Bull. Math. Soc. Sci. Math. Roumanie
Tome 65 (113), No. 1, 2022, 73–96

Tykhonov triples and convergence analysis of an inclusion problem
by
Mircea Sofonea

Abstract

We consider an inclusion problem governed by a strongly monotone Lipschitz continuous operator defined on a real Hilbert space. We list the assumption on the data and recall the existence of a unique solution to the problem. Then we introduce several Tykhonov triples, compare them and prove the corresponding well-posedness results. Moreover, using the approximating sequences generated by these triples, we obtain various convergence results. In particular, with a specific choice of the Tykhonov triple, we deduce a criterion of convergence to the solution of the inclusion. The proofs of our results are based on arguments of compactness, pseudomonotonicity, convexity, fixed point and the Mosco convergence of sets.

Key Words: Normal cone, monotone operator, inclusion, Tykhonov triple, Tykhonov well-posedness, approximating sequence, fixed point, Mosco convergence.
2010 Mathematics Subject Classification: Primary 47J22; Secondary 49J40, 49J21, 34G25.

1 Introduction

Convergence results to the solution of a given problem is a fundamental topic in both Functional Analysis, Numerical Analysis and their applications. Some typical examples are the convergence of the solution of a penalty problem to the solution of the original problem when the penalty parameter converges to zero, the convergence of the solution of a regularized problem to the solution of a nonsmooth problem when the regularization parameter converges, the convergence of the solution of a discrete problem to the solution of the continuous problem when the time-step or the spatial discretization parameter converges to zero. Details on these topics can be found in [1, 2, 5, 7], for instance.

Convergence results allow us to establish the continuous dependence of the solution of a problem with respect to the data and parameters. They also allow to establish the link between different models used in Mechanics, Physics and Engineering Sciences and to justify some assumptions made in the modelling of different settings. Some few examples are the following: a viscoelastic problem can be approached by an elastic problem for a small viscosity coefficient, a frictional contact problem can be approached by a frictionless contact problem when the coefficient of friction converges to zero, a contact problem with a rigid foundation can be approached by a contact problem with a deformable foundation for a large stiffness coefficient. All these approaches can be justified based on convergence results concerning the solutions of the corresponding problems. References in the field include the books [3, 8, 11, 16, 19].

For the reasons above, a considerable effort was done to obtain convergence results in the study of various mathematical problems including nonlinear equations, inequality problems, inclusions, fixed point problems, optimization problems, among others. The literature in the field is extensive and the corresponding results have been obtained by using different methods and functional arguments, Nevertheless, most of these results are stated in the following functional framework: given a space X, a problem $\mathcal{P}$ which has a unique solution $u \in X$ and a sequence $\{u_\theta\} \subset X$ or a family of problems $\{\mathcal{P}_\theta\}$ such that u_θ is the solution of Problem $\mathcal{P}_\theta$, the aim consists to prove that u_θ converge to u in X, as θ converges.

A carefully analysis of this description reveals that, in practice, the functional framework above has to be completed by providing details on the following three items: i) the set I to which the parameter θ belongs; ii) the problem $\mathcal{P}_\theta$ or its sets of solutions, denoted by $\Omega(\theta)$, for each $\theta \in I$; iii) the meaning we give to the convergence of the parameter θ. Collecting these three ingredients we arrive in a natural way to the concept of Tykhonov triple, denoted by $\mathcal{T} = (I, \Omega, \mathcal{C})$, where $\mathcal{C}$ is a set which governs the convergence of the parameter θ. Note that this concept was introduced in [29] in the functional framework of metric spaces.

Tyknonov triples represent a useful mathematical tool in the analysis of various problems. Indeed, as we shall see in Section 2, if a Problem $\mathcal{P}$ is well-posed with respect a Tyknonov triple $\mathcal{T}$, then all the approximating sequences generated by $\mathcal{T}$, the so-called $\mathcal{T}$-approximating sequences, converge to the unique solution of $\mathcal{P}$. Among these sequences we may identify some remarkable ones and, in this way we implicitly deduce their convergence. To conclude, the interest in using the mathematical tool provided by the Tyknonov triples arises in the fact that it allows to obtain general convergence results and to unify convergence results previously obtained by different functional arguments. The difficulty in using this tool consists in the choice of the appropriate Tykhonov triple, which needs to be large enough in order to be used to recover a specific convergence result, but small enough in order to guarantee the well-posedness of the considered problem. To overcome with this difficulty, some elementary properties concerning the equivalence and the comparison of Tykhonov triples have been provided in [29].

Tykhonov triples can be used to extend various well-posedness concepts previously studied in the literature, including the concept of well-posedness in the sense of Tykhonov for a minimization problem introduced in [26] and the concept of well-posednes in the sense of Levitin-Polyak for a constrainted optimization problem introduced in [14]. They can be used to reformulate the well-posedness results obtained in [4, 10, 12, 13], in the study of variational inequalities, as well as the well-posedness results obtained in [6, 27, 28], in the study of hemivariational inequalities and inclusion problems. Recently, Tykhonov triples have been employed in the study hemivariational inequalities and minimization problems in [9] and [23], respectively. Moreover, the have been used in [21, 22] in the analysis and control of two elliptic problems describing the antiplane shear of an elastic body and the heat transfer with unilateral constraints, in [24] to prove the well-posedness of a quasistatic contact problem with elasto-viscoplastic materials and in [20] to provide various convergence results for contact problems with elastic materials.

The current paper represents a continuation of our previous works [15, 17]. The paper [15] dealt with existence and uniqueness results for several classe of inclusions while [17] dealt with their optimal control and their application in Contact Mechanics. In contrast, in this current paper we deal with convergence analysis of one of the inclusions considered in [15, 17]. The novelty of our results arises in the fact that the analysis is carried out by using the concepts of Tykhonov well-posedness and Tykhonov triples. Our aim in this paper is twofold. The first one is to obtain the continuous dependence of the solution with respect to the problem data, including the set of constraints. Our second aim is to introduce a general criterion of convergence to the solution of the corresponding inclusion. The results we present in this paper find application in the study of boundary value problems which, in a variational formulation, lead to such kind of inclusions.

The rest of the manuscript is organized as follows. In Section 2 we introduce the inclusion we are interested in, list the assumption on the data and recall an existence and uniqueness result obtained in [15], together with some preliminary material. Then, in Section 3 we introduce the concepts of Tykhonov triple and well-posedness, provide some examples and state and prove our first well-posedness result, Theorem 2. In Section 4 we state and prove our second well-posedness result, Theorem 3. As a consequence we deduce a continuous dependence result of the solution with respect to the data, Corollary 2. In Section 5 we provide a criterion of convergence to the solution of the inclusion problem, Theorem 4. Its proof is inspired from the equivalence of the inclusion problem with a fixed point problem for contractive operators.

2 Problem statement and preliminaries

Everywhere in this paper X represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and its associated norm $\|\cdot\|_X$. We denote by 0_X the zero element of X, by I_X the identity map on X and by 2^X the set of parts of X. The symbols "$\rightharpoonup$" and "$\to$" represent the weak and the strong convergence in X, respectively. All the limits, upper and lower limits will be considered as $n\to\infty$, even if we do not mention it explicitely. Everywhere in this paper we assume the following.

$$K \text{ is a nonemply closed convex subset of } X. \tag{2.1}$$

$$\left\{\begin{array}{l} A : X \to X \text{ is a strongly monotone and Lipschitz continuous} \\ \text{ operator, i.e., there exist } m_A > 0 \text{ and } L_A > 0 \text{ such that} \\ \text{(a)} \quad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X. \\ \text{(b)} \quad \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall\, u, v \in X \end{array}\right. \tag{2.2}$$

$$f \in X. \tag{2.3}$$

We denote by $P_K : X \to K$ the projection operator on K and by $N_K : X \to 2^X$ the outward normal cone of K in the sense of convex analysis. It is well known that the following equivalences hold, for all $\eta,\ \xi, \sigma \in X$:

$$\xi \in N_K(\eta) \iff \eta \in K, \quad (\xi, v - \eta)_X \leq 0 \quad \forall\, v \in K. \tag{2.4}$$

$$\sigma = P_K \xi \iff \sigma \in K, \quad (\xi - \sigma, v - \sigma)_X \leq 0 \quad \forall\, v \in K. \tag{2.5}$$

With the above notation and assumptions, the inclusion we consider in this paper is the following.

Problem $\mathcal{P}$. *Find an element* $u \in X$ *such that*

$$-u \in N_K\big(Au + f\big). \tag{2.6}$$

The unique solvability of Problem $\mathcal{P}$ is provided by the following existence and uniqueness result.

Theorem 1. *Assume* (2.1)–(2.3). *Then there exists a unique element $u \in X$ such that* (2.6) *holds.*

A proof of Theorem 1 can be found in [15] and, for this reason we skip it. Nevertheless, for the convenience of the reader we recall the following ingredients which have been used in the proof and which will be used in Section 5 of the manuscript.

$$\left\{\begin{array}{l} \text{The operator } A : X \to X \text{ is invertible and its inverse} \\ A^{-1} : X \to X \text{ is strongly motonone and Lipschitz continuous} \\ \text{with constants } m' = \frac{m_A}{L_A^2} \text{ and } L' = \frac{1}{m_A}. \end{array}\right. \tag{2.7}$$

$$\left\{\begin{array}{l} \text{The element } u \in X \text{ is a solution of Problem } \mathcal{P} \text{ if and only if} \\ \sigma = Au + f \text{ is a fixed point of the operator } \Lambda_\rho : X \to X \text{ defined by} \\ \Lambda_\rho \xi = P_K\big(\xi - \rho A^{-1}(\xi - f)\big) \quad \forall\, \xi \in X, \text{ for any } \rho > 0. \end{array}\right. \tag{2.8}$$

$$\left\{\begin{array}{l} \text{The operator } \Lambda_\rho \text{ defined in (2.8) is a contraction on } X, \\ \text{for any real number } \rho \text{ such that } 0 < \rho < \frac{2m'}{L'^2} = \frac{2m_A^3}{L_A^2}, \text{ that is} \\ \|\Lambda_\rho \xi - \Lambda_\rho \eta\|_X \le k(\rho)\|\xi - \eta\|_X \quad \forall\, \xi, \eta \in X \\ \text{with } k(\rho) = \sqrt{1 - 2\rho m' + \rho^2 L'^2} < 1. \end{array}\right. \tag{2.9}$$

Everywhere below we denote by u the solution of Problem $\mathcal{P}$ provided by Theorem 1. Consider a sequence of elements $\{u_n\} \subset X$. Our aim in what follows is to provide conditions which guarantee the convergence

$$u_n \to u \quad \text{in} \quad X, \quad \text{as } n \to \infty. \tag{2.10}$$

In order to provide an answer to the question above, we use a two steps strategy:

a) First, we identify a set $\mathcal{S}_\mathcal{T}$ of sequences in X with the property that each element of $\mathcal{S}_\mathcal{T}$ converge to u in X. This set will be associated to a Tykhonov triple $\mathcal{T}$ and, for this reason, it is denoted by $\mathcal{S}_\mathcal{T}$. The Tykhonov triple $\mathcal{T}$ is choosed in such a way that Problem $\mathcal{P}$ is well-posed with $\mathcal{T}$. The details will be introduced in the next section, together with relevant examples.

b) Second, we prove that the given sequence $\{u_n\}$ belongs to $\mathcal{S}_\mathcal{T}$ and, using the step a) we conclude that (2.10) holds.

Our interest is to apply this strategy in order to establish convergence result of the solution u with respect to the data K, A and f. To this end we consider three sequences $\{K_n\}$, $\{A_n\}$ and $\{f_n\}$ such that, for each $n \in \mathbb{N}$, K_n, A_n and f_n represent a perturbation of K, A and f, respectively, assumed to satisfy conditions (2.1)–(2.3). Then, using Theorem 1 it follows that for each $n \in \mathbb{N}$ the exists a unique solution to the following inclusion problem.

Problem $\mathcal{P}_n$. *Find an element $u_n \in X$ such that*

$$-u_n \in N_{K_n}\big(A_n u_n + f_n\big). \tag{2.11}$$

To deduce the convergence of the solution u_n to u we use the notion of Mosco convergence that we recall in what follows.

Definition 1. *Let $\{K_n\}$ be a sequence of nonempty sets of X and let K a nonempty set of X. We say that the sequence $\{K_n\}$ converge to K in the sense of Mosco if the following properties hold.*

$$\left\{\begin{array}{l}\text{(i) For every } v \in K, \text{ there exists a sequence } \{v_n\} \subset X \text{ such that} \\ \quad v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \to v \text{ in } X. \\ \text{(ii) For each sequence } \{v_n\} \text{ such that } v_n \in K_n \text{ for each } n \in \mathbb{N} \\ \quad \text{and } v_n \rightharpoonup v \text{ in } X, \text{ we have } v \in K.\end{array}\right. \tag{2.12}$$

For the convergence defined above we shall use notation $K_n \xrightarrow{M} K$ in X. Moreover, we recall following equivalence result, proved in [25].

Proposition 1. *Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of X and let K be a nonempty closed convex subset of X. Then, $K_n \xrightarrow{M} K$ in X if and only if $P_{K_n}\xi \to P_K\xi$ in X, for any $\xi \in X$.*

In addiditon, we recall the following classical pseudomonotonicity property of the operator A, proved in [18, p.20], for instance.

Proposition 2. *Assume* (2.2) *and let $\{u_n\}$ be a sequence of elements in X such that $u_n \rightharpoonup \widetilde{u}$ in X and $\limsup\,(Au_n, u_n - \widetilde{u})_X \le 0$. Then*

$$\liminf\,(Au_n, u_n - v)_X \ge (A\widetilde{u}, \widetilde{u} - v)_X \quad \forall\, v \in X.$$

We end this section with the following elementary result which will be used in Section 3 below.

Proposition 3. *Let K be a closed convex nonempty subset of X and let $A = I_X$. Then, for each $f \in X$ the solution of the inclusion* (2.6) *is given by*

$$u = P_K f - f. \tag{2.13}$$

In addition, if K is the ball of radious 1 centred on 0_X, then

$$u = \left\{\begin{array}{ll} \Big(\frac{1}{\|f\|_X} - 1\Big) f & \text{if } \ \|f\|_X > 1, \\ \quad 0 & \text{if } \ \|f\|_X \le 1. \end{array}\right. \tag{2.14}$$

Proof. We use (2.4) to see that, in the particular case when $A = I_X$, u is a solution to (2.6) if and only if

$$u + f \in K, \quad (u + f - v, u)_X \le 0 \qquad \forall\, v \in K$$

or, equivalently,

$$u + f \in K, \quad ((u + f) - v, (u + f) - f)_X \le 0 \qquad \forall\, v \in K. \tag{2.15}$$

We now combine (2.15) and (2.5) to see that $u + f = P_K f$ which proves (2.13).

Assume now that K is is the closed ball of radious 1 centred on 0_X, i.e.,

$$K = \Big\{\, v \in V \,:\, \|v\|_X \le 1 \,\Big\}$$

Then, using (2.5) it is easy to see that

$$P_K f = \begin{cases} \frac{f}{\|f\|_X} & \text{if } \ \|f\|_X > 1, \\ f & \text{if } \ \|f\|_X \le 1. \end{cases}$$

and, using (2.13) we deduce (2.14). □

3 Tykhonov triples and Tykhonov well-posedness

We now recall the concepts of Tykhonov triple and Tykhonov well-posedness (well-posedness, for short) introduced in [29]. Note that these concepts have been introduced for a general problem which could be an equation, an inequality, an inclusion, a fixed point or an optimization problem, in the framework of metric spaces. Nevertheless, for the convenience of the reader, we restrict below to recall them in the particular setting of inclusion (2.6) where, recall, X is a Hilbert space.

Definition 2. a) *A Tykhonov triple is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$ where I is a given nonempty set, $\Omega : I \to 2^X$ is a set-valued mapping such that $\Omega(\theta) \neq \emptyset$ for each $\theta \in I$ and $\mathcal{C}$ is a nonempty subset of sequences with elements in I.*

b) *Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{u_n\} \subset X$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\theta_n\} \in \mathcal{C}$, such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.*

c) *Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, Problem $\mathcal{P}$ is said to be $\mathcal{T}$-well-posed (or, equivalently, well-posed with $\mathcal{T}$) if it has a unique solution and every $\mathcal{T}$-approximating sequence converges in X to this solution.*

Let $\mathcal{T} = (I, \Omega, \mathcal{C})$ be a Tykhonov triple. Below in this paper we refer to I as the set of parameters. A typical element of I will be denoted by θ. We refer to the family of sets $\{\Omega(\theta)\}_{\theta \in I}$ as the family of approximating sets and, moreover, we say that the set $\mathcal{C}$ defines the criterion of convergence. Note that approximating sequences always exist since, by assumption, $\mathcal{C} \neq \emptyset$ and, moreover, for any sequence $\{\theta_n\} \in \mathcal{C}$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty. We also remark that the concept of approximating sequence above depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, we use the terminology "$\mathcal{T}$-approximating sequence". As a consequence, the concept of well-posedness for Problem $\mathcal{P}$ depends on the Tykhonov triple $\mathcal{T}$ and, therefore, we refer to it as "well-posedness with $\mathcal{T}$" or "$\mathcal{T}$-well-posedness", as mentionned in Definition 2 c).

Next, we denote by $\mathcal{S}_{\mathcal{P}}$ the set of sequences of X which converge to the solution u of Problem $\mathcal{P}$ and, given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, we use the notation $\mathcal{S}_{\mathcal{T}}$ for set of $\mathcal{T}$-approximating sequences, that is,

$$\mathcal{S}_{\mathcal{P}} = \Big\{ \{u_n\} \subset X \ : \ u_n \to u \ \text{ in } \ X \Big\}, \tag{3.1}$$

$$\mathcal{S}_{\mathcal{T}} = \Big\{ \{u_n\} \subset X \ : \{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence} \Big\}. \tag{3.2}$$

To avoid any confusion, we underlie that in (3.1), (3.2) and below in this paper we use the notation $\{\omega_n\}$ for a sequence of elements ω and use big parenthesis for sets, i.e., for instance, we write $A = \Big\{a, b, c\Big\}$ for the set A with elements a, b, c. Next, we use Definition 2 c) and equalities (3.1), (3.2) to see that

$$\text{Problem } \mathcal{P} \text{ is } \mathcal{T}\text{-well-posed if and only if } \mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{P}}. \tag{3.3}$$

Moreover, the set $\mathcal{S}_{\mathcal{T}}$ of $\mathcal{T}$-approximating sequences suggests us to introduce the following definition.

Definition 3. *Given two Tykhonov triples $\mathcal{T} = (I, \Omega, \mathcal{C})$ and $\mathcal{T}' = (I', \Omega', \mathcal{C}')$, we say that:*

a) *$\mathcal{T}$ and $\mathcal{T}'$ are equivalent if their sets of approximating sequences are the same, i.e., $\mathcal{S}_{\mathcal{T}} = \mathcal{S}_{\mathcal{T}'}$. In this case we write $\mathcal{T} \approx \mathcal{T}'$.*

b) *$\mathcal{T}$ is smaller than $\mathcal{T}'$ if $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}'}$. In this case we write $\mathcal{T} \leq \mathcal{T}'$.*

c) *$\mathcal{T}$ is stricly smaller than $\mathcal{T}'$ if $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}'}$ and $\mathcal{S}_{\mathcal{T}'} \not\subset \mathcal{S}_{\mathcal{T}}$. In this case we write $\mathcal{T} < \mathcal{T}'$.*

It is easy to see that " $\approx$ " represents an equivalence relation on the set of Tykhonov triples while " $\leq$ " defines a relation of order on the same set.

We now provide two special exemples Tykhonov triples associated to Problem $\mathcal{P}$.

Example 1. *Assume that $\{u_n\}$ is a given sequence of elements in X. Moreover, consider the Tykhonov triple $\mathcal{T}_0 = (I_0, \Omega_0, \mathcal{C}_0)$ defined as follows :*

$$I_0 = \mathbb{N} = \Big\{1, 2, , \ldots, n, \ldots\Big\},$$

$$\Omega_0 : I_0 \to 2^X, \quad \Omega_0(n) = \Big\{u_n\Big\} \quad \forall\, n \in \mathbb{N},$$

$$\mathcal{C}_0 = \Big\{ \{k_n\} \subset I_0 \ :\ \ k_1 < k_2 < \ldots < k_n \ldots \Big\}.$$

Then, using Definition 2 a) *it is easy to see that a sequence $\{\widetilde{u}_n\} \subset X$ is a $\mathcal{T}_0$-approximating sequence if and only if $\{\widetilde{u}_n\}$ is a subsequence of the sequence $\{u_n\}$. Therefore, by Definition* 2 c) *we deduce that $u_n \to u$ in X if and only if Problem $\mathcal{P}$ is well posed with the Tykhonov triple $\mathcal{T}_0$.*

Example 2. *Let $\mathcal{T}_{\mathcal{P}} = (I_{\mathcal{P}}, \Omega_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}})$ where*

$$I_{\mathcal{P}} = \mathbb{R}_+ = [0, +\infty), \tag{3.4}$$

$$\Omega_{\mathcal{P}} : I_{\mathcal{P}} \to 2^X, \qquad \Omega_{\mathcal{P}}(\theta) = \Big\{ \widetilde{u} \in X \ :\ \|\widetilde{u} - u\|_X \leq \theta \Big\} \quad \forall\, \theta \geq 0, \tag{3.5}$$

$$\mathcal{C}_{\mathcal{P}} = \Big\{ \{\theta_n\} \subset I_{\mathcal{P}} \ :\ \theta_n \to 0 \Big\}. \tag{3.6}$$

Then, Problem $\mathcal{P}$ is $\mathcal{T}_{\mathcal{P}}$-well-posed. To prove this statement, we shall prove that

$$\mathcal{S}_{\mathcal{T}_{\mathcal{P}}} = \mathcal{S}_{\mathcal{P}}. \tag{3.7}$$

Let $\{u_n\}$ be a $\mathcal{T}_{\mathcal{P}}$-approximating sequence, i.e., $\{u_n\} \in \mathcal{S}_{\mathcal{T}_{\mathcal{P}}}$. Then, using (3.4)–(3.6) *we deduce that there exists $\{\theta_n\} \subset \mathbb{R}_+$ such that $\theta_n \to 0$ and $\|u_n - u\|_X \leq \theta_n$ for each $n \in \mathbb{N}$. This* *implies that* $u_n \to u$ *in* X *and, using by definition* (3.1)*, we deduce that* $\{u_n\} \subset \mathcal{S}_{\mathcal{P}}$*. Conversely, assume that* $\{u_n\} \subset \mathcal{S}_{\mathcal{P}}$*. Then,* (3.1) *implies that* $u_n \to u$ *in* X*. Denote* $\theta_n = \|u_n - u\|_X$*, for each* $n \in \mathbb{N}$*. It follows from here that* $\{\theta_n\} \in \mathcal{C}_{\mathcal{P}}$ *and, moreover,* $\{u_n\} \in \Omega_{\mathcal{P}}(\theta_n)$*, for each* $n \in \mathbb{N}$*. This shows that* $\{u_n\}$ *is a* $\mathcal{T}_{\mathcal{P}}$*-approximating sequence and, using definition* (3.2) *we deduce that* $\{u_n\} \subset \mathcal{S}_{\mathcal{T}_{\mathcal{P}}}$*. It results from above that equality* (3.7) *holds, as claimed. We now use the equivalence* (3.3) *to deduce that Problem* $\mathcal{P}$ *is* $\mathcal{T}_{\mathcal{P}}$*-well-posed.*

We proceed with several comments related to Examples 1–2.

Remark 1. *It follows from Exemple* 1 *that a specific convergence result to the solution of Problem* $\mathcal{P}$ *is equivalent with the well-posedness of* $\mathcal{P}$ *with a specific Tykhonov triple. Nevertheless, in practice we construct Tykhonov triples* $\mathcal{T} = (I, \Omega, \mathcal{C})$ *for which the approximating sets* $\Omega(\theta)$ *are large enough and, therefore, the set of approximating sequences* $\mathcal{S}_{\mathcal{T}}$ *is quite large. Then, the well-posedness of Problem* $\mathcal{P}$ *with such a Tykhonov triple implicitly provides more than one convergence result since, by definition, all the* $\mathcal{T}$*-approximating sequences converge to the solution* u *of* $\mathcal{P}$*. We conclude that the well-posedness concept introduced in Definition* 2 *provides a framework which allows to unify various convergence results (usually obtained by using different functional arguments) and, as already mentioned, this represents the main interest in this concept.*

Remark 2. *We denote in what follows by* $(\mathcal{A}_{\mathcal{P}}, \leq)$ *the set of Tykhonov triples with whom Problem* $\mathcal{P}$ *is well-posed, endowed with the relation of order in Definition* 3 b)*. Then, it follows from Example* 2 *that the Tyhonov triple* $\mathcal{T}_{\mathcal{P}}$ *is a maximal element of the set* $(\mathcal{A}_{\mathcal{P}}, \leq)$*. Next, equivalence* (3.3) *and equality* (3.7) *show that among all the elements in* $\mathcal{A}_{\mathcal{P}}$*, the Tyhonov triple* (3.4)–(3.6) *is a triple which generates the largest set of approximating sequences, since all the sequences which converge to the solution* u *of problem* $\mathcal{P}$ *are* $\mathcal{T}_{\mathcal{P}}$*-approximating sequences. Even if these properties could seem interesting, the choice of the Tykhonov triple* $\mathcal{T}_{\mathcal{P}}$ *is not convenient to study the well-posedness for Problem* $\mathcal{P}$*. Indeed, the definition* (3.5) *uses the solution* u *of Problem* $\mathcal{P}$ *which is a priori unknown. A reasonable definition of the approximating sets would use the problem itself, or some of its perturbation, not its solution. For this reason it is important to introduce Tykhonov triples which are equivalent with* $\mathcal{T}_{\mathcal{P}}$*, defined without mention to the solution* u *and this is what we shall do in Section* 5 *of this manuscript, see Theorem* 4 *and Remark* 3*.*

We now introduce three relevant Tykhonov triples associated to Problem $\mathcal{P}$. To this end we use equivalence (2.4) to see that an element $u \in X$ is solution to Problem $\mathcal{P}$ if and only if

$$Au + f \in K, \quad (Au + f - v, u)_X \leq 0 \qquad \forall\, v \in K. \tag{3.8}$$

Relaxing this inequality suggests us to consider the approximating sets

$$\Omega_1(\theta) = \Big\{\, \widetilde{u} \in X \;:\; A\widetilde{u} + f \in K, \;\; (A\widetilde{u} + f - v, \widetilde{u})_X \le \theta \;\;\forall\, v \in K \,\Big\}, \tag{3.9}$$

$$\Omega_2(\theta) = \Big\{\, \widetilde{u} \in X \;:\; A\widetilde{u} + f \in K, \\ (A\widetilde{u} + f - v, \widetilde{u})_X \le \theta\big(\|\widetilde{u}\|_X + 1\big) \;\;\forall\, v \in K \,\Big\}, \tag{3.10}$$

$$\Omega_3(\theta) = \Big\{\, \widetilde{u} \in X \;:\; A\widetilde{u} + f \in K, \\ (A\widetilde{u} + f - v, \widetilde{u})_X \le \theta\big(\|A\widetilde{u} + f - v\|_X + 1\big) \;\;\forall\, v \in K \,\Big\}, \tag{3.11}$$

for all $\theta \ge 0$. Moreover, we consider the sets I and $\mathcal{C}$ defined by

$$I = \mathbb{R}_+ = [0, +\infty), \tag{3.12}$$

$$\mathcal{C} = \Big\{\, \{\theta_n\} \subset I \;:\; \theta_n \to 0 \,\Big\}. \tag{3.13}$$

With these ingredients we introduce the triples $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ defined by

$$\mathcal{T}_1 = (I, \Omega_1, \mathcal{C}), \qquad \mathcal{T}_2 = (I, \Omega_2, \mathcal{C}), \qquad \mathcal{T}_3 = (I, \Omega_3, \mathcal{C}). \tag{3.14}$$

Note that, since we assume (2.1)–(2.3), Theorem 1 guarantees that Problem $\mathcal{P}$ has a unique solution u. Then, using (3.8) it is easy to see that $u \in \Omega_1(\theta)$, $u \in \Omega_2(\theta)$ and $u \in \Omega_3(\theta)$, for each $\theta \in I$. This implies that $\Omega_1(\theta) \ne \emptyset$, $\Omega_2(\theta) \ne \emptyset$ and $\Omega_3(\theta) \ne \emptyset$ for each $\theta \in I$ and, therefore, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are Tykhonov triples in the sense of Definition 2.

The properties of these triples can be resumed as follows.

Proposition 4. *Assume* (2.1)–(2.3). *Then, the following statement hold.*

a) *The Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent, i.e., $\mathcal{T}_1 \approx \mathcal{T}_2$.*

b) *The Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$ are smaller than the Tykhonov triple $\mathcal{T}_3$, i.e., $\mathcal{T}_1 \le \mathcal{T}_3$ and $\mathcal{T}_2 \le \mathcal{T}_3$. Moreover, unless additional assumptions, these inequalities are strict, i.e., $\mathcal{T}_1 < \mathcal{T}_3$ and $\mathcal{T}_2 < \mathcal{T}_3$.*

c) *If $K \subset X$ is bounded, then the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are equivalent, i.e., $\mathcal{T}_1 \approx \mathcal{T}_2 \approx \mathcal{T}_3$.*

Proof. a) Let $\{u_n\}$ be a $\mathcal{T}_2$-approximating sequence, i.e. $\{u_n\} \subset \mathcal{S}_{\mathcal{T}_2}$. Then there exists a sequence $\{\theta_n\} \subset \mathbb{R}_+$ such that $\theta_n \to 0$ and, moreover,

$$Au_n + f \in K, \quad (Au_n + f - v, u_n)_X \le \theta_n\big(\|u_n\|_X + 1\big) \quad \forall\, v \in K, \tag{3.15}$$

for each $n \in \mathbb{N}$. We fix $n \in \mathbb{N}$ and $v \in K$. Then, using (2.2)(a) and (3.15) we find that

$$\begin{aligned} m_A\|u_n\|_X^2 &\le (Au_n - A0_X, u_n)_X = (Au_n u_n)_X - (A0_X, u_n)_X \\ &\le \theta_n\big(\|u_n\|_X + 1\big) + (v - f, u_n)_X - (A0_X, u_n)_X \\ &\le \big(\theta_n + \|v - f\|_X + \|A0_X\|_X\big)\|u_n\|_X + \theta_n. \end{aligned}$$

This implies that there exists $D > 0$, which does not depend on n, such that

$$\|u_n\| \le D. \tag{3.16}$$

We now use (3.15) and (3.16) to see that $u_n \in \Omega_1(\widetilde{\theta}_n)$ with $\widetilde{\theta}_n = \theta_n(D+1)$ and, since $\theta_n \to 0$, we deduce that $\{u_n\} \subset \mathcal{S}_{\mathcal{T}_1}$. It follows from here that $\mathcal{S}_{\mathcal{T}_2} \subset \mathcal{S}_{\mathcal{T}_1}$. On the other hand, it is easy to see that $\Omega_1(\theta) \subset \Omega_2(\theta)$ for each $\theta \in I$ which implies that $\mathcal{S}_{\mathcal{T}_1} \subset \mathcal{S}_{\mathcal{T}_2}$. We deduce from above that $\mathcal{S}_{\mathcal{T}_1} = \mathcal{S}_{\mathcal{T}_1}$ and, therefore, $\mathcal{T}_1 \approx \mathcal{T}_2$.

b) Note that $\Omega_1(\theta) \subset \Omega_3(\theta)$ for each $\theta \in I$ which shows that $\mathcal{S}_{\mathcal{T}_1} \subset \mathcal{S}_{\mathcal{T}_3}$. We deduce from here that $\mathcal{T}_1 \le \mathcal{T}_3$ and, since $\mathcal{T}_1 \approx \mathcal{T}_2$ we obtain that $\mathcal{T}_2 \le \mathcal{T}_3$, too.

In order to prove that these inequalities are strict we consider the following counter-example. Let $K = X$, $A = I_X$, $f = 0_X$, $u_0 \in X$, $u_0 \ne 0_X$ and let $u_n = \frac{1}{n}\,u_0$, $\theta_n = \frac{1}{n}\,\|u_0\|_X$, for each $n \in \mathbb{N}$. Then, using the inequality

$$(u_n - v, u_n)_X \le \|u_n - v\|_X \|u_n\|_X = \frac{1}{n}\,\|u_0\|_X \|u_n - v\|_X \quad \forall\, v \in K,\ n \in \mathbb{N},$$

we deduce that $u_n \in \Omega_3(\theta_n)$ for each $n \in \mathbb{N}$, which shows that $\{u_n\}$ is a $\mathcal{T}_3$-approximating sequence.

Assume now that $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. Then Definition 2 b) guarantees that there exists a sequence $\{\widetilde{\theta}_n\} \in \mathcal{C}$ such that $(u_n - v, u_n)_X \le \widetilde{\theta}_n$ for each $v \in X$ and $n \in \mathbb{N}$ or, equivalently,

$$\frac{1}{n^2}\,\|u_0\|_X^2 - \frac{1}{n}(v, u_0)_X \le \widetilde{\theta}_n \qquad \forall\, v \in X,\ n \in \mathbb{N}. \tag{3.17}$$

We now that choose $n \in \mathbb{N}$ arbitrary and take $v = -nu_0$ in inequality (3.17) to deduce that $\widetilde{\theta}_n \ge \big(1 + \frac{1}{n^2}\big)\|u_0\|_X^2$ for each $n \in \mathbb{N}$. This shows that the sequence $\{\widetilde{\theta}_n\}$ does not converge to zero which is in contradiction with the inclusion $\{\widetilde{\theta}_n\} \in \mathcal{C}$.

We conclude from above that there exists $\mathcal{T}_3$-approximating sequences which are not $\mathcal{T}_1$-approximating sequences and, therefore, $\mathcal{T}_1 < \mathcal{T}_3$. Moreover, since $\mathcal{T}_1 \approx \mathcal{T}_2$ we deduce that $\mathcal{T}_2 < \mathcal{T}_3$, too.

c) Assume now that set K is bounded. Let $\{u_n\}$ be a $\mathcal{T}_3$-approximating sequence. Then there exists a sequence $\{\theta_n\} \subset \mathbb{R}_+$ such that $\theta_n \to 0$ and, moreover,

$$Au_n + f \in K, \quad (Au_n + f - v, u_n)_X \le \theta_n\big(\|Au_n + f - v\|_X + 1\big) \quad \forall\, v \in K,\ n \in \mathbb{N}. \tag{3.18}$$

Now, since K is a bounded set, the inclusions $Au_n + f \in K$, $v \in K$ in (3.18) show that there exists a constant $E > 0$ such that

$$\|Au_n + f - v\|_X \le E \qquad \forall\, v \in K,\ n \in \mathbb{N}. \tag{3.19}$$

We now combine (3.18) and (3.19) to see that

$$Au_n + f \in K, \quad (Au_n + f_n - v, u_n)_X \le \theta_n(E + 1) \quad \forall\, v \in K,\ n \in \mathbb{N},$$

which shows that $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence. We conclude from here that $\mathcal{S}_{\mathcal{T}_3} \subset \mathcal{S}_{\mathcal{T}_1}$ and, since we already proved that $\mathcal{S}_{\mathcal{T}_1} = \mathcal{S}_{\mathcal{T}_2} \subset \mathcal{S}_{\mathcal{T}_3}$, we deduce that $\mathcal{S}_{\mathcal{T}_1} = \mathcal{S}_{\mathcal{T}_2} = \mathcal{S}_{\mathcal{T}_3}$ which completes the proof. □

We now state and prove the following well-posedness result.

Theorem 2. *Assume* (2.1)–(2.3). *Then Problem $\mathcal{P}$ is $\mathcal{T}_1$-, $\mathcal{T}_2$- and $\mathcal{T}_3$-well posed.*

Proof. Let $\{u_n\}$ be a $\mathcal{T}_3$-approximating sequence. Then there exists a sequence $\{\theta_n\} \subset \mathbb{R}_+$ such that $\theta_n \to 0$ and, moreover, (3.18) holds. We fix $n \in \mathbb{N}$ and then we take $v = Au_n + f$ in (3.8) and $v = Au + f$ in (3.18). Adding the resulting inequalities we find that

$$(Au_n - Au, u_n - u)_X \leq \theta_n\big(\|Au_n - Au\|_X + 1\big).$$

Next, we use the properties (2.2) of the operator A to find that

$$\|u_n - u\|_X^2 \leq \frac{\theta_n L_A}{m_A}\|u_n - u\|_X + \frac{\theta_n}{m_A}.$$

We now use the elementary inequality

$$x^2 \leq ax + b \implies x \leq a + \sqrt{b} \qquad \forall\, x,\, a,\, b \geq 0 \tag{3.20}$$

to see that

$$\|u_n - u\|_X \leq \frac{\theta_n L_A}{m_A} + \sqrt{\frac{\theta_n}{m_A}}$$

and, therefore, the convergence $\theta_n \to 0$ implies that $u_n \to u$ in X. It follows from here that $\mathcal{S}_{\mathcal{T}_3} \subset \mathcal{S}_{\mathcal{P}}$. We now use the equivalence (3.3) to deduce that Problem $\mathcal{P}$ is $\mathcal{T}_3$-well-posed.

On the other hand, using Proposition 4 it follows that $\mathcal{S}_{\mathcal{T}_1} = \mathcal{S}_{\mathcal{T}_2} \subset \mathcal{S}_{\mathcal{T}_3}$ and, therefore, $\mathcal{S}_{\mathcal{T}_1} = \mathcal{S}_{\mathcal{T}_2} \subset \mathcal{S}_{\mathcal{P}}$. This implies the $\mathcal{T}_1$- and $\mathcal{T}_2$-well-posedness of Problem $\mathcal{P}$ and concludes the proof. □

We now wonder if the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ can be used to prove the convergence of the solution of the perturbed inclusion (2.11) to the solution of the original inclusion (2.6). To this end we start with the following consequence of Theorem 2.

Corollary 1. *Assume* (2.1)–(2.3). *For each $n \in \mathbb{N}$, let $f_n \in X$, denote by u_n the solution of the inclusion* (2.6) *with $f = f_n$ and assume that $Au_n + f \in K$. Then, the convergence $f_n \to f$ in X implies the convergence* (2.10).

Proof. Let $n \in \mathbb{N}$ be fixed. We have $-u_n \in N_K(Au_n + f_n)$ and, using (3.8), we deduce that

$$(Au_n + f_n - v, u_n)_X \leq 0 \quad \forall\, v \in K. \tag{3.21}$$

Therefore

$$(Au_n + f - v, u_n)_X \leq (f - f_n, u_n)_X \quad \forall\, v \in K. \tag{3.22}$$

We now prove that the sequence $\{u_n\}$ is bounded in X. To this end we fix an element $v \in K$ and, using (3.21) we write

$$(Au_n + f_n - v, u_n)_X = (Au_n - A0_X, u_n)_X + (A0_X + f_n - v, u_n)_X \leq 0.$$

Therefore, using assumption (2.2)(b) it follows that

$$\begin{aligned} m_A\|u_n\|_X^2 &\leq (Au_n - A0_X, u_n)_X \leq (v - A0_X - f_n, u_n)_X \\ &\leq \|A0_X + f_n - v\|_X\|u_n\|_X. \end{aligned}$$

This shows that there exists $D > 0$, which does not depend on n, such that (3.16) holds. Next, we use (3.16) and (3.22) to see that

$$(Au_n + f - v, u_n)_X \le D\|f_n - f\|_X \quad \forall\, v \in K. \tag{3.23}$$

Inequality (3.23), the convergence $f_n \to f$ in X and the regularity $Au_n + f \in K$ guarantee that $\{u_n\}$ is a $\mathcal{T}_1-$ approximating sequence for Problem $\mathcal{P}$. We now use Theorem 2 and Definition 2(c) to deduce the convergence (2.10), which concludes the proof. □

Note that Corollary 1 provides the convergence of the solution u_n of the inclusion $\mathcal{P}_n$ with $K_n = K$ and $A_n = A$ to the solution u of the inclusion $\mathcal{P}$, under the very restrictive condition $Au_n + f \in K$, for each $n \in \mathbb{N}$. This condition is satisfied in Exemple 3 but fails to be satisfied in the Example 4 we present below.

Example 3. *Assume* (2.1)*,* $A = I_X$ *and* $f \in \mathrm{int}(K)$ *where* $\mathrm{int}(K)$ *represents the interior of* K *in the strong topology of* X*. We claim that if* $f_n \to f$ *in* X*, then the regularity condition* $Au_n + f \in K$ *is satisfied. Indeed, since* $f \in \mathrm{int}(K)$*, the convergence* $f_n \to f$ *in* X *implies that, for* n *large enough, we have* $f_n \in K$*. Therefore, Proposition* 3 *implies that* $u_n = P_K f_n - f_n = 0_X$*, hence* $Au_n + f = u_n + f = f \in \mathrm{int}(K) \subset K$*.*

Example 4. *Assume that* K *is the ball of radious* 1 *centred on* 0_X*,* $A = I_X$ *and* $f \in X$*,* $\|f\|_X = 2$*. Then using* (2.14) *we obtain that the solution of the inclusion* (2.6) *is* $u = -\frac{f}{2}$*. Next, for each* $n \ge 2$ *let* $f_n = \left(1 - \frac{1}{n}\right)f$*. Using again* (2.14) *it is easy to see that the solution of the inclusion* (2.11) *is* $u_n = \left(\frac{1}{n} - \frac{1}{2}\right)f$*. This implies that* $u_n + f = \left(\frac{1}{n} + \frac{1}{2}\right)f$ *and, therefore* $\|u_n + f\|_X = 1 + \frac{2}{n} > 1$*. We conclude from here that* $u_n + f \notin K$ *which shows that the regularity condition* $Au_n + f \in K$ *is not satisfied in this case, as claimed.*

It follows from Exemple 4 that the well-posedness of Problem $\mathcal{P}$ with the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ cannot be used in order to prove the convergence of the solution of Problem $\mathcal{P}$ with respect to the data. For this reason, in the next section we consider an additional Tykhonov triple, $\mathcal{T}_M$, such that the sequence $\{u_n\}$ in Exemple 4 is a $\mathcal{T}_M$ -approximating sequence. Moreover, this will allow us to extend the convergence result in Corollary 1 by describing the convergence of the solution with respect to the set of all the data K, A and f.

4 A well-posedness result

In this section we use the Tykhonov triple $\mathcal{T}_M = (I_M, \Omega_M, \mathcal{C}_M)$ defined as follows:

$$I_M = \{\, \theta = (\widetilde{K}, \varepsilon) : \widetilde{K} \text{ is closed nonempty convex subset of } X,\ \ \varepsilon \ge 0 \,\}, \tag{4.1}$$

$$\Omega_M(\theta) = \Big\{\, \widetilde{u} \in X \ :\ A\widetilde{u} + f \in \widetilde{K},\ (A\widetilde{u} + f - v, \widetilde{u})_X \le \varepsilon \quad \forall\, v \in \widetilde{K} \,\Big\} \tag{4.2}$$
$$\forall\, \theta = (\widetilde{K}, \varepsilon) \in I_M,$$

$$\mathcal{C}_M = \Big\{\, \{\theta_n\} \subset I_M \ :\ K_n \xrightarrow{M} K \text{ in } X,\ \varepsilon_n \to 0 \,\Big\}. \tag{4.3}$$

Our main result, based on pseudomonotonicity and Mosco convergence ingredients, is the following.

Theorem 3. *Assume* (2.1)–(2.3). *Then Problem* $\mathcal{P}$ *is* $\mathcal{T}_M$*-well posed.*

Proof. Consider a $\mathcal{T}_M$-approximating sequence of Problem $\mathcal{P}$, denoted by $\{u_n\}$. Then, Definition 2 a) and (4.2) show that there exists a sequence $\{\theta_n\} \in \mathcal{C}_M$ with $\theta_n = (K_n, \varepsilon_n)$ such that

$$Au_n + f \in K_n, \quad (Au_n + f - v, u_n)_X \le \varepsilon_n \qquad \forall\, v \in K_n, \tag{4.4}$$

for each $n \in \mathbb{N}$. Recall also that inclusion $\{\theta_n\} \in \mathcal{C}_M$ implies the folowing convergences:

$$\varepsilon_n \to 0, \tag{4.5}$$

$$K_n \xrightarrow{M} K \quad \text{in } X. \tag{4.6}$$

We shall prove that $u_n \to u$ in X and, to this end, we divide the proof in three steps, described below.

Step i) The sequences $\{u_n\}$ *and* $\{Au_n\}$ *are bounded in* X.

Let $v \in K$ be a given element. Then, (4.6) implies that there exists a sequence $\{v_n\} \subset X$ such that $v_n \in K_n$ for all $n \in \mathbb{N}$ and $v_n \to v$ in X. Let $n \in \mathbb{N}$. We write

$$\begin{aligned}(Au_n + f - v_n, u_n) &= (Au_n - Av_n, u_n - v_n)\\ &+(Av_n + f - v_n, u_n - v_n) + (Au_n + f - v_n, v_n),\end{aligned}$$

then we use (4.4) with $v = v_n$ to see that

$$(Au_n - Av_n, u_n - v_n) + (Av_n + f - v_n, u_n - v_n) + (Au_n + f - v_n, v_n) \le \varepsilon_n.$$

Therefore, assumption (2.2)(b) yields

$$\begin{aligned}m_A\|u_n - v_n\|_X^2 &\le \|Av_n + f - v_n\|_X \|u_n - v_n\|_X\\ &+\|Au_n + f - v_n\|_X \|v_n\|_X + \varepsilon_n\end{aligned}$$

and, using assumption (2.2)(a), we find that

$$\begin{aligned}m_A\|u_n - v_n\|_X^2 &\le \|Av_n + f - v_n\|_X \|u_n - v_n\|_X\\ &+L_A\|u_n - v_n\|_X\|v_n\|_X + \|Av_n + f - v_n\|_X \|v_n\|_X + \varepsilon_n.\end{aligned}$$

Now, since the sequence $\{v_n\}$ is bounded in X and A is a bounded operator, the convergence (4.5) and the previous inequality imply that there exists two positive constants C_1 and C_2 which do not depend on n such that

$$\|u_n - v_n\|_X^2 \le C_1\|u_n - v_n\|_X + C_2.$$

This inequality combined with (3.20) show that the sequence $\{u_n - v_n\}$ is bounded in X and, therefore, $\{u_n\}$ is a bounded sequence in X, too. We conclude this step by using the property (2.2) (a) of the operator A.

Step ii) The sequence $\{u_n\}$ *converges weakly to the solution* u *of Problem* $\mathcal{P}$.

Using the step i) and the reflexivity of the space X we deduce that, passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup \widetilde{u} \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty, \tag{4.7}$$

$$Au_n \rightharpoonup z \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty, \tag{4.8}$$

with some $\widetilde{u}$, $z \in X$. Our aim in what follows is to prove that $\widetilde{u}$ is a solution of Problem $\mathcal{P}$. To this end, we remark that the regularity $Au_n + f \in K_n$ combined with the convergences (4.6) and (4.8) imply that

$$z + f \in K. \tag{4.9}$$

Next, we use (4.4) to see that

$$(Au_n + f - v_n, u_n)_X \le \varepsilon_n \quad \forall\, n \in \mathbb{N}.$$

Then, passing to the upper limit and using the convergence (4.5), we find that

$$\limsup\, (Au_n + f - v_n, u_n)_X \le 0. \tag{4.10}$$

We now use the convergences (4.7), $v_n \to v$ in X and inequality (4.10) to deduce that

$$\limsup\, (Au_n, u_n)_X \le (v - f, \widetilde{u})_X \qquad \forall\, v \in K. \tag{4.11}$$

On the other hand, (4.9) allows us to take $v = z + f$ in (4.11) in order to find that

$$\limsup\, (Au_n, u_n)_X \le (z, \widetilde{u})_X. \tag{4.12}$$

Inequality (4.12) and the convergence (4.8) yield

$$\limsup\, (Au_n, u_n - \widetilde{u})_X \le 0$$

and, therefore, Proposition 2 implies that

$$(A\widetilde{u}, \widetilde{u} - v)_X \le \liminf\, (Au_n, u_n - v)_X \quad \forall\, v \in X. \tag{4.13}$$

Moreover, the convergence (4.8) implies that

$$\limsup\, (Au_n, u_n - v)_X = \limsup\, (Au_n, u_n)_X - (z, v)_X,$$

hence inequality (4.12) shows that

$$\limsup\, (Au_n, u_n - v)_X \le (z, \widetilde{u} - v)_X \quad \forall\, v \in X. \tag{4.14}$$

We now combine inequalities (4.13) and (4.14) to find that

$$(A\widetilde{u}, \widetilde{u} - v)_X \le (z, \widetilde{u} - v)_X \quad \forall\, v \in X \tag{4.15}$$

which implies that

$$A\widetilde{u} = z. \tag{4.16}$$

Next, we use (4.8), (4.16) to see that

$$\limsup\,(Au_n, u_n - v)_X = \limsup\,(Au_n, u_n)_X - (A\widetilde{u}, v)_X$$

and, therefore, (4.11) yields

$$\limsup\,(Au_n, u_n - v)_X \le (v - f, \widetilde{u})_X - (A\widetilde{u}, v)_X \qquad \forall\, v \in K. \tag{4.17}$$

We now combine (4.13) and (4.17) to see that

$$(A\widetilde{u}, \widetilde{u} - v)_X \le (v - f, \widetilde{u})_X - (A\widetilde{u}, v)_X \qquad \forall\, v \in K$$

or, equivalently,

$$(A\widetilde{u} + f - v, \widetilde{u})_X \le 0 \qquad \forall\, v \in K. \tag{4.18}$$

Next, from (4.9), (4.16) and (4.18) we obtain that $\widetilde{u}$ is a solution to Problem $\mathcal{P}$, as claimed. Thus, by the uniqueness of the solution of $\mathcal{P}$, we find that $\widetilde{u} = u$.

A careful analysis of the results presented above indicates that every subsequence of $\{u_n\}$ which converges weakly in X has the same weak limit u. On the other hand, Step i) guarantees that $\{u_n\}$ is bounded in X. Therefore, we deduce that the whole sequence $\{u_n\}$ converges weakly to u in X, as $n \to \infty$, which concludes the proof of this step.

Step iii) The sequence $\{u_n\}$ converges strongly to the solution u of Problem $\mathcal{P}$.

We take $v = \widetilde{u}$ in (4.13) and (4.14), then we use equality $\widetilde{u} = u$ to obtain

$$0 \le \liminf\,(Au_n, u_n - u)_X \le \limsup\,(Au_n, u_n - u)_X \le 0,$$

which shows that $(Au_n, u_n - u)_X \to 0$, as $n \to \infty$. Therefore, using the strong monotonicity of the operator A and the convergence $u_n \rightharpoonup u$ in X, we have

$$m_A \|u_n - u\|_X^2 \le (Au_n - Au, u_n - u)_X = (Au_n, u_n - u)_X - (Au, u_n - u)_X \to 0,$$

as $n \to \infty$. Hence, it follows that $u_n \to u$ in X, which concludes the proof of this step.

To resume, we proved that any $\mathcal{T}_M$-approximating sequence converges to the solution of Problem $\mathcal{P}$. Therefore, using Definition 2 c) it follows that Problem $\mathcal{P}$ is $\mathcal{T}_M$-well-posed, which concludes the proof of the theorem. □

We end this section with a corollary of Theorem 3 which extends Corollary 1, since it represents a continuous dependence result of the solution of Problem $\mathcal{P}$ with respect to the data K, A and f. So, consider three sequences $\{K_n\}$, $\{A_n\}$ and $\{f_n\}$ such that, for each $n \in \mathbb{N}$, the following hold.

K_n is a nonempty closed convex subset of K. (4.19)

$A_n : X \to X$ satisfies condition (2.2) with some constants m_n and L_n. (4.20)

$f_n \in X$. (4.21)

Then, using Theorem 1 it follows that for each $n \in \mathbb{N}$ the exists a unique solution to the inclusion problem $\mathcal{P}_n$. Consider now the following additional assumptions.

$$\left\{\begin{array}{l}\text{For each } n \in \mathbb{N} \text{ there exists } a_n \geq 0 \text{ such that}\\ \text{(a) } \|A_n v - Av\|_X \leq a_n(\|v\|_X + 1) \quad \text{for all } v \in X.\\ \text{(b) } a_n \to 0 \quad \text{as} \quad n \to \infty.\end{array}\right. \tag{4.22}$$

$$\left\{\begin{array}{l}\text{There exist } m_0 > 0 \text{ and } L_0 > 0 \text{ such that}\\ m_0 \leq m_n \leq L_n \leq L_0 \quad \forall\, n \in \mathbb{N}.\end{array}\right. \tag{4.23}$$

$$K_n \xrightarrow{M} K \quad \text{in } X. \tag{4.24}$$

$$f_n \to f \quad \text{in} \quad X. \tag{4.25}$$

We have the following result.

Corollary 2. *Assume* (2.1)–(2.3), (4.19)–(4.25). *Then the solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$, that is $u_n \to u$ in X.*

Proof. Let $n \in \mathbb{N}$ and note that inclusion (2.11) implies that

$$A_n u_n + f_n \in K_n, \quad (A_n u_n + f_n - v, u_n)_X \leq 0 \quad \forall\, v \in K_n. \tag{4.26}$$

We first prove that the sequence $\{u_n\}$ is bounded in X. To this end we denote

$$w_n = Au_n - A_n u_n + f - f_n, \tag{4.27}$$

which implies that

$$Au_n + f = A_n u_n + f_n + w_n. \tag{4.28}$$

Moreover, using assumptions (4.22) we have

$$\|w_n\|_X \leq \|Au_n - A_n u_n\|_X + \|f_n - f\|_X \leq a_n\big(\|u_n\|_X + 1\big) + \|f_n - f\|_X$$

and, using notation

$$\widetilde{a}_n = a_n + \|f_n - f\|_X, \tag{4.29}$$

we find that

$$\|w_n\|_X \leq \widetilde{a}_n\big(\|u_n\|_X + 1\big). \tag{4.30}$$

We now fix an element $v \in K$ and, using (4.24) we know that there exists a sequence $\{v_n\}$ such that

$$v_n \in K_n \quad \forall\, n \in \mathbb{N}, \quad v_n \to v \ \text{ in } \ X. \tag{4.31}$$

Moreover, (4.26) implies that

$$(A_n u_n + f_n - v_n, u_n)_X \leq 0 \tag{4.32}$$

or, equivalently,

$$(A_n u_n - A_n 0_X, u_n)_X \leq (v_n - f_n - A_n 0_X, u_n)_X.$$

Therefore, using assumption (4.20) and (4.23) it follows that

$$m_0\|u_n\|_X^2 \le m_n\|u_n\|_X^2 \le (A_nu_n - A_n0_X, u_n)_X \le (v_n - f_n - A_n0_X, u_n)_X,$$

which implies that

$$m_0\|u_n\|_X \le \|A_n0_X + f_n - v_n\|_X. \tag{4.33}$$

We now use assumption (4.22) to see that $A_n0_X \to A0_X$ in X which, combined with condition (4.21) shows that the sequence $\{A_n0_X + f_n - v\}$ is bounded in X. Therefore, the bound (4.33) implies that there exists $D > 0$, which does not depend on n, such that (3.16) holds.

Let

$$\delta_n = \widetilde{a}_n(D+1), \tag{4.34}$$

denote by B_n the closed ball of center 0_X and radius δ_n and let $\widetilde{K}_n$ be the subset of X given by

$$\widetilde{K}_n = K_n + B_n. \tag{4.35}$$

We use (4.30), (3.16) and notation (4.34) to see that $w_n \in B_n$ and, since (4.26) shows that $A_nu_n + f_n \in K_n$, (4.28) and (4.35) yield

$$Au_n + f \in \widetilde{K}_n. \tag{4.36}$$

Let $\widetilde{v} \in \widetilde{K}_n$. Using (4.35) we can write $\widetilde{v} = v + z$ where $v \in K_n$ and $z \in B_n$. Then, using (4.28), (4.26) and inequalities $\|w_n\|_X \le \delta_n$, $\|z\|_X \le \delta_n$, we find that

$$\begin{aligned}(Au_n + f - \widetilde{v}, u_n)_X &= (A_nu_n + f_n + w_n - v - z, u_n)_X\\ &= (A_nu_n + f_n - v, u_n)_X + (w_n - z, u_n)_X \le (w_n - z, u_n)_X\\ &\le \big(\|w_n\|_X + \|z\|_X\big)\|u_n\|_X \le 2\delta_n\|u_n\|_X.\end{aligned}$$

Now, using the bound (3.16) it follows that

$$(Au_n + f - \widetilde{v}, u_n)_X \le 2\delta_nD. \tag{4.37}$$

We now gather relations (4.36) and (4.37) to conclude that

$$u_n \in \Omega(\theta_n) \quad \text{with} \quad \theta_n = (\widetilde{K}_n, 2\delta_nD). \tag{4.38}$$

On the other hand, assumptions (4.22)(b), (4.25) and (4.29) show that $\widetilde{a}_n \to 0$ and, therefore, (4.34) implies that $\delta_n \to 0$. In addition, using this convergence, definition (4.35) and assumption (4.24), it is easy to see that $\widetilde{K}_n \xrightarrow{M} K$. It follows from here that

$$\theta_n \in \mathcal{C}_M. \tag{4.39}$$

We now use (4.38) and (4.39) to see that $\{u_n\}$ is a $\mathcal{T}_M$- approximating sequence for inclusion $\mathcal{P}$. Therefore, the $\mathcal{T}_M$-well-posedness of $\mathcal{P}$, guaranteed by Theorem 3, implies that $u_n \to u$ in X, which concludes the proof. □

We end this section with the remark that Corollary 2 can be used to provide the convergence of the sequence $\{u_n\}$ in Example 4 to the solution of the corresponding Problem $\mathcal{P}$. This arises from the fact that, in the framework of Example 4, the sequence $\{u_n\}$ is a $\mathcal{T}_M$-approximating sequence. Recall that the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_3$ and $\mathcal{T}_3$ cannot be use in order to prove this convergence. This illustrates the importance of the choice of the Tykhonov triple in employing the stategy presented in Section 2.

5 A convergence criterion

In this section we construct two Tykhonov triples which are equivalent with the Tykhonov triple $\mathcal{T}_{\mathcal{P}}$ introduced in Section 2 that, recall, is a maximal element of the set $(\mathcal{A}_{\mathcal{P}}, \leq)$ introduced in Remark 2. This will allow us to formulate a critetion of convergence to the solution of the inclusion (2.6). To introduce these triples we need some preliminaries.

First, eveywhere in this section we assume (2.1)–(2.3) and use the short hand notation Λ for the opertor Λ_ρ defined in (2.9) with any ρ arbitrary fixed in the interval $\Big(0, \frac{m'}{L'^2}\Big)$. Moreover, we introduce the approximating sets

$$\Omega_a(\theta) = \Big\{\, \widetilde{u} \in X \,:\, \|A\widetilde{u} + f - \Lambda(A\widetilde{u} + f)\|_X \leq \theta \,\Big\}, \tag{5.1}$$

$$\Omega_b(\theta) = \Big\{\, \widetilde{u} \in X \,:\, \|A\widetilde{u} + f - \Lambda(A\widetilde{u} + f)\|_X \leq \theta\big(\|A\widetilde{u} + f\|_X + 1\big) \Big\}, \tag{5.2}$$

for all $\theta \geq 0$. In addition, with the notation (3.12) and (3.13) for I and $\mathcal{C}$, respectively, we introduce the triples $\mathcal{T}_a$ and $\mathcal{T}_b$ defined by

$$\mathcal{T}_a = (I, \Omega_a, \mathcal{C}), \qquad \mathcal{T}_b = (I, \Omega_b, \mathcal{C}). \tag{5.3}$$

Note that Theorem 1 guarantees that Problem $\mathcal{P}$ has a unique solution u. Then, using (2.8) it is easy to see that $u \in \Omega_a(\theta)$ and $u \in \Omega_b(\theta)$, for each $\theta \in I$. This implies that $\Omega_a(\theta) \neq \emptyset$ and $\Omega_b(\theta) \neq \emptyset$ for each $\theta \in I$ and, therefore, $\mathcal{T}_a$ and $\mathcal{T}_b$ are Tykhonov triples in the sense of Definition 2.

Our first result in this section is the following.

Theorem 4. *Assume* (2.1)–(2.3)*. Then the Tykhonov triples* $\mathcal{T}_a$*,* $\mathcal{T}_b$ *and* $\mathcal{T}_{\mathcal{P}}$ *are equivalent.*

Proof. We use notation (3.1) and (3.2). We start by proving the inclusion

$$\mathcal{S}_{\mathcal{P}} \subset \mathcal{S}_{\mathcal{T}_a}. \tag{5.4}$$

Assume that $\{u_n\}$ is a sequence with converge to u in X, that is $\{u_n\} \in \mathcal{S}_{\mathcal{P}}$. Denote by σ and σ_n the elements of X given by

$$\sigma = Au + f, \tag{5.5}$$

$$\sigma_n = Au_n + f \qquad \forall\, n \in \mathbb{N}. \tag{5.6}$$

Then, using (2.7) it follows that

$$u = A^{-1}(\sigma - f), \tag{5.7}$$

$$u_n = A^{-1}(\sigma_n - f) \qquad \forall\, n \in \mathbb{N} \tag{5.8}$$

and, moreover,

$$\sigma_n \to \sigma \quad \text{in} \quad X. \tag{5.9}$$

Fix $n \in \mathbb{N}$. Using (2.9) it follows that there exists $k \in [0, 1)$ such that

$$\|\Lambda\tau - \Lambda\omega\|_X \leq k\, \|\tau - \omega\|_X \qquad \forall\, \tau,\, \omega \in X \tag{5.10}$$

and, using (2.8) we deduce that

$$\Lambda\sigma = \sigma. \tag{5.11}$$

We now write

$$\|\sigma_n - \Lambda\sigma_n\|_X \le \|\sigma_n - \sigma\|_X + \|\sigma - \Lambda\sigma_n\|_X,$$

then we use (5.11) and (5.10) to deduce that

$$\|\sigma_n - \Lambda\sigma_n\|_X \le (1+k)\|\sigma_n - \sigma\|_X.$$

This implies that $u_n \in \Omega_a(\theta_n)$ with $\theta_n = (1+k)\|\sigma_n - \sigma\|_X$ and, since (5.9) guarantees that $\theta_n \to 0$ we deduce that $\{u_n\}$ is a $\mathcal{T}_a$-approximating sequence, that is $\{u_n\} \subset \mathcal{S}_{\mathcal{T}_a}$. It follows from above that the inclusion (5.4) holds.

Next, we prove the inclusion

$$\mathcal{S}_{\mathcal{T}_b} \subset \mathcal{S}_{\mathcal{P}}. \tag{5.12}$$

To this end we consider an approximating sequence $\{u_n\} \in \mathcal{S}_{\mathcal{T}_b}$. We keep the notation (5.5), (5.6) and use (5.2) to see that that there exists a sequence $\{\theta_n\} \subset \mathbb{R}_+$ such that $\theta_n \to 0$ and, moreover,

$$\|\sigma_n - \Lambda\sigma_n\|_X \le \theta_n\big(\|\sigma_n\|_X + 1\big) \qquad \forall\, n \in \mathbb{N}. \tag{5.13}$$

We now write

$$\|\sigma_n - \sigma\|_X \le \|\sigma_n - \Lambda\sigma_n\|_X + \|\Lambda\sigma_n - \sigma\|_X$$

then use (5.13), (5.11) and (5.10) to deduce that

$$\|\sigma_n - \sigma\|_X \le \theta_n\big(\|\sigma_n\|_X + 1\big) + k\,\|\sigma_n - \sigma\|_X$$

or, equivalently,

$$(1-k)\|\sigma_n - \sigma\|_X \le \theta_n\big(\|\sigma_n\|_X + 1\big). \tag{5.14}$$

On the other hand, writing

$$\|\sigma_n\|_X \le \|\sigma_n - \Lambda\sigma_n\|_X + \|\Lambda\sigma_n - \Lambda 0_X\|_X + \|\Lambda 0_X\|_X$$

and using (5.13), (5.10) yields

$$\|\sigma_n\|_X \le \theta_n\big(\|\sigma_n\|_X + 1\big) + k\,\|\sigma_n\|_X + \|\Lambda 0_X\|_X$$

or, equivalently,

$$(1-k-\theta_n)\|\sigma_n\|_X \le \theta_n + \|\Lambda 0_X\|_X. \tag{5.15}$$

Now, since $\theta_n \to 0$ and $k \in [0,1)$, for n large enough we may assume that $\theta_n \le \frac{1-k}{2}$ which implies that $1 - k - \theta_n \ge \frac{1-k}{2}$ and $\theta_n \le \frac{1}{2}$. So, inequality (5.15) shows that

$$\|\sigma_n\|_X \le \frac{1}{1-k}(1 + 2\|\Lambda 0_X\|_X). \tag{5.16}$$

We now combine inequalities (5.14) and (5.16), then use the convergence $\theta_n \to 0$ to deduce that $\sigma_n \to \sigma$ in X. This convergence, (5.8), (5.7) and (2.7) imply that $u_n \to u$ in X and, therefore, $\{u_n\} \in \mathcal{S}_{\mathcal{P}}$. We conclude from above that inclusion (5.12) holds.

On the other hand, it is easy to see that $\Omega_a(\theta) \subset \Omega_b(\theta)$, for each $\theta \in I$ which implies that

$$\mathcal{S}_{\mathcal{T}_a} \subset \mathcal{S}_{\mathcal{T}_b}. \tag{5.17}$$

We now gather the inclusions (5.4), (5.17) and (5.12) to see that $\mathcal{S}_{\mathcal{T}_a} = \mathcal{S}_{\mathcal{T}_a} = \mathcal{S}_{\mathcal{P}}$ which concludes the proof. □

We now use Theorem 4 in order to provide a different proof of Corollary 2 and, to this end, we assume in what follows that (2.1)–(2.3), (4.19)–(4.25) hold.

Proof of Corollary 2. The proof is structured in thereee steps, as follows.

Step i) Preliminaries.

We use assumptions (2.2), (4.20) and (4.23), denote $m = \min\{m_A, m_0\}$, $L = \max\{L_A, L_0\}$ and deduce that the operators A and A_n are stongly monotone Lipschitz continuous operators with the same constants m and L, respectively, for each $n \in \mathbb{N}$. We now take $\rho_0 = \frac{m^3}{L^2}$ and use (2.9) to see that the operators

$$\Lambda_n \xi = P_{K_n}\big(\xi - \rho_0 A_n^{-1}(\xi - f_n)\big), \quad \Lambda \xi = P_K\big(\xi - \rho_0 A^{-1}(\xi - \eta)\big) \quad \forall\, \xi \in X \tag{5.18}$$

satisfy the inequalities

$$\|\Lambda_n \xi - \Lambda_n \eta\|_X \le k_0 \|\xi - \eta\|_X, \tag{5.19}$$

$$\|\Lambda \xi - \Lambda \eta\|_X \le k_0 \|\xi - \eta\|_X, \tag{5.20}$$

for all ξ, $\eta \in X$, $n \in \mathbb{N}$, where $k_0 \in [0,1)$ depends only on m and L.

Let $n \in \mathbb{N}$ be fixed and let

$$\sigma_n = A_n u_n + f_n, \tag{5.21}$$

which implies that

$$u_n = A^{-1}(\sigma_n - f_n). \tag{5.22}$$

Moreover, following (2.8) and (2.9), we deduce that $\Lambda_n \sigma_n = \sigma_n$ and $\Lambda \sigma = \sigma$.

ii) Proof of the convergence

$$\sigma_n \to \sigma \qquad \text{in} \ \ X, \quad \text{as} \ \ n \to \infty. \tag{5.23}$$

Let $n \in \mathbb{N}$. We use equalities $\Lambda_n \sigma_n = \sigma_n$, $\Lambda \sigma = \sigma$ and (5.19) to write

$$\begin{aligned} \|\sigma_n - \sigma\|_X = \|\Lambda_n \sigma_n - \Lambda \sigma\|_X &\le \|\Lambda_n \sigma_n - \Lambda_n \sigma\|_X + \|\Lambda_n \sigma - \Lambda \sigma\|_X \\ &\le k_0 \|\sigma_n - \sigma\|_X + \|\Lambda_n \sigma - \Lambda \sigma\|_X \end{aligned}$$

which implies that

$$\|\sigma_n - \sigma\|_X \le \frac{1}{1 - k_0}\, \|\Lambda_n \sigma - \Lambda \sigma\|_X \tag{5.24}$$

On the other hand, using the definition (5.18) of the operators Λ_n and Λ we have

$$\Lambda_n \sigma = P_{K_n}\big(\sigma - \rho_0 A_n^{-1}(\sigma - f_n)\big), \quad \Lambda \sigma = P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)$$

where, here and below, P_{K_n} represents the projection operator on K_n. This implies that

$$\|\Lambda_n\sigma - \Lambda\sigma\|_X = \|P_{K_n}\big(\sigma - \rho_0 A_n^{-1}(\sigma - f_n)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X$$
$$\leq \|P_{K_n}\big(\sigma - \rho_0 A_n^{-1}(\sigma - f_n)\big) - P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X$$
$$+\|P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X$$

and, using the nonexpansivity of the operator P_{K_n}, we find that

$$\|\Lambda_n\sigma - \Lambda\sigma\|_X \leq \rho_0\|A_n^{-1}(\sigma - f_n) - A^{-1}(\sigma - f)\|_X$$
$$+\|P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f_n)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X$$
$$\leq \rho_0\|A_n^{-1}(\sigma - f_n) - A^{-1}(\sigma - f_n)\|_X + \rho_0\|A^{-1}(\sigma - f_n) - A^{-1}(\sigma - f)\|_X$$
$$+\|P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X.$$

Next, the Lipschitz continuity of th operator A^{-1} yields

$$\|\Lambda_n\sigma - \Lambda\sigma\|_X \leq \rho_0\|A_n^{-1}(\sigma - f_n) - A^{-1}(\sigma - f_n)\|_X + \rho_0 L'\|f_n - f\|_X \tag{5.25}$$
$$+\|P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X.$$

We shall prove in what follows that

$$\|A_n^{-1}(\sigma - f_n) - A^{-1}(\sigma - f_n)\|_X \to 0. \tag{5.26}$$

To this end denote

$$v_n = A_n^{-1}(\sigma - f_n), \qquad w_n = A^{-1}(\sigma - f_n) \tag{5.27}$$

which implies that $A_n v_n = A w_n$. Using this equality and assumptions (4.23), (4.20), (4.22) we have

$$m_0\|v_n - w_n\|_X^2 \leq m_n\|v_n - w_n\|_X^2 \leq (A_n v_n - A_n w_n, v_n - w_n)_X$$
$$= (A w_n - A_n w_n, v_n - w_n)_X \leq \|A w_n - A_n w_n\|_X \|v_n - w_n\|_X$$
$$\leq a_n(\|w_n\|_X + 1)\|v_n - w_n\|_X,$$

and, therefore,

$$m_0\|v_n - w_n\|_X \leq a_n(\|w_n\|_X + 1).$$

Now, since the convergence (4.25) guarantees that the sequence $\{w_n\}$ is bounded in X, assumption (4.22)(b) implies that $\|v_n - w_n\|_X \to 0$ which, together with (5.27), shows that (5.26) holds.

Next, we use assumption (4.24) and Proposition 1 to see that

$$\|P_{K_n}\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big) - P_K\big(\sigma - \rho_0 A^{-1}(\sigma - f)\big)\|_X \to 0. \tag{5.28}$$

Finally, note that assumption (4.25) implies that

$$\|f_n - f\|_X \to 0. \tag{5.29}$$

We now use inequalities (5.24), (5.25) and convergences (5.26), (5.28), (5.29) to deduce that (5.23) holds.

iii) Proof of the convergence (2.10).

Let $n \in \mathbb{N}$. Using (5.21), (5.5) and equality $\sigma = \Lambda\sigma$ we find that

$$\begin{aligned} &\|Au_n + f - \Lambda(Au_n + f)\|_X \\ &\leq \|Au_n - A_n u_n\|_X + \|\sigma_n + f - f_n - \sigma\|_X \\ &+\|\Lambda\sigma - \Lambda(Au_n - A_n u_n + \sigma_n + f - f_n)\|_X \\ &\leq \|Au_n - A_n u_n\|_X + \|\sigma_n - \sigma\|_X + \|f_n - f\|_X \\ &+\|\Lambda\sigma - \Lambda(Au_n - A_n u_n + \sigma_n + f - f_n)\|_X \end{aligned}$$

and, since Λ is a contraction with constant k_0, we deduce that

$$\|Au_n + f - \Lambda(Au_n + f)\|_X \leq \theta_n \tag{5.30}$$

where

$$\theta_n = (1 + k_0)\big(\|Au_n - A_n u_n\|_X + \|\sigma_n - \sigma\|_X + \|f_n - f\|_X\big). \tag{5.31}$$

Now, inequality (5.30) combined with definition (5.1) shows that $u_n \in \Omega_a(\theta)$. Moreover, assumption (4.22), the bound (3.16) and the convergences (5.23), (4.25) guarantee that $\theta_n \to 0$, i.e. $\{\theta_n\} \subset \mathcal{C}$ where, recall, $\mathcal{C}$ is defined by (3.13). We conclude from here that $\{u_n\}$ is a $\mathcal{T}_a$-approximating sequence. The, using Theorem 4 we deduce that the convergence (2.10) holds, which ends the proof. □

We end this section with the following remark.

Remark 3. *Note that Theorem* 4 *states that* $\mathcal{S}_{\mathcal{T}_a} = \mathcal{S}_{\mathcal{T}_b} = \mathcal{S}_{\mathcal{P}}$ *and, therefore, the definition* (3.1) *shows that a sequence* $\{u_n\}$ *converges to* u *in* X *if and only if* $\{u_n\} \in \mathcal{S}_{\mathcal{T}_a} = \mathcal{S}_{\mathcal{T}_b}$ *or, equivalently, if and only if there exists a sequence* $\{\theta_n\} \subset \mathbb{R}_+$ *such that for any* $n \in \mathbb{N}$ *one of the two inequalities below hold.*

$$\begin{aligned} &\|Au_n + f - \Lambda(Au_n + f)\|_X \leq \theta_n, \\ &\|Au_n + f - \Lambda(Au_n + f)\|_X \leq \theta_n\big(\|Au_n + f\|_X + 1\big). \end{aligned}$$

We conclude from here that Theorem 4 *provides necessary and sufficent conditions which guarantee the convergence* (2.10)*, i.e., it represents a criterion of convergence. This criterion is intrinsic, since no reference to the solution* u *of Problem* $\mathcal{P}$ *is made on the inequalities above.*

Acknowledgement. This work has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No 823731 CONMECH.

References

[1] K. Atkinson, W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Third edition, Springer-Verlag, New York (2009).

[2] F. Chouly, P. Hild, On convergence of the penalty method for unilateral contact problems, *Applied Numerical Mathematics*, **65**, 27–48 (2013).

[3] G. Duvaut, J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin (1976).

[4] Y. P. Fang, H. J. Huang, J. C. Yao, Well-posedness by perturbations of mixed variational inequalities in Banach spaces, *Eur. J. Oper. Res.*, **201**, 682–692 (2010).

[5] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York (1984).

[6] D. Goeleven, D. Mentagui, Well-posed hemivariational inequalities, *Numerical Functional Analysis and Optimization*, **16**, 909–921 (1995).

[7] W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, *Mathematics and Mechanics of Solids*, **23**, 279–293 (2018).

[8] W. Han, M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics, **30**, American Mathematical Society, Providence, RI–International Press, Somerville, MA (2002).

[9] R. Hu, M. Sofonea, Y. B. Xiao, Tykhonov triples and convergence results for hemivariational inequalities, *Nonlinear Analysis: Modelling and Control*, **26**, 271-292 (2021).

[10] X. X. Huang, X. Q. Yang, D. L. Zhu, Levitin-Polyak well-posedness of variational inequality problems with functional constraints, *J. Glob. Optim.*, **44**, 159–174 (2009).

[11] N. Kikuchi, J. T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia (1988).

[12] R. Lucchetti, F. Patrone, A characterization of Tychonov well-posedness for minimum problems with applications to variational inequalities, *Numer. Funct. Anal. Optim.*, **3**, 461–476 (1981).

[13] R. Lucchetti, F. Patrone, Some properties of "well-posedness" variational inequalities governed by linear operators, *Numer. Funct. Anal. Optim.*, **5**, 349–361 (1983).

[14] E. S. Levitin, B. T. Polyak, Convergence of minimizing sequences in conditional extremum problem, *Soviet Math. Dokl.*, **7**, 764–767 (1966).

[15] F. Nacry, M. Sofonea, A class of nonlinear inclusions and sweeping processes in Solid Mechanics, *Acta Applicandae Mathematicae*, **171** (2021), https://doi.org/10.1007/s10440-020-00380-4, to appear.

[16] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston (1985).

[17] M. Sofonea, Analysis and control of stationary inclusions in contact mechanics, *Nonlinear Analysis Series B, Real Word Applications*, (2021), https://doi.org/10.1016/j.nonrwa.2021.103335, to appear.

[18] M. Sofonea, A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series, **398**, Cambridge University Press (2012).

[19] M. Sofonea, S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London (2018).

[20] M. Sofonea, M. Shillor, Tykhonov well-posedness and convergence results for contact problems with unilateral constraints, *Technologies*, **9**, 1–25 (2021).

[21] M. Sofonea, D. A. Tarzia, On the Tykhonov well-posedness of an antiplane shear problem, *Mediterranean Journal of Mathematics*, https://doi.org/10.1007/s00009-020-01577-5, to appear.

[22] M. Sofonea, D. A. Tarzia, Tykhonov well-posedness of a heat transfer problem with unilateral constraints, *Applications of Mathematics*, (2021), https://doi.org/10.21136/AM.2021.0172-20, to appear.

[23] M. Sofonea, Y. B. Xiao, On the well-posedness concept in the sense of Tykhonov, *Journal of Optimization Theory and Applications*, **183**, 139–157 (2019).

[24] M. Sofonea, Y. B. Xiao, Tykhonov well-posedness of a viscoplastic contact problem, *Journal of Evolution Equations and Control Theory*, **9**, 1167–1185 (2020).

[25] M. Tsukada, Convergence of best approximations in a smooth Banach space, *J. Approx. Theory*, **40**, 301–309 (1984).

[26] A. N. Tykhonov, On the stability of functional optimization problems, *USSR Comput. Math. Math. Phys.*, **6**, 631–634 (1966).

[27] Y. M. Wang, Y. B. Xiao, X. Wang, Y. J. Cho, Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, *Journal of Nonlinear Sciences and Applications*, **9**, 1178–1192 (2016).

[28] Y. B. Xiao, N. J. Huang, M. M. Wong, Well-posedness of hemivariational inequalities and inclusion problems, *Taiwanese Journal of Mathematics*, **15**, 1261–1276 (2011).

[29] Y. B. Xiao, M. Sofonea, Tykhonov triples, well-posedness and convergence results, *Carpathian Journal of Mathematics*, **37**, 135–143 (2021).

Received: 18.08.2021
Accepted: 03.09.2021

Laboratory of Mathematics and Physics (LAMPS),
University of Perpignan Via Domitia,
52 Avenue Paul Alduy, 66860 Perpignan, France
E-mail: `sofonea@univ-perp.fr`