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On a problem related to Egyptian fractions

by

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Abstract

Let S be a sum of irreducible fractions with distinct denominators. In this note we give necessary and sufficient conditions for the sum S to be an integer. The method can be a starting point for generalizations of some problems connected to Egyptian fractions. In this direction we propose a generalization of two classical results of Erdős and Oblath.

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1 Introduction

Egyptian fractions are finite sums of distinct fractions with the numerator 1. The first proof that every positive rational number (in particular, every positive integer) can be represented by Egyptian fractions was given by Fibonacci, in 1202. As a matter of fact, a positive rational number has infinitely many representations by Egyptian fractions. If

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{1}{x_i}, \tag{1.1}$$

with $1 < x_1 < x_2 < \cdots < x_n$ positive integers, is a representation of length n of the rational number $\frac{a}{b}$, then, since $\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$, we can conclude that $\frac{a}{b}$ has a representation by Egyptian fractions of any length $k \geq n$.

Several algorithms have been constructed to find Egyptian fraction representations for rational numbers (see [3]) and different problems related to Egyptian fractions continue to attract the interest of mathematicians, especially in the domain of number theory (see [5], [7], [9]). Among these, we mention the problem of bounding the length or the maximum denominator in Egyptian fraction representations, finding representations of certain special forms or in which all the denominators are of some special type (see Graham [9]). For example, in one of his earliest papers, Paul Erdős proved that it is not possible for a harmonic progression to form an Egyptian fraction representation of an integer (see [6]). In 2015, Butler, Erdős and Graham, the authors of the latest publication of Erdős [4], which was published almost twenty years after his death, show that any natural number can be represented as a sum of Egyptian fractions, where each denominator is the product of three distinct odd primes.

In the present paper we study the following problem related to Egyptian fractions, but in our case the numerators of the fractions are arbitrary integers.

Problem. Given a set of irreducible fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$, where a_j, b_j, $j = 1, 2, \ldots, n$ are nonzero integers and $b_1, b_2, \ldots, b_n$ are distinct, denote $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. In what conditions the sum

$$S(\mathbf{a}, \mathbf{b}) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \tag{1.2}$$

is an integer?

The outline of the paper is as follows. In Section 2 we introduce Theorem 1, the main result of the paper, and then discuss some examples of how it can be applied in order to decide if a sum of fractions is an integer or not. Next, we prove Theorem 1 in Section 3. In Section 4 we state and prove Theorem 2, our second result on the particular Egyptian fractions whose denominators are products of two distinct primes. Finally, in Section 5, we conclude by presenting a generalization of two classical results of Erdős [6] and Oblath [11].

2 The main result

We say that a set of irreducible fractions, $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$ is in *standard form* if a_j, b_j are nonzero integers and the denominators b_j are distinct for $j = 1, 2, \ldots, n$.

Denote $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$. Let $\mathcal{P}_{\mathbf{b}}$ be the finite set of the prime numbers which divide the product $b_1 b_2 \cdots b_n$ and $\operatorname{lcm}(\mathbf{b})$ be the least common multiple of $b_1, b_2, \ldots, b_n$. For each $p \in \mathcal{P}_{\mathbf{b}}$, we denote by $k(p)$ the p-adic order of $\operatorname{lcm}(\mathbf{b})$, that is, the highest exponent such that $p^{k(p)} \mid \operatorname{lcm}(\mathbf{b})$. We say that $p^{k(p)}$ *exactly divides* $\operatorname{lcm}(\mathbf{b})$ and denote $p^{k(p)} \| \operatorname{lcm}(\mathbf{b})$:

$$p^{k(p)} \mid \operatorname{lcm}(\mathbf{b}) \quad \text{and} \quad p^{k(p)+1} \nmid \operatorname{lcm}(\mathbf{b}).$$

Note that $k(p) \geq 1$. We also introduce the following notation for each $p \in \mathcal{P}_{\mathbf{b}}$:

$$\check{p}_{\mathbf{b}} = \prod_{q \in \mathcal{P}_{\mathbf{b}} \setminus \{p\}} q^{k(q)} = \frac{\operatorname{lcm}(\mathbf{b})}{p^{k(p)}}. \tag{2.1}$$

For every $p \in \mathcal{P}_{\mathbf{b}}$ and each denominator b_j, $j = 1, 2, \ldots, n$, we denote by $k_j(p)$ the p-adic order of b_j (that is, $p^{k_j(p)} \| b_j$) and by $b_j(p)$ the quotient $b_j(p) = b_j / p^{k_j(p)}$. Thus,

$$b_j = p^{k_j(p)} b_j(p) \quad \text{and} \quad p \nmid b_j(p). \tag{2.2}$$

Note that $k_j(p) \in \{0, 1, \ldots, k(p)\}$. For any $p \in \mathcal{P}_{\mathbf{b}}$ and $s = 0, 1, \ldots, k(p)$, we denote by $\mathcal{I}_s(p)$ the subset of indices

$$\mathcal{I}_s(p) = \{j \in \{1, 2, \ldots, n\} : p^s \| b_j\} \tag{2.3}$$

and by $N_s(p)$ the number

$$N_s(p) = \check{p}_{\mathbf{b}} \sum_{j \in \mathcal{I}_s(p)} \frac{a_j}{b_j(p)}. \tag{2.4}$$

If $\mathcal{I}_s(p) = \emptyset$, we set $N_s(p) = 0$.

Assuming the above definitions, notations and hypotheses, we now state our main result.

Theorem 1. *Let* $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}$ *be a set of fractions in standard form. Then, the sum* $S(\mathbf{a}, \mathbf{b}) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$ *is an integer if and only if*

$$\sum_{j=0}^{s} \frac{N_{k(p)-j}(p)}{p^{s-j}} \equiv 0 \pmod{p}, \tag{2.5}$$

for any $p \in \mathcal{P}_{\mathbf{b}}$ *and* $s = 0, 1, \dots, k(p) - 1$.

If we take $s = 0$ in (2.5), we obtain that $N_{k(p)}(p) \equiv 0 \pmod{p}$ is a necessary condition for the sum $S(\mathbf{a}, \mathbf{b})$ to be an integer. If the set $\mathcal{I}_{k(p)}$ has only one element, say $\mathcal{I}_{k(p)} = \{j'\}$, then $N_{k(p)}(p) = \check{p}_{\mathbf{b}} \frac{a_{j'}}{b_{j'}(p)}$ cannot be divisible by p (because according to the assumption made at the beginning the fraction is irreducible) and the next consequence follows.

Corollary 1. *If there exists a prime number* $p \in \mathcal{P}_{\mathbf{b}}$ *such that* $\left|\mathcal{I}_{k(p)}\right| = 1$, *then the sum* $S(\mathbf{a}, \mathbf{b})$ *is not an integer.*

The next examples show how Theorem 1 and Corollary 1 can be applied in order to decide whether a sum of irreducible fractions in standard form is an integer.

Example 1. The sums of the type $S(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (1, \pm 1, \dots, \pm 1)$ and $\mathbf{b} = (2, 3, \dots, n)$,

$$S_{\pm} = \frac{1}{2} \pm \frac{1}{3} \pm \cdots \pm \frac{1}{n}, \tag{2.6}$$

are not integers for any natural number $n \geq 2$. In particular, the sum

$$S_{+} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not an integer for any integer $n \geq 2$ (see also [13]). Indeed, if 2^{h_0} is the greatest power of 2 in the denominators of $S_{\pm}$, that is,

$$h_0 = \max\left\{h \in \mathbb{N} \mid 2^h \leq n\right\},$$

then $1/2^{h_0}$ is the unique fraction in the sum $S_{\pm}$ whose denominator is a multiple of 2^{h_0}. By applying Corollary 1 for $p = 2$, we find that the sum in (2.6) is not an integer.

The same reasoning can be applied to prove that

$$S_{odd} = \frac{1}{3} \pm \frac{1}{5} \pm \cdots \pm \frac{1}{2n+1}, \tag{2.7}$$

is not an integer for any integer $n > 0$. In that case we consider the greatest power of 3 in the denominators of the sum and, because all of them are odd numbers, we find that $1/3^{h_0}$ is the unique fraction in the sum (2.7) whose denominator is a multiple of 3^{h_0}.

In connection with the results obtained in [4] we give the following example.

Example 2. We consider the sums of the form

$$S(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{n} \frac{1}{p_{1,k} p_{2,k} p_{3,k}},$$

where $\mathbf{a} = (1, 1, \ldots, 1)$ and $\mathbf{b} = (p_{1,1}p_{2,1}p_{3,1}, \ldots, p_{1,n}p_{2,n}p_{3,n})$ and $p_{i,k}$ are prime numbers for each $k = 1, 2, \ldots, n$ and $i = 1, 2, 3$. If there exists a prime number q which appears in the denominator of only one fraction, then, by Corollary 1, the sum $S(\mathbf{a}, \mathbf{b})$ cannot be an integer.

Example 3. Let $\{1/3, 1/4, 1/8, 1/10, 1/15, 1/24, 1/30, 1/40, 1/60, 1/120\}$ be a set of ten Egyptian fractions in standard form. Let

$$S(\mathbf{a}, \mathbf{b}) = \frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{10} + \frac{1}{15} + \frac{1}{24} + \frac{1}{30} + \frac{1}{40} + \frac{1}{60} + \frac{1}{120}, \tag{2.8}$$

where $\mathbf{a} = (1, \ldots, 1)$ and $\mathbf{b} = (3, 4, 8, 10, 15, 24, 30, 40, 60, 120)$ A simple calculation shows that $S(\mathbf{a}, \mathbf{b}) = 1 \in \mathbb{Z}$, but we want to show, in this particular instructive case, how the mechanism of Theorem 1 can be applied to find out whether or not $S(\mathbf{a}, \mathbf{b})$ is an integer.

We have: $\operatorname{lcm}(\mathbf{b}) = 120 = 2^3 \cdot 3 \cdot 5$, so that $\mathcal{P}_{\mathbf{b}} = \{2, 3, 5\}$.

Next, leaving out for simplicity the subscript $\mathbf{b}$ in the notation of the complementary parts of the primes in the prime factorization of $\operatorname{lcm}(\mathbf{b})$, we have $\check{2} = 15$, $\check{3} = 40$, $\check{5} = 24$, and $k(2) = 3$, $k(3) = k(5) = 1$.

Further, we gather in the Table 1 the quotients $b_j(p)$ introduced by relation (2.2). Afterwards, we use these numbers to check out the congruences (2.5) to see whether $S(\mathbf{a}, \mathbf{b})$ is an integer.

Table 1: The values of b_j and the quotients $b_j(p)$ for $p = 2, 3, 5$ and $j = 1, \ldots, 10$ in Example 3.

j	1	2	3	4	5	6	7	8	9	10
b_j	3	4	8	10	15	24	30	40	60	120
$b_j(2)$	3	1	1	5	15	3	15	5	15	15
$b_j(3)$	1	4	8	10	5	8	10	40	20	40
$b_j(5)$	3	4	8	2	3	24	6	8	12	24

If $p = 2$, the partition of the set of indices is

$$\begin{aligned}
\mathcal{I}_0(2) &= \{j \in \{1, 2, \ldots, 10\} : 2 \nmid b_j\} = \{1, 5\} \\
\mathcal{I}_1(2) &= \{j \in \{1, 2, \ldots, 10\} : 2 \| b_j\} = \{4, 7\} \\
\mathcal{I}_2(2) &= \left\{j \in \{1, 2, \ldots, 10\} : 2^2 \| b_j\right\} = \{2, 9\} \\
\mathcal{I}_3(2) &= \left\{j \in \{1, 2, \ldots, 10\} : 2^3 \| b_j\right\} = \{3, 6, 8, 10\}
\end{aligned}$$

Now we calculate the numerators of the fractions on the left-hand side of (2.5). We have:

$$\begin{aligned}
N_3(2) &= \check{2}\left(\frac{1}{b_3(2)} + \frac{1}{b_6(2)} + \frac{1}{b_8(2)} + \frac{1}{b_{10}(2)}\right) = 15\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right) = 24; \\
N_2(2) &= \check{2}\left(\frac{1}{b_2(2)} + \frac{1}{b_9(2)}\right) = 15\left(\frac{1}{1} + \frac{1}{15}\right) = 16; \\
N_1(2) &= \check{2}\left(\frac{1}{b_4(2)} + \frac{1}{b_7(2)}\right) = 15\left(\frac{1}{5} + \frac{1}{15}\right) = 4.
\end{aligned}$$

These numbers verify relations (2.5) for $s = 0, 1, 2$, respectively. Indeed, $k(2) = 3$ and we have:

$$\begin{aligned} N_3(2) &= 24 \equiv 0 \pmod 2; \\ \frac{N_3(2)}{2} + N_2(2) &= 12 + 16 \equiv 0 \pmod 2; \\ \frac{N_3(2)}{2^2} + \frac{N_2(2)}{2} + N_1(2) &= 6 + 8 + 4 \equiv 0 \pmod 2. \end{aligned} \tag{2.9}$$

If $p = 3$, then

$$\mathcal{I}_1(3) = \{j \in \{1, 2, \dots, 10\} : 3 \| b_j\} = \{1, 5, 6, 7, 9, 10\},$$

and since $k(3) = 1$, we have to check the validity of (2.5) only for $s = 0$, which holds true, since

$$\begin{aligned} N_1(3) &= \check{3}\left(\frac{1}{b_1(3)} + \frac{1}{b_5(3)} + \frac{1}{b_6(3)} + \frac{1}{b_7(3)} + \frac{1}{b_9(3)} + \frac{1}{b_{10}(3)}\right) \\ &= 40\left(1 + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40}\right) = 60 \equiv 0 \pmod 3. \end{aligned} \tag{2.10}$$

Finally, similarly, if $p = 5$, then $k(5) = 1$, and relation (2.5) is also satisfied:

$$\mathcal{I}_1(5) = \{j \in \{1, 2, \dots, 10\} : 5 \| b_j\} = \{4, 5, 7, 8, 9, 10\},$$

and

$$\begin{aligned} N_1(5) &= \check{5}\left(\frac{1}{b_4(5)} + \frac{1}{b_5(5)} + \frac{1}{b_7(5)} + \frac{1}{b_8(5)} + \frac{1}{b_9(5)} + \frac{1}{b_{10}(5)}\right) \\ &= 24\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \frac{1}{24}\right) = 30 \equiv 0 \pmod 5. \end{aligned} \tag{2.11}$$

Together, relations (2.9), (2.10) and (2.11) complete the verification of the conditions required for integrity by Theorem 1, and consequently, the sum $S(\mathbf{a}, \mathbf{b})$ in (2.8) must be an integer.

To conclude, bounding from above the terms of the sum by dyadic fractions, we find that

$$0 < S(\mathbf{a}, \mathbf{b}) < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{64} = \frac{85}{64} < 2,$$

therefore, the sum in (2.8) is the only possible integer, so that $S(\mathbf{a}, \mathbf{b}) = 1$.

3 Proof of Theorem 1

Proof. Let $p \in \mathcal{P}_{\mathbf{b}}$ be a prime factor of the least common denominator $\operatorname{lcm}(\mathbf{b})$. We rearrange the terms of the sum $S(\mathbf{a}, \mathbf{b})$ in the following form:

$$S(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{k(p)} \left(\frac{1}{p^{k(p)-i}} \sum_{j \in \mathcal{I}_{k(p)-i}} \frac{a_j}{b_j(p)}\right) \tag{3.1}$$

"$\Longrightarrow$" Let us assume that $S(\mathbf{a},\mathbf{b})$ is an integer. In order to prove (2.5) for $s = 0,1,\dots,k(p)-1$ we multiply both sides of the equality (3.1) by $\check{p}_{\mathbf{b}}p^{k(p)-s-1}$. Thus, we obtain:

$$\begin{aligned} S(\mathbf{a},\mathbf{b})\cdot\check{p}_{\mathbf{b}}p^{k(p)-s-1} &= \sum_{i=0}^{k(p)}\frac{1}{p^{s-i+1}}N_{k(p)-i}(p) \\ &= \frac{1}{p}\sum_{i=0}^{s}\frac{1}{p^{s-i}}N_{k(p)-i}(p)+\sum_{i=s+1}^{k(p)}p^{i-s-1}N_{k(p)-i}(p). \end{aligned} \tag{3.2}$$

Since the last sum in (3.2) is an integer, we obtain that $\frac{1}{p}\sum_{i=0}^{s}\frac{1}{p^{s-i}}N_{k(p)-i}(p)$ is also an integer and hence (2.5) follows.

"$\Longleftarrow$" Suppose that the condition (2.5) is verified for all $p\in\mathcal{P}_{\mathbf{b}}$ and the sum $S(\mathbf{a},\mathbf{b})$ from (1.2) equals the irreducible fraction $S(\mathbf{a},\mathbf{b})=\frac{a}{b}$. If $b>1$, then there exists p a prime divisor of b. Let k be the p-adic order of b: $b=p^k b(p)$ and $p\nmid b(p)$. Obviously, $p\in\mathcal{P}_{\mathbf{b}}$, $1\le k\le k(p)$ and $S(\mathbf{a},\mathbf{b})=\frac{a}{p^k b(p)}$. Then, we multiply both sides of (3.1) by $\check{p}_{\mathbf{b}}p^{k-1}$ and obtain:

$$\begin{aligned} \frac{a\check{p}_{\mathbf{b}}}{pb(p)} &= \sum_{i=0}^{k(p)}\frac{1}{p^{k(p)-k-i+1}}N_{k(p)-i}(p) \\ &= \frac{1}{p}\sum_{i=0}^{k(p)-k}\frac{1}{p^{k(p)-k-i}}N_{k(p)-i}(p)+\sum_{j=1}^{k}p^{j-1}N_{k-j}(p). \end{aligned} \tag{3.3}$$

Since (2.5) is verified for $s=k(p)-k$, it follows that $\frac{a\check{p}_{\mathbf{b}}}{pb(p)}pb(p)$ is an integer, so p should be a divisor of a, which contradicts the irreducibility of the fraction $\frac{a}{b}$. Hence $b=1$ and the proof is complete. ∎

4 Egyptian fractions with each denominator having two distinct prime divisors

Let $\mathcal{P}=\{2,3,5,\dots\}$ be the set of prime numbers.

Definition 1. Let A be a finite set of distinct integers containing at least 3 elements. We say that A is a circular set if its elements can be put on a circle such that each one is a divisor of the sum of its neighbors. This means that we can order the elements of A, $a_1,a_2,\dots,a_n$ such that

$$a_i \mid (a_{i-1}+a_{i+1}),\ \forall\, i=1,2,\dots,n,$$

where $a_0=a_n$ and $a_{n+1}=a_1$.

Lemma 1. *The set of prime numbers contains no circular subsets.*

Proof. Suppose that such a circular subset $\mathcal{Q} = \{q_1, q_2, \dots, q_n\} \subset \mathcal{P}$ does exist. We assume that the elements of $\mathcal{Q}$ are indexed such that

$$q_i \mid (q_{i-1} + q_{i+1}), \ \forall\, i = 1, 2, \dots, n,$$

(where $q_0 = q_n$ and $q_{n+1} = q_1$) and q_n is the largest number in $\mathcal{Q}$. Since $q_{n-1} + q_1 < 2q_n$ and $q_n \mid (q_{n-1} + q_1)$, we get that $q_{n-1} + q_1 = q_n$, so one of the prime numbers q_{n-1} or q_1 equals 2; suppose that $q_1 = 2$ (the case $q_{n-1} = 2$ can be discussed in a similar way). Thus, $q_{n-1} = q_n - 2$, and $(q_n - 2) \mid (q_{n-2} + q_n)$. We can see that $q_n \geq 7$ (if $q_n = 5$, then $\mathcal{Q} = \{2, 3, 5\}$, which is not a circular set), so we have:

$$q_n - 2 < q_{n-2} + q_n < 2q_n < 3(q_n - 2).$$

It follows that $q_{n-2} + q_n = 2(q_n - 2)$, hence $q_{n-2} = q_n - 4$. Since q_{n-2}, q_{n-1}, q_n form an arithmetic progression with common difference 2, one of them must be divisible by 3, so $q_{n-1} = 3$ or $q_{n-2} = 3$. But $q_n \geq 7$, hence $q_{n-2} = 3$ and $\mathcal{Q} = \{2, 3, 5, 7\}$, which is not a circular set and the proof is complete. □

Butler et al. [4] (Theorem 1) state that any positive integer can be written as a sum of unit fractions where each denominator is the product of three distinct primes. The authors note that the result can be extended by similar arguments to prove that any positive integer can be written as a sum of unit fractions where each denominator is the product of $\omega \geq 4$ distinct primes and conjecture that a similar result holds for $\omega = 2$.

The next theorem establishes that a sum of unit fractions where each denominator is the product of two distinct primes and each prime factor appears exactly in two fractions is not an integer.

Theorem 2. *Let $\mathcal{Q} = \{q_1, \dots, q_n\} \subset \mathcal{P}$ and let*

$$S_{\mathcal{Q}} := \frac{1}{q_1 q_2} + \frac{1}{q_2 q_3} + \cdots + \frac{1}{q_{n-1} q_n} + \frac{1}{q_n q_1}. \tag{4.1}$$

Then $S_{\mathcal{Q}}$ is not an integer.

Proof. The statement is obvious for $n = 2$. Let us assume that $S_{\mathcal{Q}}$ is an integer for some $n \geq 3$. We apply Theorem 1 for $S_{\mathcal{Q}} = S(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (1, 1, \dots, 1)$ and $\mathbf{b} = (q_1 q_2, \dots, q_n q_1)$. The set of primes is $\mathcal{P}_{\mathbf{b}} = \mathcal{Q}$ and $k(q_j) = 1$ for $j = 1, 2, \dots, n$. Then,

$$N_1(q_i) = \check{q}_i \left(\frac{1}{q_{i-1}} + \frac{1}{q_{i+1}} \right) = (q_{i-1} + q_{i+1}) \prod_{\substack{j=1 \\ j \neq i, i\pm 1}}^{n} q_j \equiv 0 \pmod{q_i}.$$

It follows that $q_i \mid (q_{i-1} + q_{i+1})$, for every $i = 1, 2, \dots, n$, which is not possible, by Lemma 1. Hence $S_{\mathcal{Q}}$ is not an integer. □

5 A generalization of two results of Oblath and Erdős

Egyptian fractions whose denominators form an arithmetic progression is a subject investigated by many mathematicians. The first result in this domain was obtained by Theisinger [13], who proved that the sum $\sum_{k=2}^{n} \frac{1}{k}$ is never an integer, for any natural number $n > 1$ (see Example 1). Kürschák [10] established that

$$\sum_{k=m}^{n} \frac{1}{k} \tag{5.1}$$

is not an integer, for any natural numbers $n > m > 1$. This result was generalized in 1932 by Erdős [6], who proved that the sum

$$\sum_{k=0}^{n-1} \frac{1}{a+kb} \tag{5.2}$$

is never an integer, for any choice of the natural numbers $a, b, n > 0$. Recently, Belbachir and Khelladi [2] have shown that any sum of the form

$$\sum_{k=0}^{n-1} \frac{1}{(a+kb)^{\alpha_k}}, \tag{5.3}$$

where $\alpha_k \geq 1$ are natural numbers, is not an integer. As can be noticed, all these results concern sums of unit fractions. Oblath [11] investigated sums with arbitrary numerators and proved that a sum of the form

$$\sum_{k=m}^{n} \frac{a_k}{k}, \tag{5.4}$$

where $n > m > 1$ and $(a_k, k) = 1$ for each $k = m, \ldots, n$, is never an integer.

In this section we generalize the results of Erdős and Oblath by applying Bertrand-Chebyshev theorem, a theorem of Shorey and Tijdeman [12], and Corollary 1.

Bertrand-Chebyshev theorem (also known as Bertrands postulate) states that, for every integer $n > 1$, there exists at least one prime number p such that $n < p < 2n$. It was conjectured and verified for $n < 3 \cdot 10^6$ by Joseph Bertrand in 1845. The first proof was given by Pafnuty Chebyshev in 1850. In 1932, in his first published paper, Paul Erdős presented a beautiful elementary proof of this theorem (see [1], Chapter 2).

Lemma 2. *Let $a \geq 2$ and $s_0, s_1, \ldots, s_{n-1}$, be positive integers, $n \geq 2$. Let $a_0, a_1, \ldots, a_{n-1}$ be nonzero integers such that $(a_i, a+i) = 1$ for $i = 0, 1, \ldots, n-1$. Denote $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\mathbf{b} = (a^{s_0}, (a+1)^{s_1}, \ldots, (a+n-1)^{s_{n-1}})$. Then, the sum:*

$$S(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{n-1} \frac{a_i}{(a+i)^{s_i}} \tag{5.5}$$

is not an integer.

Proof. The idea of the proof is to find a prime divisor of the number

$$P = a(a+1)\cdots(a+n-1)$$

such that it divides only one factor of this product. For $n = 2, 3$ it is easy to find such a prime number, namely, it is sufficient to take any prime divisor p of $a+1$. This one is not a divisor either of a or of $a+2$.

Assume that $n \geq 4$. By Bertrand-Chebyshev theorem, there exists a prime number p such that

$$\lfloor (a+n-1)/2 \rfloor < p < 2\lfloor (a+n-1)/2 \rfloor .$$

Thus, there is a $j \in \{0, 1, \ldots, n-1\}$ with $p = a+j$ and

$$\lfloor (a+n-1)/2 \rfloor + 1 \leq a+j \leq 2\lfloor (a+n-1)/2 \rfloor - 1.$$

If another denominator, say $a+l$, was divisible by p, then $a+l$ should be at least $2p$. But $2p \geq 2\lfloor (a+n-1)/2 \rfloor + 2 > a+n-1$, so it follows that $l > n-1$, a contradiction. Thus, this prime number p is a divisor of only one denominator and, by Corollary 1, we can conclude that the sum $S(\mathbf{a}, \mathbf{b})$ in (5.5) is not an integer. ∎

Note that for $s_0 = \cdots = s_{n-1} = 1$, we obtain the result of Oblath [11] regarding the sum (5.4). If, in addition, all the numerators are 1, $a_0 = \cdots = a_{n-1} = 1$, the sum obtained is (5.1) and we find the main proposition of Kürschák [10].

The following result is a direct consequence of a theorem of Shorey and Tijdeman [12].

Lemma 3. *Let a, b and n be positive integers such that $a, b \geq 2$, $(a, b) = 1$ and $n \geq 4$. Then, the product*

$$P = a(a+b)(a+2b)\cdots(a+(n-1)b)$$

has at least one prime divisor p which is greater than n.

Theorem 3. *Let $a, b, s_0, s_1, \ldots, s_{n-1}$, be positive integers such that $(a, b) = 1$, and $a, n \geq 2$. Let $a_0, a_1, \ldots, a_{n-1}$ be nonzero integers such that $(a_i, a+ib) = 1$ for $i = 0, 1, \ldots, n-1$. Then, the sum:*

$$S(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{n-1} \frac{a_i}{(a+ib)^{s_i}}, \tag{5.6}$$

where $\mathbf{a} = (a_0, a-1, \ldots, a_{n-1})$ and $\mathbf{b} = (a^{s_0}, (a+b)^{s_1}, \ldots, (a+(n-1)b)^{s_{n-1}})$, is not an integer.

Proof. For $b = 1$, we get the statement of Lemma 2. Let us assume that $b > 1$ and consider the product

$$P = a(a+b)(a+2b)\cdots(a+(n-1)b).$$

If $n = 2, 3$, we take a prime divisor p of $a+b$, as in the proof of Lemma 2. Obviously, p is not a divisor of a, nor of $a+2b$, hence, by Corollary 1, we obtain that the sum $S(\mathbf{a}, \mathbf{b})$ is not an integer.

If $n \geq 4$, by Lemma 3 we know that there exists p a prime divisor of P that is greater than n. We shall see that this prime number p cannot divide two distinct denominators in (5.6). Indeed, if $i < j$, $i, j \in \{0, 1, \ldots, n-1\}$ are such that $p \mid (a+ib)$ and $p \mid (a+jb)$, then $p \mid (j-i)b$. But p cannot divide b because $(a, b) = 1$, so $p \mid (j-i)$, a contradiction, because $p > n \geq j - i$. Now we can apply Corollary 1 and find that the sum $S(\mathbf{a}, \mathbf{b})$ is not an integer. □

For $a_0 = \cdots = a_{n-1} = 1$ we obtain the sum (5.3) from [2] which generalizes the result of Erdős [6] regarding the sum (5.2).

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