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Componentwise linearity and the gcd condition are preserved by the polarization

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Abstract

A graded ideal I of a polynomial ring over a field is componentwise linear if for every nonnegative integer j, the ideal generated by all homogeneous polynomials of degree j belonging to I admits a linear resolution. In this paper, we show that the componentwise linearity of monomial ideals is preserved by the polarization. As an application, we give a condition to guarantee that none of the powers of a monomial ideal is componentwise linear.

Key Words: Monomial ideal, linear resolution, componentwise linear, the gcd condition, polarization.
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1 Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. One of the fastest developing branches of algebraic combinatorics is combinatorial commutative algebra. It has evolved into one of the most active and vibrant branches of mathematical research during the past several decades. In this paper, we deal with a topic related to the minimal free resolutions of monomial ideals, which is one of the widely studied topics in combinatorial commutative algebra. In this direction, one important question is to figure out whether a monomial ideal or its powers have a linear resolution. Fröberg [4] has given a combinatorial characterization of squarefree monomial ideals of degree two, namely edge ideals, which admit a linear resolution. A generalization of this result to general monomial ideals of degree two is given afterward by Herzog et al. [7]. Nevo and Peeva [9] have given a criterion for when some of powers of edge ideals of graphs admit a linear resolution, and after then, Altafi et al. [1] have generalized this latter result to general monomial ideals.

In 1999, Herzog and Hibi [5] introduced a notion which is related to admitting a linear resolution. This notion is the componentwise linearity, which we deal with here. It is known that there exist notions of the subject which are or are not preserved by the polarization. For example, the Cohen–Macaulayness is preserved by the polarization, whereas the normality of ideals is not. In this paper, we prove that the componentwise linearity of monomial ideals is preserved by the polarization. As an application, we give a condition to guarantee that none of the powers of a monomial ideal is componentwise linear.

2 Preliminaries

In this section, we recall some preliminaries which are needed later on. We refer the reader to the book by Herzog and Hibi [6] for any undefined terms in combinatorial commutative algebra. Other important references on the subject are the books by Bruns and Herzog [2], Stanley [11], and Villarreal [12].

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{K}$. For a finitely generated graded S-module M, a *minimal graded free resolution* of M is an exact sequence

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0,$$

where $p \leq n$, every F_i is a graded free S-module of the form

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$$

with the minimal number of basis elements, and every map is graded. Here, $S(-j)$ denotes the S-module obtained by shifting the degrees of S by j, and the value $\beta_{i,j}(M)$ is called the ith *graded Betti number* of M of degree j. Note that the minimal graded free resolution of M is unique up to isomorphism, which implies that the graded Betti numbers are uniquely determined.

Let I be a monomial ideal of S and let $G(I)$ denote the set of minimal monomial generators of I. One says that I has a *linear resolution* if for some integer d, $\beta_{i,i+t}(S/I) = 0$ for all $i > 0$ and all $t \neq d$. It is easy to see that if I has a linear resolution, then all the minimal generators of I have the same degree. Herzog and Hibi [5] have generalized the notion of admitting a linear resolution for the case where I is not generated in a single degree. The *Castelnuovo–Mumford regularity* (or the *regularity* for short) of S/I and I are also as follows:

$$\begin{aligned} \operatorname{reg}(S/I) &= \max\left\{j - i \mid \beta_{i,j}(S/I) \neq 0\right\}, \\ \operatorname{reg}(I) &= \operatorname{reg}(S/I) + 1. \end{aligned}$$

2.1 The componentwise linearity

For a monomial ideal I of S, the *truncation of I in degree j*, denoted by $I_{\geq j}$, is the ideal generated by all monomials of degree $\geq j$ in I, that is, $I_{\geq j} = \oplus_{i \geq j} I_i$. The following notations are also used throughout this paper: $I_{\leq j}$ and $I_{\langle j \rangle}$. The first one refers to the ideal generated by all monomials of degree $\leq j$ in I and the second one refers to the ideal generated by all monomials of degree j in I. For example, if $I = (x^2, xz^2, y^2z^2, z^5)$ is an ideal of $\mathbb{K}[x, y, z]$, then

$$\begin{aligned} &I_{\geq 3} = (x^3, x^2y, x^2z, xz^2, y^2z^2, z^5), \\ &I_{\leq 3} = (x^2, xz^2), \\ &I_{\langle 3 \rangle} = (x^3, x^2y, x^2z, xz^2). \end{aligned}$$

A monomial ideal I of S is called *componentwise linear* if for every nonnegative integer j, the ideal $I_{\langle j \rangle}$ has a linear resolution.

2.2 The gcd condition

Monomial ideals which satisfy the gcd condition are defined by Jöllenbeck [8]. He has given a special acyclic matching on the Taylor resolution for this type of ideals, which are in connection with the Golod property of monomial rings. For a monomial ideal I of S, one says that I satisfies the gcd *condition* if for every two monomials $u, v \in G(I)$ with $\gcd(u, v) = 1$, there exists a monomial $w \neq u, v$ in $G(I)$ such that $w|uv$. As an example, the monomial ideal $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5)$ of $\mathbb{K}[x_1, x_2, x_3, x_4, x_5]$ satisfies the gcd condition.

2.3 The polarization

There is a process, called polarization, which is used to reduce the monomial ideals to the squarefree ones. The advantage of reducing the squarefree monomials is that one can use the Stanley–Reisner correspondence. This provides a nice correlation between squarefree monomial ideals and simplicial complexes. Let us introduce this process according to Peeva [10]. For this, we need new variables, which are $t_{j,i}$'s. Consider the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ and let $q_j = x_j^{a_j}$, $1 \leq j \leq n$. The *polarization* of q_j, denoted by q_j^{pol}, is defined as follows:

$$q_j^{\text{pol}} = \begin{cases} 1 & \text{if } a_j = 0, \\ x_j \displaystyle\prod_{i=1}^{a_j-1} t_{j,i} & \text{if } a_j \neq 0. \end{cases}$$

Note that by convention the product of the empty set of terms is equal to one. Now, let u be a monomial in S and set $u = q_1 \ldots q_n$, where for every $1 \leq j \leq n$, $q_j = x_j^{a_j}$. The *polarization* of u, denoted by u^{pol}, is defined as $u^{\text{pol}} = q_1^{\text{pol}} \ldots q_n^{\text{pol}}$. If we consider $A = \{1 \leq j \leq n \mid a_j \neq 0\}$, then the latter expression can be written as

$$u^{\text{pol}} = \prod_{j \in A} \left(x_j \prod_{i=1}^{a_j-1} t_{j,i} \right).$$

Finally, if $I = (u_1, \ldots, u_r)$ is a monomial ideal of S, its *polarization* I^{pol} is defined as $I^{\text{pol}} = (u_1^{\text{pol}}, \ldots, u_r^{\text{pol}})$. Note that

$$S^{\text{pol}} = S[t_{1,1}, \ldots, t_{1,p_1}, t_{2,1}, \ldots, t_{2,p_2}, \ldots, t_{n,1}, \ldots, t_{n,p_n}]$$

is the polynomial ring in which the monomial ideal I^{pol} lives. Here, p_j's are as follows:

$$p_j = \max\left\{a \mid x_j^{a+1} \text{ divides some monomials among } u_1, \ldots, u_r\right\}.$$

The following isomorphism explains the relation between the ideal I and its polarized ideal I^{pol}, where $\alpha = \left\{t_{j,i} - x_j \mid 1 \leq j \leq n, 1 \leq i \leq p_j\right\}$:

$$S^{\text{pol}}/(I^{\text{pol}} + (\alpha)) \cong S/I.$$

Let us conclude this section with an example. Let $I = (x_1^3, x_2x_3^2x_4, x_3^3, x_1x_4)$ be an ideal of $\mathbb{K}[x_1, x_2, x_3, x_4]$. The polarization of x_1^3 is $x_1t_{1,1}t_{1,2}$. The polarization of $x_2x_3^2x_4$ is

$x_2x_3t_{3,1}x_4$. The polarization of x_3^3 is $x_3t_{3,1}t_{3,2}$. The polarization of x_1x_4 is x_1x_4. Hence, the polarization of I is $I^{\text{pol}} = (x_1t_{1,1}t_{1,2}, x_2x_3t_{3,1}x_4, x_3t_{3,1}t_{3,2}, x_1x_4)$, which is an ideal of $\mathbb{K}[x_1, t_{1,1}, t_{1,2}, x_2, x_3, t_{3,1}, t_{3,2}, x_4]$.

3 Preserving by the polarization

In this section, we state and prove our results. In the sequel, $S = \mathbb{K}[x_1, \dots, x_n]$ is the polynomial ring over a field $\mathbb{K}$. Let us start by the componentwise linearity.

3.1 Polarization and the componentwise linearity

In the following proposition, we prove that polarization preserves the componentwise linearity. For this, we need the following two lemmas.

Lemma 1. *Let $I = (u_1, \dots, u_r)$ be a monomial ideal of S, where for every $1 \le \ell \le r$, $\deg(u_\ell) = d_\ell$. Then I is componentwise linear if and only if for every $1 \le \ell \le r$, $I_{\langle d_\ell \rangle}$ has a linear resolution.*

Proof. The "only if" part is trivial by the definition of componentwise linearity. For the "if" part, let j be a nonnegative integer. If $j < d_\ell$ for every $1 \le \ell \le r$, then $I_{\langle j \rangle} = 0$, and so, $I_{\langle j \rangle}$ has a linear resolution. If $j = d_\ell$ for some $1 \le \ell \le r$, then $I_{\langle j \rangle} = I_{\langle d_\ell \rangle}$, and so, by the assumption, $I_{\langle j \rangle}$ has again a linear resolution. Otherwise, there exists $1 \le \ell \le r$ such that I has no generator of degree bigger than d_ℓ and smaller than j. This means that $I_{\langle j \rangle}$ is equal to the truncation of $I_{\langle d_\ell \rangle}$ in degree j. Since the truncation of an ideal in a degree preserves having a linear resolution and, by the assumption, $I_{\langle d_\ell \rangle}$ has a linear resolution, $I_{\langle j \rangle}$ also has a linear resolution. All in all, $I_{\langle j \rangle}$ has a linear resolution, which means that I is componentwise linear, as required. ∎

Lemma 2. *Let $I = (u_1, \dots, u_r)$ be a monomial ideal of S, where for every $1 \le \ell \le r$, $\deg(u_\ell) = d_\ell$. Then I is componentwise linear if and only if for every $1 \le \ell \le r$, $\operatorname{reg}\left(I_{\le d_\ell}\right) \le d_\ell$.*

Proof. It is known that the regularity of a monomial ideal is upper bounded by d exactly whenever the truncation of the ideal in degree d has a linear resolution (see [3, Proposition 1.1]). Also, note that for a given j, the ideal $I_{\langle j \rangle}$ is equal to the truncation of $I_{\le j}$ in degree j, that is, $I_{\langle j \rangle} = \left(I_{\le j}\right)_{\ge j}$. Based on these observations and applying Lemma 1,

$$\begin{aligned} I \text{ is componentwise linear} &\Longleftrightarrow \text{for every } 1 \le \ell \le r,\ I_{\langle d_\ell \rangle} \text{ has a linear resolution} \\ &\Longleftrightarrow \text{for every } 1 \le \ell \le r,\ \left(I_{\le d_\ell}\right)_{\ge d_\ell} \text{ has a linear resolution} \\ &\Longleftrightarrow \text{for every } 1 \le \ell \le r,\ \operatorname{reg}\left(I_{\le d_\ell}\right) \le d_\ell, \end{aligned}$$

which completes the proof. ∎

We are now in the position to prove our first main result.

Proposition 1. *Let I be a monomial ideal of S. Then I is componentwise linear if and only if I^{pol} is componentwise linear.*

Proof. Let $I = (u_1, \ldots, u_r)$, where for every $1 \leq \ell \leq r$, $\deg(u_\ell) = d_\ell$. Firstly, note that the polarization preserves the set of degrees of monomial generators of an ideal, and so, $\{d_1, \ldots, d_r\}$ is a set of degrees of monomial generators of the polarized ideal I^{pol}. Thus, by Lemma 2, I^{pol} is componentwise linear if and only if for every $1 \leq \ell \leq r$, $\operatorname{reg}\big((I^{\mathtt{pol}})_{\leq d_\ell}\big) \leq d_\ell$. Secondly, note that for a given j, defining $I_{\leq j}$ is based on choosing generators of I associated with some specific degrees, which means that $\big(I_{\leq j}\big)^{\mathrm{pol}} = \big(I^{\mathrm{pol}}\big)_{\leq j}$. Finally, it is known that the polarization preserves Betti numbers. Based on these observations and again applying Lemma 2,

$$\begin{aligned} I \text{ is componentwise linear} &\Longleftrightarrow \text{for every } 1 \leq \ell \leq r,\ \operatorname{reg}\big(I_{\leq d_\ell}\big) \leq d_\ell \\ &\Longleftrightarrow \text{for every } 1 \leq \ell \leq r,\ \operatorname{reg}\big((I_{\leq d_\ell})^{\mathtt{pol}}\big) \leq d_\ell \\ &\Longleftrightarrow \text{for every } 1 \leq \ell \leq r,\ \operatorname{reg}\big((I^{\mathtt{pol}})_{\leq d_\ell}\big) \leq d_\ell \\ &\Longleftrightarrow I^{\mathtt{pol}} \text{ is componentwise linear,} \end{aligned}$$

which completes the proof. ◻

3.2 Polarization and the gcd condition

In the next proposition, we prove that the polarization preserves the gcd condition. For this, we need the following two lemmas.

Lemma 3. *Let u and v be two monomials in S. Then $\gcd(u, v) = 1$ if and only if $\gcd(u^{\mathrm{pol}}, v^{\mathrm{pol}}) = 1$.*

Proof. If $\gcd(u, v) = 1$, then the variables appearing in u and v are different, and thus, all the new variables $t_{j,i}$'s appearing in u^{pol} and v^{pol} are so. This means that $\gcd(u^{\mathrm{pol}}, v^{\mathrm{pol}}) = 1$. The same argument works for the other direction. ◻

Lemma 4. *Let u, v and w be three monomials in S with $\gcd(u, v) = 1$. Then $w|uv$ if and only if $w^{\mathrm{pol}}|u^{\mathrm{pol}}v^{\mathrm{pol}}$.*

Proof. Let $u = x_1^{a_1} \ldots x_n^{a_n}$, $v = x_1^{b_1} \ldots x_n^{b_n}$ and $w = x_1^{c_1} \ldots x_n^{c_n}$. For the "only if" part, note that $w|uv$ implies that for every $1 \leq j \leq n$, $c_j \leq a_j + b_j$. But $\gcd(u, v) = 1$ means that for every $1 \leq j \leq n$, $a_j b_j = 0$, and so, $a_j + b_j$ is equal to either a_j or b_j. Therefore, we obtain that for every $1 \leq j \leq n$, c_j is less than or equal to either a_j or b_j. Hence, for every $1 \leq j \leq n$, all the new variables $t_{j,i}$'s appearing in w^{pol} also appear in either u^{pol} or v^{pol}, and thus, $w^{\mathrm{pol}}|u^{\mathrm{pol}}v^{\mathrm{pol}}$. For the "if" part, note that $w^{\mathrm{pol}}|u^{\mathrm{pol}}v^{\mathrm{pol}}$ implies that for every $1 \leq j \leq n$, the new variables $t_{j,i}$'s appearing in w^{pol}, appear in $u^{\mathrm{pol}}v^{\mathrm{pol}}$, and since, by Lemma 3, $\gcd(u^{\mathrm{pol}}, v^{\mathrm{pol}}) = 1$, they should appear in either u^{pol} or v^{pol}. This means that for every $1 \leq j \leq n$, c_j is less than or equal to either a_j or b_j, and so, $w|uv$. ◻

We are now in the position to prove our second main result.

Proposition 2. *Let I be a monomial ideal of S. Then I satisfies the* gcd *condition if and only if I^{pol} satisfies the* gcd *condition.*

Proof. For the "only if" part, let $u^{\text{pol}}, v^{\text{pol}} \in G(I^{\text{pol}})$ with $\gcd(u^{\text{pol}}, v^{\text{pol}}) = 1$. Thus, $u, v \in G(I)$, and by Lemma 3, $\gcd(u, v) = 1$. Therefore, by the assumption, there exists $w \neq u, v$ in $G(I)$ for which $w|uv$, and so, by Lemma 4, $w^{\text{pol}}|u^{\text{pol}}v^{\text{pol}}$. Since $w^{\text{pol}} \neq u^{\text{pol}}, v^{\text{pol}}$ is a monomial in $G(I^{\text{pol}})$, I^{pol} satisfies the gcd condition. The "if" part can be derived similarly as above. □

3.3 An application of the results

Finally, we conclude the paper by generalizing a known result which states that if I is a *squarefree* monomial ideal of S which contains no variable and it does not satisfy the gcd condition, then none of the powers of I is componentwise linear (see [1, Corollary 2.3]). As an application of our two main propositions, we show that the above-mentioned result is valid too for general monomial ideals. For this, we need the following two lemmas.

Lemma 5. *Let u and v be two monomials in S. Then $\left(u^{\text{pol}}v\right)^{\text{pol}} = \left(uv\right)^{\text{pol}}$.*

Proof. Let $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$. Let also $A = \{1 \leq j \leq n \mid a_j \neq 0\}$ and $B = \{1 \leq j \leq n \mid b_j \neq 0\}$. Since $u^{\text{pol}} = \prod_{j \in A}(x_j \prod_{i=1}^{a_j-1} t_{j,i})$ and $v = \prod_{j \in B} x_j^{b_j}$, we obtain that

$$
\begin{aligned}
u^{\text{pol}} v &= \prod_{j\in A}\left(x_j \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in B} x_j^{b_j} \\
&= \prod_{j\in A\setminus(A\cap B)}\left(x_j \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in A\cap B}\left(x_j \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in A\cap B} x_j^{b_j} \prod_{j\in B\setminus(A\cap B)} x_j^{b_j} \\
&= \prod_{j\in A\setminus(A\cap B)}\left(x_j \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in A\cap B}\left(x_j^{b_j+1} \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in B\setminus(A\cap B)} x_j^{b_j}.
\end{aligned}
$$

This implies that

$$
\left(u^{\text{pol}} v\right)^{\text{pol}} = \prod_{j\in A\setminus(A\cap B)}\left(x_j \prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in A\cap B}\left(x_j \prod_{i=1}^{b_j} T_{j,i}\prod_{i=1}^{a_j-1} t_{j,i}\right)\prod_{j\in B\setminus(A\cap B)}\left(x_j \prod_{i=1}^{b_j-1} t_{j,i}\right).
$$

In order to obtain a closed form of the above expression, we define $\widehat{t}_{j,i}$'s as follows. For $j \in A \setminus (A \cap B)$, if $1 \leq i \leq a_j - 1$, we define $\widehat{t}_{j,i} = t_{j,i}$, and otherwise we set $\widehat{t}_{j,i} = 0$. Also, for $j \in A \cap B$, if $1 \leq i \leq b_j$, we define $\widehat{t}_{j,i} = T_{j,i}$, whereas if $b_j + 1 \leq i \leq a_j + b_j - 1$, we set $\widehat{t}_{j,i} = t_{j,i-b_j}$. Finally, for $j \in B \setminus (A \cap B)$, if $1 \leq i \leq b_j - 1$, we define $\widehat{t}_{j,i} = t_{j,i}$, and otherwise we set $\widehat{t}_{j,i} = 0$. By this setting and noting that $C = \{1 \leq j \leq n \mid a_j + b_j \neq 0\}$ is equal to $A \cup B$, the latter expression can be written as

$$\begin{aligned}
\left(u^{\text{pol}} v\right)^{\text{pol}} &= \prod_{j \in A \setminus (A \cap B)} \left(x_j \prod_{i=1}^{a_j+b_j-1} \widehat{t}_{j,i} \right) \prod_{j \in A \cap B} \left(x_j \prod_{i=1}^{a_j+b_j-1} \widehat{t}_{j,i} \right) \prod_{j \in B \setminus (A \cap B)} \left(x_j \prod_{i=1}^{a_j+b_j-1} \widehat{t}_{j,i} \right) \\
&= \prod_{j \in C} \left(x_j \prod_{i=1}^{a_j+b_j-1} \widehat{t}_{j,i} \right) \\
&= \left(uv\right)^{\text{pol}}.
\end{aligned}$$

□

Lemma 6. *Let I be a monomial ideal of S and $\ell \geq 1$ be an integer. Then $\left(I^{\ell}\right)^{\text{pol}} = \left(\left(I^{\text{pol}}\right)^{\ell}\right)^{\text{pol}}$.*

Proof. It is suffices to prove that for ℓ monomials $u_1, \ldots, u_\ell$,

$$\left(u_1 \ldots u_\ell\right)^{\text{pol}} = \left(u_1^{\text{pol}} \ldots u_\ell^{\text{pol}}\right)^{\text{pol}}.$$

This is easily done inductively by using Lemma 5 as follows:

$$\begin{aligned}
\left(u_1^{\text{pol}} \ldots u_\ell^{\text{pol}}\right)^{\text{pol}} &= \left(u_1 u_2^{\text{pol}} \ldots u_\ell^{\text{pol}}\right)^{\text{pol}} \\
&= \left(u_1 u_2 u_3^{\text{pol}} \ldots u_\ell^{\text{pol}}\right)^{\text{pol}} \\
&\;\;\vdots \\
&= \left(u_1 \ldots u_\ell\right)^{\text{pol}}.
\end{aligned}$$

□

Proposition 3. *Let I be a monomial ideal of S which contains no variable. If I does not satisfy the* gcd *condition, then none of the powers of I is componentwise linear.*

Proof. Suppose, on the contrary, that I^ℓ is componentwise linear for some integer $\ell \geq 1$. Thus, by Proposition 1, $\left(I^{\ell}\right)^{\text{pol}}$ is also componentwise linear. Hence, by Lemma 6, $\left(\left(I^{\text{pol}}\right)^{\ell}\right)^{\text{pol}}$ is componentwise linear. Again, applying Proposition 1 implies that $\left(I^{\text{pol}}\right)^{\ell}$ is componentwise linear. Since I^{pol} is squarefree, by [1, Corollary 2.3], it satisfies the gcd condition. Now, Proposition 2 implies that I also satisfies the gcd condition, a contradiction. □

Let us close the paper by a remark. It is known that the lexsegment ideals, the strongly stable ideals, the stable ideals and the **a**-stable ideals (**a** is an n-tuple) are all componentwise linear. Based on this point, Proposition 3 implies that if a monomial ideal does not satisfy the gcd condition, then none of its powers is lexsegment, strongly stable, stable or **a**-stable. Thus, if, for example, we let $I = (x^3y^2, x^2y^3, z^2)$ be an ideal in $\mathbb{K}[x, y, z]$, then none of the powers of I is lexsegment, strongly stable, stable, **a**-stable or componentwise linear, since I does not satisfy the gcd condition. However it is not easy to achieve these observations by using the definitions, we believe that the importance of Proposition 3 is something else. Indeed, Proposition 3 expresses that failing the gcd condition is not a good sign rather than to say that if the gcd condition occurs then it would be good.

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References

[1] N. Altafi, N. Nemati, S. A. Seyed Fakhari, S. Yassemi, Free resolution of powers of monomial ideals and Golod rings, *Math. Scand.*, **120**, 59–67 (2017).

[2] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press (1993).

[3] D. Eisenbud, S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra*, **88**, 89–133 (1984).

[4] R. Fröberg, On Stanley–Reisner rings, in: *Topics in Algebra, Part 2 (Warsaw, 1988)*, 57–70, Banach Center Publ., **26**, Part 2, PWN, Warsaw (1990).

[5] J. Herzog, T. Hibi, Componentwise linear ideals, *Nagoya Math. J.*, **153**, 141–153 (1999).

[6] J. Herzog, T. Hibi, *Monomial Ideals*, Springer-Verlag London (2011).

[7] J. Herzog, T. Hibi, X. Zheng, Monomial ideals whose powers have a linear resolution, *Math. Scand.*, **95**, 23–32 (2004).

[8] M. Jöllenbeck, On the multigraded Hilbert and Poincaré–Betti series and the Golod property of monomial rings, *J. Pure Appl. Algebra*, **207**, 261–298 (2006).

[9] E. Nevo, I. Peeva, C_4-free edge ideals, *J. Algebraic Combin.*, **37**, 243–248 (2013).

[10] I. Peeva, *Graded Syzygies*, Springer-Verlag London (2011).

[11] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser Boston (1996).

[12] R. H. Villarreal, *Monomial Algebras*, CRC Press (2015).

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