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Signed partitions - A 'balls into urns' approach *

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Abstract

Using Reiner's definition of Stirling numbers of the second kind for the group of signed permutations, we provide a 'balls into urns' approach for proving a generalization of a well-known identity concerning the classical Stirling numbers $S(n,k)$ of the second kind:

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot x(x-1)\cdots(x-k+1).$$

We also present a combinatorial proof (based on Feller's coupling) of the defining identity for the Stirling numbers of the first kind in the group of signed permutations.

Our proofs are self-contained and accessible also for non-experts.

Key Words: Stirling number, signed partitions, 'balls into urns' approach.
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1 Introduction

A *partition* of a set A is a set of non-empty pairwise disjoint subsets of A (called *blocks*) whose union is A. The partitions of the set $[n] = \{1, \ldots, n\}$ having k blocks are enumerated by the *Stirling numbers of the second kind*, denoted by $S(n,k)$ or $\left\{ n \atop k \right\}$ (see [11, page 81]). These numbers arise in a variety of problems in enumerative combinatorics (e.g. moments of Poisson's distribution); they have many combinatorial interpretations, and have been generalized in various contexts and in many ways (see e.g. the survey of Boyadzhiev [4]).

One of the celebrated results concerning Stirling numbers of the second kind is the following: Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:

$$x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} [x]_k, \tag{1}$$

where $[x]_k := x(x-1)\cdots(x-k+1)$ is the *falling factorial of degree k* and $[x]_0 := 1$.

There are some known proofs for this identity. A combinatorial one, realizing x^n as the number of functions from the set $\{1, \ldots, n\}$ to the set $\{1, \ldots, x\}$ (for x integer), is presented by Stanley [11, Eqn. (1.94d); its proof is in page 83], and we quote it here (where $\#N = n$ and $\#X = x$):

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"The left-hand side is the total number of functions $f : N \to X$. Each such function is surjective onto a unique subset $Y = f(N)$ of X satisfying $\#Y \leq n$. If $\#Y = k$, then there are $k!S(n,k)$ such functions, and there are $\binom{x}{k}$ choices of subsets Y of X with $\#Y = k$. Hence:

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k} = \sum_{k=0}^{n} S(n,k)[x]_k."$$

Note that this equality is a polynomial identity, consisting of polynomials of degree n. By the fundamental theorem of Algebra, the two sides of the equation will be identical if the polynomials coincide on at least $n+1$ values, and it was shown by Stanley for any natural number x.

There is a nice generalization of Identity (1), which appears in Remmel and Wachs [9] and Bala [2], see also [6]. In order to demonstrate this generalization combinatorially, we use the Stirling numbers of type B of the second kind, denoted by $S_B(n,k)$ or $\left\{ n \atop k \right\}_B$, which are related to the group of signed permutations, denoted by B_n (see exact definitions in Section 2). The generalization is:

Theorem 1.1. *Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:*

$$x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\}_B [x]_k^B, \tag{2}$$

where $[x]_k^B := (x-1)(x-3)\cdots(x-2k+1)$ and $[x]_0^B := 1$.

Remmel and Wachs [9] proved this equality using their interpretation of $S_B(n,k)$ as counting certain configurations in Rook theory (specifically, this is $S_{n,k}^{0,2}(1,1)$ in their notation). Bala [2] proved this equality using a generating-functions technique ($S_{(2,0,1)}$ in his notation).

In [1], a geometric way to prove Equation (2), interpreting x^n as counting the number of points in an n-dimensional cubical lattice, is presented.

The purpose of this note is to introduce a simple combinatorial proof of Equation (2), which interprets its both sides using a 'balls into urns' approach. Note that our proof is actually a generalization of the proof for Identity (1), that we have quoted above from Stanley, for the group of signed permutations.

2 Signed permutations and signed partitions

In this section, we introduce the group of signed permutations. Then, we define the objects which the Stirling numbers of type B of the second kind count, introduced by Reiner [8].

2.1 The group of signed permutations

We start with a combinatorial motivation: assume that we have a pack of n cards numbered by $\{1,\ldots,n\}$ such that each card is colored by two colors: red and black, one on each side of the card. It is easy to observe that the number of ways to put these cards in a row is

$2^n \cdot n!$. The set of such configurations carries a structure of a group, known as the *group of signed permutations*, also known as the *Coxeter group of type* B or the *hyperoctahedral group*.

Here is its formal definition:

Definition 2.1. Denote $[\pm n] := \{\pm 1, \dots, \pm n\}$. A *signed permutation* is a bijective function:

$$\pi : [\pm n] \to [\pm n],$$

satisfying: $\pi(-i) = -\pi(i)$ for all $1 \le i \le n$. The group of signed permutations of the set $[\pm n]$ (with respect to composition of functions) is denoted by B_n.

Example 2.2. Here is an example of a signed permutation:

$$\pi = \left(\begin{array}{ccccc|ccccc} -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 \\ 4 & -5 & 1 & -2 & 3 & -3 & 2 & -1 & 5 & -4 \end{array} \right) \in B_5.$$

Note that it is sufficient to know the values of $\pi(1), \dots, \pi(n)$, so that the above permutation π can also be written (in window notation) as:

$$\pi = [-3, 2, -1, 5, -4].$$

Considering B_n as a subgroup of the symmetric group on $2n$ elements, S_{2n}, in the natural way, we can also write every signed permutation as a product of disjoint cycles. We consider the pairs of cycles C and $-C = \{-x \mid x \in C\}$ (if $-C \neq C$) as one unit, i.e. although C and $-C$ are two disjoint cycles, we consider them as two parts of the same cycle (since they act on the same set of absolute values).

Moreover, we distinguish between two types of cycles: a cycle C will be called a *non-split cycle* (or an *odd cycle*) if the following condition holds: "$i \in C$ if and only if $-i \in C$", and will be called a *split cycle* (or an *even cycle*) otherwise. The names *odd* (and *even*, respectively) are coined according to the number of negative elements in the image of $\{1, \dots, n\}$ under that cycle.

We say that a permutation written as a sequence of disjoint cycles is presented in *standard form*, if its cycles are ordered in such a way that the sequence composed by the smallest positive elements of each cycle increases.

Example 2.3. The permutation π in Example 2.2 can be written as a product of disjoint cycles as follows:

$$\pi = (\mathbf{1}, -3)(3, -1)(\mathbf{2})(-2)(\mathbf{4}, 5, -4, -5).$$

Note that this permutation is indeed presented in standard form, as the sequence composed by the smallest positive elements of each cycle (marked) is $1, 2, 4$, which indeed increases.

Its even cycles are $(1, -3)(3, -1)$ and $(2)(-2)$, as the images of $\{1, \dots, n\}$ under the cycles $(1, -3)(3, -1)$ and $(2)(-2)$ are $\{-1, -3\}$ and $\{2\}$, respectively, so the number of negative images in each cycle is even.

On the other hand, its odd cycle is $(4, 5, -4, -5)$, and the image of $\{1, \dots, n\}$ by this cycle has one negative element $\{-4\}$.

2.2 Signed partitions

Following Reiner [8], we define:

Definition 2.4. A *signed partition* is a set partition of $[\pm n]$ into blocks, which satisfies the following conditions:

- There exists at most one block satisfying $-B = B$, called the *zero-block*. It is a subset of $[\pm n]$ of the form $\{\pm i \mid i \in S\}$ for some $S \subseteq [n]$.
- If B appears as a block in the partition, then $-B$ also appears in that partition.

Example 2.5. The following partitions

$$P_1 = \{\{3, -3\}, \{-2, 1\}, \{2, -1\}, \{-4, 5\}, \{4, -5\}\},$$

$$P_2 = \{\{3\}, \{-3\}, \{-2, 1\}, \{2, -1\}, \{-4, 5\}, \{4, -5\}\},$$

are respectively a signed partition of $[\pm 5]$ with a zero block $\{3, -3\}$ and a signed partition of $[\pm 5]$ without a zero-block.

We denote by $S_B(n, k)$ or $\left\{ {n \atop k} \right\}_B$ the number of signed partitions of $[\pm n]$ having exactly k pairs of nonzero blocks. These numbers are called *Stirling numbers of type B of the second kind*. They form the sequence A039755 in OEIS [10]. Table 1 records these numbers for small values of n and k.

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	4	1				
3	1	13	9	1			
4	1	40	58	16	1		
5	1	121	330	170	25	1	
6	1	364	1771	1520	395	36	1

Table 1: Stirling numbers of type B of the second kind $S_B(n, k)$

3 The combinatorial proof

In this section, we present a direct combinatorial proof for Theorem 1.1, where x^n is interpreted as the number of assignments of n numbered balls into x distinguishable urns.

Note that Equation (2) is a polynomial identity of degree n. As mentioned above, it is sufficient to prove it for at least $n + 1$ values. We prove it for any odd natural number, and then the equality follows.

Direct combinatorial proof. Let $m \in \mathbb{N}$ be an odd number. We will show:

$$m^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_B [m]_k^B. \tag{3}$$

The left-hand side of Equation (3) is the number of assignments of n numbered balls into m distinguishable urns. In the right-hand side, we associate for each k, $0 \le k \le n$,

$$[m]_k^B = (m-1)(m-3)\cdots(m-2k+1)$$

assignments to each one of the signed partitions enumerated by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_B$ as will be presented in the squeul, and then we sum them up to get the total number of assignments, thus proving the identity.

Let $\mathcal{B} = \{B_0, B_1, -B_1, \dots, B_k, -B_k\}$ be a signed partition, where B_0 is the zero-block (which might not exist). Note that by the definition of $[x]_k^B$ which appeared in Theorem 1.1, we have $[m]_k^B = 0$ for $m < 2k$ (since m is odd, one of the factors in $[m]_k^B$ is 0), so we may assume that $k < \frac{m}{2}$.

For the proof, we impose an order satisfying the following rules on each signed partition $\mathcal{B}$:

- The zero-block B_0, if exists, is located as the first block.
- The blocks B and $-B$ are adjacent, and the internal order between B and $-B$ is chosen in such a way that the block which contains the minimal positive element of $B \cup -B$ is located first.
- Every two pairs of blocks are ordered in such a way that the pair having a smaller minimal positive element precede.

For example, the following signed partition is properly ordered:

$$\{\{5,-5\},\{\mathbf{1},-3\},\{-1,3\},\{\mathbf{2},4\},\{-2,-4\}\}.$$

For convenience, we consider an assignment of n balls into m urns as a function $f : [n] \to [m]$, and associate with a signed partition $\mathcal{B}$ the set of ball assignments according to the following procedure:

- For any *positive* $i \in B_0$ (the zero-block), define: $f(i) = 1$, i.e. insert ball number i in urn number 1.
- Choose an urn p out of the $m-1$ remaining urns ($2 \le p \le m$), and insert the positive elements of the block B_1 to p. The absolute values of the negative elements of B_1 (i.e. the positive elements of $-B_1$, if they exist) will be inserted to the next urn in cyclical order excluding the urn number 1 (which might have already been occupied by the positive elements of the zero-block). The choice of p can be done in $m-1$ different ways.

- Pass to the pair of blocks B_2 and $-B_2$. Similarly, choose a new urn p' out of the $m-3$ remaining urns (the urn number 1 might have been occupied by the positive elements of the zero-block, and two additional urns are already occupied by the elements of the pair of blocks B_1 and $-B_1$), and insert the positive elements of B_2 to the urn p'. For each negative $i \in B_2$, the absolute value of i will be inserted to the next unoccupied urn in cyclical order. The choice of p' may be done in $m-3$ different ways.
- Proceeding this way, associate a set of $[m]_k^B$ functions from $[n]$ to $[m]$ to each signed partition having k pairs of nonzero blocks.

Conversely, we now recover the signed partition $\mathcal{B}$ from a given function $f : [n] \to [m]$. Define:

$$B_0 = \{\pm i \mid f(i) = 1\}.$$

Mark the number 1 as used. Let $k \in [n]$ be the minimal positive number such that $f(k) \neq 1$. Denote $a := f(k)$. Let $b \in [m] - \{1\}$ be the next unused number in cyclical order. Define:

$$B_1 = \{i \in [n] \mid f(i) = a\} \cup \{-i \mid f(i) = b\},$$

and add the pair of blocks B_1 and $-B_1$ to the signed partition $\mathcal{B}$. Now mark the numbers a, b as used. Proceeding along these lines, we arrive at the signed partition $\mathcal{B}$ which induces the function f. $\square$

Example 3.1. Let $n = 6$ and $m = 7$. Consider the signed partition (which is properly ordered):

$$\mathcal{B} = \{\{\pm 1\}, \{2, -3, 5\}, \{-2, 3, -5\}, \{4, -6\}, \{-4, 6\}\}$$

of $[\pm 6]$. Every function $f : [6] \to [7]$ which is induced by the signed partition $\mathcal{B}$ inserts 1 (which is the unique positive element of the zero-block) to urn number 1 (see Figure 1).

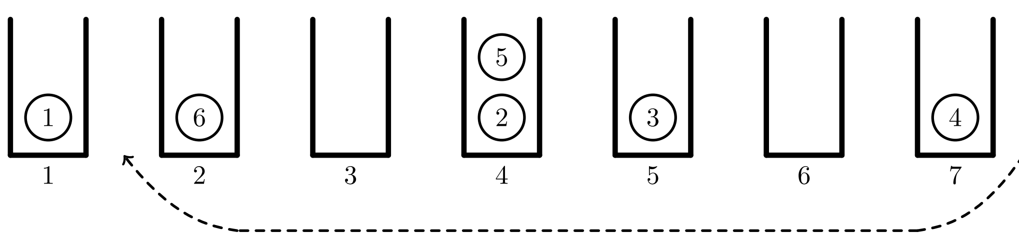

Figure 1: An assignment of 6 balls into 7 urns

Now we pass to the first block $B_1 = \{2, -3, 5\}$ (together with its negative block $-B_1 = \{-2, 3, -5\}$): we have to choose a value for the images of 2 and 5 out of 6 possibilities (or choose an urn out of urns $\{2, \ldots, 7\}$ to the balls numbered 2 and 5). Take for example $f(2) = f(5) = 4$. Then we have to assign $f(3) = 5$, which is the next free value in cyclical order.

The next block is $B_2 = \{4, -6\}$ (together with its negative block $-B_2 = \{-4, 6\}$). For this block, we are left with 4 possibilities for assigning values. Choose for instance $f(4) = 7$

and so we must assign $f(6) = 2$, which is the next free value in cyclical order. The resulting 'balls into urns' assignment is depicted in Figure 1.

Conversely, given the 'balls into urns' assignment obtained above:

$$f(1) = 1,\ f(2) = 4,\ f(3) = 5,\ f(4) = 7,\ f(5) = 4, f(6) = 2,$$

we recover the signed partition $\mathcal{B}$, which induced this assignment, as follows:

- Only 1 is sent to 1, so we have the zero-block $B_0 = \{\pm 1\}$.
- The minimal unassigned element is 2, which is sent by f to 4, so the positive part of the next block will be $f^{-1}(\{4\}) = \{2, 5\}$ and the negative part will be $f^{-1}(\{5\}) = \{3\}$ (since 5 is the next unused value after 4 in cyclical order). Hence, we get the pair of blocks: $B_1 = \{2, 5, -3\}$ and $-B_1 = \{-2, -5, 3\}$.
- Now, the current next unassigned element is 4, which is sent by f to 7, so the positive part of the block will be $f^{-1}(\{7\}) = \{4\}$ and the negative part will be $f^{-1}(\{2\}) = \{6\}$ (since 2 is the next unused value after 7 in cyclical order). Hence, we get the pair of blocks: $B_2 = \{4, -6\}$ and $-B_2 = \{6, -4\}$.

Hence, we get that the signed partition is:

$$\mathcal{B} = \{\{\pm 1\}, \{2, 5, -3\}, \{-2, -5, 3\}, \{4, -6\}, \{-4, 6\}\},$$

which indeed was our original signed partition.

4 Stirling number of the first kind

Recall that the (unsigned) Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined by the following identity:

$$t(t+1)\cdots(t+n-1) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k. \tag{4}$$

A known combinatorial interpretation for these numbers is given by considering them as counting the number of permutations of the set $[n]$ having k cycles.

Bala [2] presented a generalization of the Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_r$ for each $r \geq 0$. In the case $r = 2$, they are defined by the following equation:

$$(t+1)(t+3)\cdots(t+2n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_2 t^k. \tag{5}$$

We present now a combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_2$, based on the cycles of the group of signed permutations B_n. The idea is based on Feller coupling [5, p. 815] (see also [3, p. 96] and [7]), and is probably known among the experts. Since we have not found any explicit reference for this argument, we present it here.

Theorem 4.1. *For each $k \in \{1, \ldots, n\}$, the coefficient of t^k in Equation (5) is the number of signed permutations having exactly k non-split cycles.*

Proof. After expanding the left hand side of Equation (5), we get an expression of the form $\sum_{k=0}^{n} c_k t^k$. Note that the coefficient c_k is a sum of products of the form $1 \cdot 3 \cdot 5 \cdots (2n-1)$ such that exactly k numbers from this product are missing. For a specific product of this form, denote the missing numbers by $a_1, \ldots, a_k$. Such a product counts the number of signed permutations having $\lceil \frac{a_i}{2} \rceil$ as minimum elements (in absolute value) of the k non-split cycles (see Example 4.2 below). For each summand of the coefficient of t^k, one can construct $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{a_1 \cdot a_2 \cdots a_k}$ signed permutations having exactly k non-split cycles, and hence the equality follows. $\square$

Here, we give an explicit example of presenting the possible signed permutations associated with a specific product, based on the idea presented in the proof.

Example 4.2. Let $n = 8$. One of the summands contributing to the coefficient of t^3 in the product $(t+1)(t+3)\cdots(t+15)$ is: $1 \cdot 3 \cdot 7 \cdot 9 \cdot 15$ (with three numbers missing: $a_1 = 5,\ a_2 = 11,\ a_3 = 13$). We construct all signed permutations having three non-split cycles, such that $3, 6, 7$ are the minima of these non-split cycles.

Now, we embed the other numbers:

- First, the number 1 must be the minimum of a split cycle.
- For the number 2, we have exactly 3 options: either put 2 or -2 right after 1, or start a new split cycle with the number 2.
- For the number $3 = \lceil \frac{5}{2} \rceil = \lceil \frac{a_1}{2} \rceil$, we have only one option, since it opens the next non-split cycle.
- For the number 4, we have 7 options as follows: it can be placed right after 1 either as 4 or as -4, it can be placed after 2 as 4 or as -4, it can be placed after 3 as 4 or as -4, or it can be placed as the minimum element of a new split cycle.
- Continuing in this manner, one can see that $1 \cdot 3 \cdot 7 \cdot 9 \cdot 15$ is the number of signed permutations such that the numbers $3 = \lceil \frac{a_1}{2} \rceil = \lceil \frac{5}{2} \rceil,\ 6 = \lceil \frac{a_2}{2} \rceil = \lceil \frac{11}{2} \rceil,\ 7 = \lceil \frac{a_3}{2} \rceil = \lceil \frac{13}{2} \rceil$ are the minimum elements of the three non-split cycles.

An example of a signed permutation constructed according to these restrictions is

$$(1,-2)(-1,2)(\mathbf{3},\mathbf{4},-\mathbf{3},-\mathbf{4})(5,-8)(-5,8)(\mathbf{6},-\mathbf{6})(\mathbf{7},-\mathbf{7}),$$

where the non-split cycles are emphasized.

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References

[1] E. Bagno, R. Biagioli, D. Garber, Some identities involving second kind Stirling numbers of types B and D, *Elect. J. Combin.*, **26 (3)**, #P3.9, (2019).

[2] P. Bala, A 3-parameter family of generalized Stirling numbers (2015), Electronic version: https://oeis.org/A143395/a143395.pdf.

[3] A. T. Benjamin, J. J. Quinn, *Proofs that really count. The art of combinatorial proof*, 1st ed., vol. **27**, Mathematical Association of America (2003).

[4] K. N. Boyadzhiev, Close encounters with the Stirling numbers of the second kind, *Math. Magazine*, **85 (4)**, 252–266 (2012).

[5] W. Feller, The fundamental limit theorems in probability, *Bull. Amer. Math. Soc.*, **51 (11)**, 800–832 (1945).

[6] L. C. Hsu, P. J.-S. Shiue, A unified approach to generalized Stirling numbers, *Adv. Appl. Math.*, **20 (3)**, 366–384 (1998).

[7] J. Najnudel, J. Pitman, Feller coupling of cycles of permutations and Poisson spacings in inhomogeneous Bernoulli trials, *Electron. Commun. Probab.*, **25**, 1–11 (2020).

[8] V. Reiner, Non-crossing partitions for classical reflection groups, *Discrete Math.*, **177 (1–3)**, 195–222 (1997).

[9] J. B. Remmel, M. L. Wachs, Rook theory, generalized Stirling numbers and (p, q)-analogues, *Electron. J. Combin.*, **11**, #R84 (2004).

[10] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org.

[11] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Second edition, Cambridge University Press (2012).

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