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On the Diophantine equation $(5pn^2-1)^x + (p(p-5)n^2+1)^y = (pn)^z$

by

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Abstract

We find all solutions of the Diophantine equation in the title in positive integers (x, y, z, p, n) where $p > 3$ is a prime.

Key Words: Diophantine equations, applications of Baker's method, primitive divisors of Lucas sequences.

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1 Introduction

Here we find all positive integer solutions (x, y, z, p, n) to the title equation with $p > 3$ prime.

Theorem 1. *All positive integer solutions (p, n, x, y, z) of the title equation with $p > 3$ prime have $(x, z) = (1, 2)$. Furthermore, $y = 1$ unless $p = 5$ case in which any value of y is possible.*

The above result was obtained in [4] under some restrictions on p and n (for example, that $p \equiv 3 \pmod 4$ and $pn \equiv \pm 1 \pmod 5$). Before indicating our method of attack, let us mention a few other relevant papers on related Diophantine equations. There are many papers dealing with the Diophantine equation

$$A^x + B^y = C^z, \tag{1.1}$$

where (A, B, C) are integers belonging to some infinite families. Two relevant survey papers are [6] and [7]. The study of the case of the triples $(A, B, C) = (3pm^2-1, p(p-3)m^2+1, pm)$ was initiated by Terai and Hibino in [8]. The more general family of triples (A, B, C) given by

$$A = r\ell m^2 - 1, \quad B = (\ell - r)\ell m^2 + 1, \quad C = \ell m \tag{1.2}$$

was considered in [5]. Here is their main theorem.

Theorem 2. *Assume that $\ell > r$ is odd, that ℓm is not a multiple of 3, $3 \mid r$ and that $\min\{r\ell m^2 - 1, (\ell - r)\ell m^2 + 1\} > 30$. Then the equation (1.1) with A, B, C given at (1.2) implies that $(x, y, z) = (1, 1, 2)$.*

We note that all of the above results impose some sort of congruence restrictions on the involved variables. Our argument bypasses such restrictions so our result is not covered by the above papers. We do still however keep the condition that $p > 3$. Our method uses elementary arguments to show that if $(x, y, z) \neq (1, 1, 2)$ and $p > 5$, then $z \geq 8$. Reducing both sides of the equation modulo $(pn^2)^k$ for $k = 1, 2, 3, 4$ one gets various congruences among x, y and p, n. The upshot of these congruences is that if $X := \max\{x, y\}$, then $X > n^2p$. Then a linear form in p-adic logarithms gives a small bound on X (logarithmic in the maximum of p and n), therefore on all the variables and the problem is reduced to a finite albeit non-trivial computation. A similar analysis as ours can be done for the case when $p \in \{2, 3\}$ yielding $n < 10^4$ and $X < 10^8$ and one would need to reduce such bounds in order to be able to finish the computations. We leave this analysis of the cases $p \in \{2, 3\}$ for a future project.

2 Bounds on p, n in terms of X

When $p = 5$ the equation becomes

$$(25n^2 - 1)^x + 1 = (5n)^z.$$

If $x > 1$, then since $5n < 25n^2 - 1$, it follows that $z > 1$ so the above equation is an instance of Catalan's equation and its only solution is $2^3 + 1 = 3^2$. Hence, we get that $(25n^2 - 1, 5n, x, z) = (2, 3, 3, 2)$, which is impossible since it gives $n = 3/5$. So, $x = 1$, which implies that $z = 2$. From now on we assume that $p \neq 5$. We also assume that $(x, y, z) \neq (1, 1, 2)$.

Recall that $X := \max\{x, y\}$. The case $X = 1$ leads to $z = 2$, which we are assuming not to hold. In particular, $X \geq 2$. Observe that

$$5pn^2 - 1 < (pn)^2 \quad \text{and} \quad p(p-5)n^2 + 1 < (pn)^2.$$

Thus,

$$2(pn)^{2X} \geq (5pn^2 - 1)^x + (p(p-5)n^2 + 1)^y = (pn)^z,$$

so $z \leq 2X$. Next, $5pn^2 - 1 > pn$ and $p(p-5)n^2 + 1 > pn$, so

$$(pn)^z = (5pn^2 - 1)^x + (p(p-5)n^2 + 1)^y > (pn)^X,$$

showing that $z > X$. We record these observations as follows.

Lemma 1. *We have $X < z \leq 2X$.*

Since $X \geq 2$, we have $z \geq 3$. Reducing the equation modulo pn^2, we get

$$(-1)^x + 1 \equiv 0 \pmod{pn^2},$$

showing that x is odd. Let us show that $5pn^2 - 1$ and $p(p-5)n^2 + 1$ are coprime. If they are not, then let q be a common prime factor of them. Then q divides their sum which is $(pn)^2$, so $q = p$ or $q \mid n$ and both instances are false since the given numbers are congruent to $\pm 1 \pmod{pn}$. We put $d := \gcd(x, y)$. We show that $d = 1$. Assume for a contradiction that $d > 1$. Write $x =: dx_1,\ y =: dy_1$. Since x is odd, it follows that d, x_1 are both odd. So,

$$\begin{aligned} (pn)^z &= (5pn^2-1)^x + (p(p-5)n^2+1)^y \\ &= ((5pn^2-1)^{x_1} + (p(p-5)n^2+1)^{y_1}) \\ &\times \frac{(5pn^2-1)^{dx_1} + (p(p-5)n^2+1)^{dy_1}}{(5pn^2-1)^{x_1} + ((p(p-5)n^2-1)^{y_1}} \end{aligned} \tag{2.1}$$

Take $(\alpha, \beta) := ((5pn^2-1)^{x_1}, -(p(p-5)n^2+1)^{y_1})$. The integers α, β are coprime by a previous argument. Thus, $\alpha+\beta$ and $\alpha\beta$ are coprime. The right–hand side in (2.1) above is the dth term of the Lucas sequence $(u_m)_{m\ge 0}$ of general term

$$u_m := \frac{\alpha^m - \beta^m}{\alpha - \beta} \qquad \text{for all} \qquad m \ge 0.$$

Its discriminant is

$$(\alpha-\beta)^2 = ((5pn^2-1)^{x_1} + (p(p-5)n^2+1)^{y_1})^2.$$

Since x_1 is odd, the expression on the right is divisible by pn. Equation (2.1) shows that u_d is divisible only by primes which are either p or divide n, and all such primes divide the discriminant of the Lucas sequence $(u_m)_{m\ge 0}$. In particular, u_d does not have *primitive divisors*. It follows, from the Primitive Divisor Theorem (see [1] and [3]), that the only possibility is $d = 3$ which gives

$$\begin{aligned} u_d &= (5pn^2-1)^{2x_1} - (5pn^2-1)^{x_1}(p(p-5)n^2+1)^{y_1} + (p(p-5)n^2+1)^{2y_1} \\ &\equiv 3 \pmod{pn}. \end{aligned}$$

Thus, $pn \mid 3$, which is false.

In particular, $x \ne y$. Assume $z \le 3$. Then $\{x, y\} = \{1, 2\}$ and x is odd, so $(x, y) = (1, 2)$. Thus, the equation is

$$-5n^2p + 2n^2p^2 + 25n^4p^2 - 10n^4p^3 + n^4p^4 = p^3n^3,$$

so $p^2n^2 \mid 5pn^2$, showing that $p = 5$, a contradiction. Thus, $z \ge 4$. We now reduce our equation modulo p^2n^4 and get

$$x(5pn^2) - 1 + yp(p-5)n^2 + 1 \equiv 0 \pmod{p^2n^4},$$

which can be rewritten as

$$5pn^2(x-y) + yp^2n^2 \equiv 0 \pmod{p^2n^4}.$$

This gives $p \mid 5(x-y)$. Since $p \ne 5$, we get $p \mid x - y$. Further, we also have that $5(x-y)/p + y \equiv 0 \pmod{n^2}$. The left–hand side is $5x/p + y(1 - 5/p)$ which is a convex combination of x and y, in particular positive. Thus, we get

$$X \ge \frac{5x}{p} + y\left(1 - \frac{5}{p}\right) \ge n^2.$$

But we need to do better. We put

$$x - y =: Ap, \quad y + 5(x-y)/p = y + 5A =: Bn^2. \tag{2.2}$$

As noted above, B is positive. If $z \le 5$, then $X < 5$. Thus, $|x - y| \in \{1, 2, 3\}$, so we cannot have that $|x - y|$ is a multiple of p. Hence, $z \ge 6$. We reduce our equation modulo p^3n^6 to get

$$-\binom{x}{2}(5pn^2)^2 + \binom{y}{2}(p(p-5)n^2)^2 + x(5pn^2) + yp(p-5)n^2 \equiv 0 \pmod{p^3n^6}.$$

The above congruence is

$$p^2n^4\left(-\binom{x}{2}5^2 + \binom{y}{2}(p-5)^2 + B\right) \equiv 0 \pmod{p^3n^6}.$$

Simplifying p^2n^4, we get

$$-\binom{x}{2}5^2 + \binom{y}{2}(p-5)^2 + B \equiv 0 \pmod{pn^2}.$$

The left–hand side above is therefore of the form

$$Cpn^2 \quad \text{where} \quad C := \frac{1}{pn^2}\left(\frac{-25Ap(x+y-1)}{2} + p(p-10)\binom{y}{2} + B\right)$$

is an integer. We thus get that $p \mid B$ since $p > 2$. Hence, $B =: pB_0$ and so

$$X > y + 5(x-y)/p = Bn^2 = B_0pn^2.$$

But we need to do better. If $z \le 7$, then $X \le 6$. Thus, $|x - y| \in \{1, 2, 3, 4, 5\}$ and it is not possible for $|x - y|$ to be multiple of p. Thus, $z \ge 8$. We reduce our equation modulo p^4n^8 getting

$$\binom{x}{3}(5pn^2)^3 + \binom{y}{3}(p(p-5)n^2)^3 + Cp^3n^6 \equiv 0 \pmod{p^4n^8}.$$

The left–hand side can be rearranged as in

$$p^3n^6\left(C + 5^3\left(\binom{x}{3} - \binom{y}{3}\right) + \binom{y}{3}\left((p-5)^3 + 5^3\right)\right) \equiv 0 \pmod{p^4n^8}.$$

Since

$$p \mid x - y, \quad \text{so} \quad p \mid \binom{x}{3} - \binom{y}{3}, \quad \text{and} \quad p \mid (p-5)^3 + 5^3,$$

we get that $p \mid C$. Using $y \equiv -5A \pmod{p}$ and $x \equiv y \pmod{p}$ (both from (2.2) together with the fact that $p \mid B$), we get that

$$\begin{aligned}
Cn^2 &\equiv -\frac{25A(2y-1)}{2} - 10\binom{-5A}{2} + B_0 \pmod{p} \\
&\equiv -\frac{25A(-10A-1)}{2} - 10\left(\frac{-5A(-5A-1)}{2}\right) + B_0 \pmod{p} \\
&\equiv B_0 - \frac{25A}{2} \pmod{p}.
\end{aligned}$$

Thus, we get that $p \mid B_0 - 25A/2$. It might be that this expression is 0. In this case $B_0 = 25A/2$ and so $A/2$ is a positive integer. Since $A/2$ divides B, it divides y and x by (2.2). Since x and y are coprime, we get $(A, B_0) = (2, 25)$, so

$$(x, y) = (25pn^2 + 2p - 10, 25pn^2 - 10).$$

The above pair is obtained by solving system (2.2) for x, y when $A = 2$, $B = B_0p = 25p$. Otherwise, that is if $B_0 - 25A/2 \neq 0$, we then get that $2B_0 - 25A$ is a nonzero integer which is a multiple of p, so

$$p \leq |2B_0 - 25A|.$$

Working out $2B_0 - 25A$, we get

$$\begin{aligned} 2B_0 - 25A &= \frac{2B}{p} - 25A \\ &= \frac{1}{p}\left(\frac{10x}{pn^2} + 2\left(1 - \frac{5}{p}\right)\frac{y}{n^2}\right) - \frac{25}{p}(x - y) \\ &= \left(\frac{2}{pn^2}\left(1 - \frac{5}{p}\right) + \frac{25}{p}\right)y - \left(\frac{25}{p} - \frac{10}{p^2n^2}\right)x. \end{aligned}$$

The above calculation shows that $2B_0 - 25A$ is a linear combination of x and y with coefficients of opposite signs and absolute values at most $27/p$. Thus,

$$p \leq |2B_0 - 25A| < 27X/p, \quad \text{therefore} \quad p < \sqrt{27X}.$$

We record this as a lemma.

Lemma 2. *The following hold:*

(i) $X > pn^2$.

(ii) Either (x, y) *is*

$$(25pn^2 + 2p - 10, 25pn^2 - 10),$$

or $p < \sqrt{27X}$.

3 Bounding all the variables

Let us put some bound on X. In our equation, we look at the exponent of p in both sides. In the right–hand side it is at least $z \geq X$. On the left–hand side, putting

$$a := 5pn^2 - 1, \quad b := p(p - 5)n^2 + 1,$$

we have that a and b are multiplicatively independent (otherwise, they will be powers of the same integer in particular not coprime, which is false by a previous argument). Further,

$$\min\{\log a, \log b\} \geq \log(pn) \quad \text{and} \quad \max\{\log a, \log b\} < 2\log(pn).$$

In addition, letting g be the minimal positive integer such that $a^g \equiv b^g \equiv 1 \pmod{p}$, we have $g = 2$. We put

$$b' := \frac{x}{\log b} + \frac{y}{\log a} \leq \frac{2X}{\log(pn)}. \tag{3.1}$$

With Theorem 3 in [2], we get

$$\begin{aligned} z &\le \operatorname{ord}_p(a^x+b^y)=\operatorname{ord}_p(b^y-(-a)^x) \\ &< \frac{48p}{(p-1)(\log p)^4}\max\{10\log p, \log b'+\log\log p+0.4\}^2(2\log(pn))^2. \end{aligned} \tag{3.2}$$

Assume first that the maximum in the right–hand side of (3.2) is in the term involving $10\log p$. We then get

$$X<z<\frac{19200p(\log pn)^2}{(p-1)(\log p)^2}. \tag{3.3}$$

If $n^2<p$, then Lemma 2 (i) and inequality (3.3) imply

$$pn^2<X<z<\frac{19200(3/2)^2p}{p-1},$$

so $(p-1)n^2<43200$. Since $n^2\le p-1$, we get $n<43200^{1/4}$. Thus,

$$n\le 14 \qquad \text{and} \qquad n^2\le p-1<\frac{43200}{n^2}. \tag{3.4}$$

If $n^2>p$, then Lemma 2 (i) and inequality (3.3) again imply

$$n^2p<X<z<19200(3\log n)^2\left(\frac{p}{(p-1)(\log p)^2}\right)=\frac{172800p(\log n)^2}{(p-1)(\log p)^2}.$$

This gives

$$\left(\frac{n}{\log n}\right)^2(p-1)(\log p)^2<172800.$$

Since $n^2>p$ and the function $t\mapsto t/\log t$ is increasing for $t>e$, it follows that for $p\ge 11$ we have $n>\sqrt{p}>e$ so

$$\left(\frac{\sqrt{p}}{\log\sqrt{p}}\right)^2(p-1)(\log p)^2<172800.$$

This gives

$$p(p-1)<\frac{172800}{4}=43200,$$

so $p\le 203$. Thus,

$$p\le 203 \qquad \text{and} \qquad \left(\frac{n}{\log n}\right)^2<\frac{172800}{(p-1)(\log p)^2}. \tag{3.5}$$

Assume next that the maximum in the right–hand side of (3.2) is in the term involving b'. We then get using also (3.1) that

$$z<\frac{192p(\log pn)^2}{(p-1)(\log p)^4}\left(\log\left(\frac{2X}{\log(pn)}\right)+\log\log p+0.4\right)^2.$$

We have

$$\log\left(\frac{2X}{\log(pn)}\right)+\log\log p+0.4<\log\left(\frac{3X\log p}{\log(pn)}\right).$$

Thus,

$$X<\frac{192p(\log pn)^2}{(p-1)(\log p)^4}\left(\log\left(\frac{3X\log p}{\log(pn)}\right)\right)^2. \tag{3.6}$$

If $n<p$, then $1\le(\log pn)/(\log p)<2$. Hence, (3.6) implies

$$X<\frac{768p(\log 3X)^2}{(p-1)(\log p)^2}.$$

The above gives

$$\frac{3X}{(\log 3X)^2}<\frac{2304p}{(p-1)(\log p)^2}. \tag{3.7}$$

The function $t\mapsto t/(\log t)^2$ is increasing for $t>e^2$. Since we have $3X>3p\ge 21>e^2$, it follows that the above inequality implies

$$\frac{3p}{(\log 3p)^2}<\frac{2304p}{(p-1)(\log p)^2},$$

so

$$p-1<768\left(\frac{\log 3p}{\log p}\right)^2,$$

so $p<1040$. In addition, the right–hand side of (3.7) is smaller than 710, so (3.7) implies

$$\frac{3X}{(\log 3X)^2}<710,$$

which gives $X<40000$. Thus,

$$\log b'+\log\log p+0.4<\log\left(\frac{2e^{0.4}X\log p}{\log pn}\right)<\log(3X)<10\log 7\le 10\log p,$$

so in fact the maximum did not occur in the term involving X in the right–hand side of (3.2).

Assuming now that $n\ge p$, we have $\log p/\log(pn)\le 1/2$. Now (3.6) implies

$$\begin{aligned}\frac{1.5X}{(\log 1.5X)^2} &< \frac{288p}{(p-1)(\log p)^2}\left(\frac{\log(pn)}{\log p}\right)^2\\ &\le 288\times 4\left(\frac{p}{(p-1)(\log p)^4}\right)(\log n)^2\\ &< 94(\log n)^2. \end{aligned} \tag{3.8}$$

Since $1.5X\ge 1.5pn^2\ge 10.5n^2>e^2$, we get

$$\frac{10.5n^2}{(\log 10.5n^2)^2}<94(\log n)^2,$$

giving $n \leq 208$. Further, $p < n$. So,

$$n \leq 208 \quad \text{and} \quad p < n. \tag{3.9}$$

Going back to (3.6), we get

$$X < 192(\log(208^2))^2 \left(\frac{p}{(p-1)(\log p)^4}\right)(\log 3X)^2 < 1781\log(3X)^2,$$

which gives $X < 342000$. Again

$$\log b' + \log\log p + 0.4 < \log(3X) < 10\log p,$$

so in fact the maximum did not occur in the term involving b' in the right–hand side of (3.2). Thus, only the situations when the maximum in the right–hand side of (3.2) appeared in the term not involving b' are possible. To summarise, we have proved the following result.

Lemma 3. *If $(x, y, z) \neq (1, 1, 2)$ and $p \geq 7$, then one of the following holds:*

(i)

$$n \leq 14 \qquad \text{and} \qquad n^2 \leq p - 1 < \frac{43200}{n^2}.$$

(ii)

$$p \leq 203 \qquad \text{and} \qquad n^2 < \frac{172800p(\log n)^2}{(p-1)(\log p)^2}.$$

In both cases,

$$pn^2 < X < \frac{19200p(\log pn)^2}{(p-1)(\log p)^2}. \tag{3.10}$$

4 The computational part

We wrote a Mathematica code which went though all possibilities of (i) and (ii) of Lemma 3 and generated quadruples $(p, n, X_{\text{low}}, X_{\text{up}})$, where (p, n) are like in (i) or (ii) and X_{low}, X_{up} are the left and the integer part of the right–hand sides of (3.10) and kept those quadruples for which $X_{\text{low}} < X_{\text{up}}$. There are 236 such pairs (p, n). By Lemma 2, either

$$(x, y) = (25pn^2 + 2p - 10, 25pn^2 - 10), \tag{4.1}$$

or

$$p < \sqrt{27X_{\text{up}}}. \tag{4.2}$$

Let again

$$(a, b, c) := (5pn^2 - 1, p(p-5)n^2 + 1, pn).$$

Assume first that we are in instance (4.1). Since x is odd, it follows that n is odd. There are 127 pairs (p, n) with n odd. A computer program went in a few minutes through the 127 pairs (p, n) with n odd and checked whether the pair (x, y) indicated in (4.1) together with $z := \lfloor \log(a^x + b^y)/\log c \rfloor$ is a solution to $a^x + b^y = c^z$ modulo T, where T is the product of the first 20 primes. The PowerMod feature of Mathematica to compute $a^x,\ b^y,\ c^z$ modulo T was used. No solution was found. Thus, inequality (4.2) must hold. There are only 224 pairs (p, n) which satisfy the additional inequality (4.2). For these ones, we note that since $a + b = (pn)^2$, we have that

$$a^x + ((pn)^2 - a)^y = (pn)^z,$$

which reduced modulo a gives $(pn)^{2y-z} \equiv 1 \pmod{a}$. The case $2y = z$ leads to $z \le 12$. Indeed, in this case

$$a^x = (pn)^{2y} - b^y = ((pn)^2 - b)\left(\frac{(pn)^{2y} - b^y}{(pn)^2 - b}\right).$$

The factor in parenthesis on the right is the yth term the Lucas sequences with roots $(\alpha, \beta) = ((pn)^2, b)$, which does not have primitive divisors since its discriminant is given by $(\alpha - \beta)^2 = ((pn)^2 - b)^2 = a^2$. This shows that $y \le 6$, so $z = 2y \le 12$, therefore $X \le 11$. Thus, $pn^2 < X \le 11$, giving $p = 7,\ n = 1,\ (a, b, c) = (34, 15, 7)$, $(\alpha, \beta) = (49, 15)$ and one checks that the Lucas sequence of general term $u_k = (49^k - 15^k)/(49 - 15)$ has primitive divisors for all $k \in \{3, 4, 5, 6\}$ but not for $k = 2$ for which $u_2 = \alpha+\beta = 64 = 2^6$ and 2 divides $(\alpha - \beta)^2$. This leaves the possibility $y = 2$, so $z = 2y = 4 > X$, a contradiction. A similar argument shows that $(pn)^{2x-z} \equiv 1 \pmod{b}$ and $2x - z$ is not 0. Hence, $\mathrm{ord}_a(pn) \mid 2y - z$ and $\mathrm{ord}_b(pn) \mid 2x - z$, where $\mathrm{ord}_N k$ is the multiplicative order of k modulo N for coprime integers $k,\ N$. Since

$$a^x + b^y = c^z,$$

it follows that either $a^x \in (c^z/2, c^z)$ or $b^y \in (c^z/2, c^z)$. Thus,

$$c^z/a^x \in (1, 2) \quad \text{or} \quad c^z/b^y \in (1, 2).$$

Thus, one of

$$z \log c - x \log a \quad \text{or} \quad z \log c - y \log b \quad \text{is in} \quad (0, \log 2).$$

Hence,

$$z = u\alpha + \zeta, \quad (u, \alpha) \in \left\{\left(x, \frac{\log a}{\log c}\right), \left(y, \frac{\log b}{\log c}\right)\right\}, \quad \zeta \in \left(0, \frac{\log 2}{\log p}\right) \subset \left(0, \frac{1}{2}\right).$$

It thus follows that

$$z = \lfloor u\alpha \rfloor. \tag{4.3}$$

A Mathematica code went through the remaining 224 quadruples of the form $(p, n, X_{\rm low}, X_{\rm up})$ by fixing (p, n) and, assuming $\alpha := \log a/\log c$, it looped over $x \in [1, X_{\rm up}]$, computed z using the formula (4.3), then y using the formula $y := \lfloor \log(c^z - a^x)/\log b \rfloor$ (assuming $c^z > a^x$). With these numbers it tested whether x is odd, $\gcd(x, y) = 1$, $p \mid x - y$ and if putting $A := (x - y)/p$, then $y + 5A \equiv 0 \pmod{pn^2}$. It further checked if $\mathrm{ord}_a(pn) \mid 2y - z$ and $\mathrm{ord}_b(pn) \mid 2x - z$. If all these tests passed it checked the congruence $a^x + b^y \equiv c^z \pmod{T}$ using again the PowerMod feature. Assuming on the other hand that $\alpha := \log b/\log c$, the similar computation with the roles of (x, a) and (y, b) interchanged was carried on. This code ran for about 30 minutes on the second author's personal laptop until it finished without finding any instance (p, n, x, y, z) passing all the above tests. This finishes the proof.

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