(\bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, X)\}\big)/\,0$$

and

$$Y = L_L Y = \big(\bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, Y)\}\big)/\,0.$$

Since $\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, Y)$, we have

$$X = L_L X = L_L Y = Y,$$

which shows that the map $X \longmapsto \mathrm{Hom}_{\mathcal{LM}}(L, X)$ is injective.

2) $\Longrightarrow$ 1) Let $X \in \mathrm{Sub}\,(L)$. By Lemma 1.11, it follows that $L_L X$ is L-generated. Set $L_L X := X'$, so by Lemma 1.18, we deduce that X' is L-generated**. Again by Lemma 1.18, we have $L_L X' = X'$.

We claim that $\mathrm{Hom}_{\mathcal{LM}}(L, X') = \mathrm{Hom}_{\mathcal{LM}}(L, X)$. Indeed, $X' = L_L X \subseteq X$. Let $f \in \mathrm{Hom}_{\mathcal{LM}}(L, X')$. Then $f \in \mathrm{Hom}_{\mathcal{LM}}(L, X)$, so $\mathrm{Hom}_{\mathcal{LM}}(L, X') \subseteq \mathrm{Hom}_{\mathcal{LM}}(L, X)$. For $g \in \mathrm{Hom}_{\mathcal{LM}}(L, X)$ we have

$$g(L) \subseteq \bigvee_{h\in \mathrm{Hom}_{\mathcal{LM}}(L,X)} h(L) = L_L X = X',$$

and then $g \in \mathrm{Hom}_{\mathcal{LM}}(L, X')$, which implies that

$$\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, X'),$$

as claimed.

Since the map $X \longmapsto \mathrm{Hom}_{\mathcal{LM}}(L, X)$ is injective, it follows that $X = X'$. Thus $X = X' = L_L X$. So, by Lemma 1.18, X is L-*generated***. Now, by Proposition 1.10, we deduce that X is L-*generated*. Thus L is a self-generator. □

For $L \in \mathcal{M}_c$ and $X \in \sigma[L]$ we set $\psi(X) := \mathrm{Hom}_{\mathcal{LM}}(L, X)$.

Proposition 2.10. *The following assertions are equivalent for $L \in \mathcal{M}_c$.*

(1) *L is a generator in $\sigma[L]$.*

(2) *The map $\psi : \sigma[L] \longrightarrow \{\mathrm{Hom}_{\mathcal{LM}}(L, X) \mid X \in \sigma[L]\}$ is injective.*

Proof. 1) $\Longrightarrow$ 2) Let $X, Y \in \sigma[L]$ be such that $\psi(X) = \psi(Y)$, i.e., $\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, Y)$. Because L is a generator in $\sigma[L]$, then X and Y are L-generated, so L-generated** by Proposition 1.10, and by Lemma 1.18, we deduce that $L_L X = X$ and $L_L Y = Y$.

On the other hand, we have

$$X = L_L X = \left(\bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, X)\}\right)/\,0_X$$

and

$$Y = L_L Y = \left(\bigvee\{f(1) \mid f \in \mathrm{Hom}_{\mathcal{LM}}(L, Y)\}\right)/\,0_Y.$$

Since $\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, Y)$, we deduce that

$$X = L_L X = L_L Y = Y.$$

Thus the map ψ is injective.

2) $\Longrightarrow$ 1) Let $X \in \sigma[L]$. Then, $L_L X$ is L-generated by Lemma 1.11. If we set $L_L X := X'$, then by Lemma 1.18, we deduce that X' is L-generated**. Now, by Lemma 1.18, we have $L_L X' = X'$.

We claim that $\mathrm{Hom}_{\mathcal{LM}}(L, X') = \mathrm{Hom}_{\mathcal{LM}}(L, X)$. Indeed, we have $X' = L_L X \subseteq X$. Now let $f \in \mathrm{Hom}_{\mathcal{LM}}(L, X')$. As $X' \subseteq X$, it follows that $f \in \mathrm{Hom}_{\mathcal{LM}}(L, X)$, and then $\mathrm{Hom}_{\mathcal{LM}}(L, X') \subseteq \mathrm{Hom}_{\mathcal{LM}}(L, X)$. For any $g \in \mathrm{Hom}_{\mathcal{LM}}(L, X)$ we have

$$g(L) \subseteq \bigvee\nolimits_{h \in \mathrm{Hom}_{\mathcal{LM}}(L,X)} h(L) = L_L X = X',$$

so $g \in \mathrm{Hom}_{\mathcal{LM}}(L, X')$, which implies that

$$\mathrm{Hom}_{\mathcal{LM}}(L, X) = \mathrm{Hom}_{\mathcal{LM}}(L, X'),$$

as claimed.

Thus $\psi(X) = \psi(X')$, so $X = X' = L_L X$ because the map ψ is injective. Now, by Lemma 1.18, X is L-generated**, and then X is L-generated by Proposition 1.10. This shows that L is a generator $\sigma[L]$, and we are done. ∎

Proposition 2.11. *Let $L \in \mathcal{M}_c$ be a generator in $\sigma[L]$. Then every simple lattice in $\sigma[L]$ is a linear homomorphic image of L.*

Proof. Let $S \in \sigma[L]$ be a simple lattice. As L is a generator in $\sigma[L]$, then S is L-generated, so L-generated** by Proposition 1.10. Then $S = L_L S$ by Lemma 1.18. It follows that there exists a non-zero linear morphism $f : L \longrightarrow S$. Since S is a simple lattice we deduce that $f(L) = S$, which implies that S is a linear homomorphic image of L. ∎

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