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General decay of the solution of a one dimensional double porous elastic system with memory

by

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Abstract

In this paper, we consider a one dimensional elastic system with two porous structures and memory effects in both porous equations. We prove that the weak dissipation generated by the memory terms produces a general rate of decay depending on the kernels of the memory terms and the coefficients of the system. Our result improves that of [6] obtained in the presence of strong damping and thermal effect. Moreover, the general decay that we obtained generalizes the previous results in the sense that exponential and polynomial rates of decay are only special cases.

Key Words: Double porosity, memory term, exponential stability, general decay.
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1 Introduction

In the present paper we are concerned with the following double porous elastic system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_1 \varphi_{tt} = \alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi & \\ \qquad - \displaystyle\int_0^t g(t-s)\,\varphi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \\ \kappa_2 \psi_{tt} = \beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi & \\ \qquad - \displaystyle\int_0^t h(t-s)\,\psi_{xx}(s)\,ds & \text{in } (0,\pi)\times(0,\infty), \end{cases} \tag{1.1}$$

with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = \varphi(0,t) = \varphi(\pi,t) = \psi(0,t) = \psi(\pi,t) = 0 \tag{1.2}$$

and initial data

$$\begin{gathered} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x). \end{gathered} \tag{1.3}$$

Here, u is the transversal displacement of a one-dimensional elastic solid of length π, φ and ψ are the unknown porous functions, the coefficients $\rho, \kappa_1, \kappa_2, \mu, \alpha,\ \gamma, \alpha_1, \alpha_2$ are positive.

As coupling is considered, the constants b, d and β must be different from zero. It is also assumed that the internal energy density associated with the system (1.1) is a positive definite quadratic form, which may be satisfied by requiring that the matrix

$$A = \begin{pmatrix} \mu & b & d \\ b & \alpha_1 & \alpha_3 \\ d & \alpha_3 & \alpha_2 \end{pmatrix}$$

is positive definite. The functions g, h are relaxation functions that assumed to satisfy some hypotheses that will be specified later.

The system considered here, represents a thermoelastic solid with double porosity structure in the framework of the theory of elastic materials with voids developed by Nunziato and Cowin [10]. This approach has been used by Ieşan and Quintanilla [13] to derive a new theory of thermoelastic solids which have a double porosity structure. In contrast to the classical theory the new one is not based on Darcy's law, and the porosity structure in the case of equilibrium is influenced by the displacement field.

The origin of the classical theory of elastic materials with double porosity goes back to the work of Barenblatt *et al.* [4, 5]. The authors introduced two liquid pressures at each point of the material which allows the body to have a double porosity structure: a macro porosity connected to the pores in the body and a micro porosity connected to the fissures in the skeleton.

Barenblatt's theory is an important generalization of Biot's theory [7] for porous materials with single porosity. Wilson and Aifentis [23] presented a theory of consolidation for elastic materials with double porosity which unifies the earlier models of Berendlatt and Biot. However, the theory proposed by Wilson and Aifentis ignored the cross-coupling effects between the volume change of the pores and fissures in the system. Later, Khalili and Valliappan [14] modified Aifentis' theory and proposed a cross-coupling terms included in the equations of conservation of mass for the pores and fissures fluid. Separately, Barryman and coauthors [2, 3] included a cross-coupling in Darcy's law for solids with double porosity.

In the beginning of the second decade of this century, Svanadze [21, 22] established the basic properties of the plane waves in the dynamic theory of elastic solids with double porosity.

In the last few years, a great interest has been given to the analysis of the longtime behavior of the solutions of porous thermoelastic problems. A part of this interest stems from the need to have general results that explain the experimental observations of engineers. The first contribution in this direction was achieved by Quintanilla [19]. He considered the following porous elastic system

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta\varphi_x & \text{in } (0,\pi)\times(0,+\infty), \\ \rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - \beta u_x - \xi\varphi - \tau\varphi_t & \text{in } (0,\pi)\times(0,+\infty), \end{cases} \tag{1.4}$$

where u is the transversal displacement and φ is the volume fraction. He used Hurwitz theorem and showed that the porous dissipation $\tau\varphi_t$ is not powerful enough to produce an exponential stability.

Casas and Quintanilla [9] coupled system (1.4) (for $\tau = 0$) with the heat equation, and proved that the thermal dissipation alone does not produce an exponential stability either. However, the combination of thermal and porous dissipations does produce it [8]. The same result was obtained by Magaña and Quintanilla [15] for microthermal dissipation combined with viscoelasticity or porous dissipation.

The viscoelastic damping represented by a memory term was introduced first by Dafermos [11, 12]. He proved that the solution of the viscoelastic equation

$$\rho u_{tt} = c u_{xx} - \int_0^t g(t-\tau)\, u_{xx}(x,\tau)\, d\tau,\ (0,1)\times(0,\infty),$$

with Dirichlet boundary conditions is asymptotically stable, but he does not specify the rate of decay of the solution [12].

Soufyane [20] examined the porous thermoelastic system of memory type

$$\begin{cases} \rho_1 u_{tt} = k(u_x+\varphi)_x - \theta_x, \\ \rho_2 \varphi_{tt} = \alpha\varphi_{xx} - k(u_x+\varphi) + \theta - \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,t-s)ds, \\ \theta_t = \kappa\theta_{xx} - u_{tx} - \varphi_t, \end{cases} \tag{1.5}$$

where all the coefficients are equal to 1 and the kernel g satisfies

$$l = \alpha - \int_0^\infty g(s)\, ds > 0, \tag{1.6}$$

$$g'(t) \le -\xi g^p(t),\, t \ge 0. \tag{1.7}$$

He used the multiplier method and established an exponential decay rate for $p = 1$, and a polynomial decay rate for $1 < p < \frac{3}{2}$.

Messaoudi and Fareh [16, 17] considered (1.5) and assumed the relaxation function g satisfies (1.6) and $g'(t) \le -\xi(t)g(t)$ for a non increasing function ξ. They obtained a general decay result for which the exponential and the polynomial rates of decay are only special cases.

Recently, Apalara [1] considered the porous thermoelastic system of memory type

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 & in\ (0,1)\times(0,+\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^t g(t-s)\,\phi_{xx}(s)\, ds = 0 & in\ (0,1)\times(0,+\infty), \end{cases}$$

with Neumann-Dirichlet boundary conditions. He studied the case of equal wave speeds $\dfrac{\mu}{\rho} = \dfrac{\delta}{J}$ and proved, in contrary to [15], that the unique dissipation given by the memory term leads to a general decay.

In the context of double porous thermoelasticity, Fernàndez *et al.* [6] considered the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x - \beta\theta_x, \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\varphi_t - \varepsilon_2\psi_t, \\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\varphi_t - \varepsilon_4\psi_t, \\ c\theta_t = \kappa\theta_{xx} - \beta u_{tx} - \gamma_1\varphi_t - \gamma_2\psi_t, \end{cases} \tag{1.8}$$

and established an exponential stability. They also considered (1.8) with only one porous dissipation structure, and showed that generally the system lacks the exponential stability. However, they give a sufficient conditions for which the solutions decay exponentially.

Recently, Nemsi and Fareh [18] studied the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x + d\psi_x + \lambda u_{txx} & \text{in } (0,L)\times(0,\infty), \\ \kappa_1\varphi_{tt} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \tau_1\varphi_t & \text{in } (0,L)\times(0,\infty), \\ \kappa_2\psi_{tt} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \tau_2\psi_t & \text{in } (0,L)\times(0,\infty), \end{cases}$$

with the boundary conditions

$$u(t,0) = u(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0 \quad \text{in } (0,\infty),$$

and established an exponential decay result without any assumption on the wave speeds.

In the present paper we consider the isothermal case of (1.8) ($\beta = \gamma_1 = \gamma_2 = 0$), with weaker dissipations of memory type instead of the strong dampings $\varepsilon_i\varphi_t, \varepsilon_i\psi_t$. We use the multiplier method and establish a general decay result depends on the coefficients of the system and the rates of decay of the relaxation functions g and h.

To the best of our knowledge, the longtime behavior of thermoelastic systems with double porosity structures has been investigated only in [6, 18], and none of them have considered the dissipation of memory type. Our result improves the previous results, in the sense that we have weakened and minimized the dissipations and have generalized the rate of decay.

The rest of the paper is organized as follows: in Section 2, we introduce some notations and preliminaries needed for our work and prove several technical lemmas. In Section 3, we state and prove the general decay of the energy associated to the solution of system (1.1).

2 Preliminaries

We begin this section by requiring the following assumptions on the relaxation functions g and h :

(H1) $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are C^1 non increasing functions satisfying

$$g(0) > 0,\ l = \alpha - \int_0^{+\infty} g(s)\,ds > 0,$$
$$h(0) > 0,\ k = \gamma - \int_0^{+\infty} h(s)\,ds > 0,$$

and

$$lk > \beta^2. \tag{2.1}$$

(H2) There exist two non-increasing C^1 functions $\eta, \xi, : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t)\,g(t),\ h'(t) \leq -\eta(t)\,h(t),\ t \geq 0.$$

We recall the notation

$$(f \circ \phi)(t) := \int_0^{\pi}\int_0^t f(t-s)\,(\phi(t) - \phi(s))^2\,dsdx.$$

Remark 1. *From (1.1)$_1$ and (1.2) we easily show that*

$$\frac{d^2}{dt^2}\int_0^\pi u(x,t)\,dx=0. \tag{2.2}$$

By solving (2.2) and using the initial data on u, we obtain

$$\int_0^\pi u(x,t)\,dx=t\int_0^\pi u_1(x)\,dx+\int_0^\pi u_0(x)\,dx.$$

Consequently, if we set

$$\overline{u}(x,t)=u(x,t)-t\int_0^\pi u_1(x)\,dx-\int_0^\pi u_0(x)\,dx,$$

then $(\overline{u},\varphi,\psi)$ satisfies (1.1)–(1.2) and the initial data

$$\overline{u}(x,0)=u_0(x)-\int_0^\pi u_0(x)\,dx,\ \overline{u}_t(x,0)=u_1(x)-\int_0^\pi u_1(x)\,dx.$$

Moreover, we have

$$\int_0^\pi \overline{u}(x,t)\,dx=0,$$

which allows the application of Poincaré's inequality. In the sequel, we work with $(\overline{u},\varphi,\psi)$ but for convenience we write (u,φ,ψ).

Remark 2. *We notice that the matrix A of coefficients is positive definite if and only if*

$$\left(\alpha_1-\frac{b^2}{\mu}\right)\left(\alpha_2-\frac{d^2}{\mu}\right)>\left(\alpha_3-\frac{bd}{\mu}\right)^2. \tag{2.3}$$

Moreover, since any principal submatrix of a positive definite matrix is also positive definite, we have

$$\alpha_1-\frac{b^2}{\mu}>0 \text{ and } \alpha_2-\frac{d^2}{\mu}>0.$$

Now, let us prove some useful lemmas.

Lemma 1. *Let*

$$\begin{aligned}E(t)=&\frac{1}{2}\int_0^\pi\left[\rho u_t^2+\kappa_1\varphi_t^2+\kappa_2\psi_t^2+\mu u_x^2+\alpha_1\varphi^2+\alpha_2\psi^2+2\alpha_3\varphi\psi\right]dx\\&+\frac{1}{2}\left(\alpha-\int_0^t g(s)\,ds\right)\int_0^\pi\varphi_x^2dx+\frac{1}{2}\left(\gamma-\int_0^t h(s)\,ds\right)\int_0^\pi\psi_x^2dx\\&+b\int_0^\pi u_x\varphi dx+d\int_0^\pi u_x\psi dx+\beta\int_0^\pi\varphi_x\psi_x dx+\frac{1}{2}\left[g\circ\varphi_x+h\circ\psi_x\right],\end{aligned}$$

be the energy associated with the solution of (1.1)–(1.2). Assume that (H1) holds and that the matrix A is positive definite, then

$$E(t)\geq 0,\ \forall t\geq 0.$$

Proof. Let $\widetilde{\alpha}=\alpha-\int_0^t g\left(s\right)ds$ and $\widetilde{\gamma}=\gamma-\int_0^t h\left(s\right)ds$, clearly, $\widetilde{\alpha}\geq l$ and $\widetilde{\gamma}\geq k$, then using (2.1) and the fact that the matrix A is positive definite, we infer that

$$\begin{aligned}E\left(t\right)&=\left\langle A\left(u_x,\varphi,\psi\right),\left(u_x,\varphi,\psi\right)\right\rangle+\frac{1}{2}\left[g\circ\varphi_x+h\circ\psi_x\right]\\&+\frac{1}{4}\left(\widetilde{\alpha}-\frac{\beta^2}{\widetilde{\gamma}}\right)\int_0^\pi\varphi_x^2dx+\frac{1}{4}\left(\widetilde{\gamma}-\frac{\beta^2}{\widetilde{\alpha}}\right)\int_0^\pi\psi_x^2dx\\&+\frac{\widetilde{\alpha}}{2}\int_0^\pi\left(\varphi_x+\frac{\beta}{\widetilde{\alpha}}\psi_x\right)^2dx+\frac{\widetilde{\gamma}}{2}\int_0^\pi\left(\psi_x+\frac{\beta}{\widetilde{\gamma}}\varphi_x\right)^2dx\geq 0.\end{aligned}$$

□

Lemma 2. *Under the assumptions (H1)-(H2), we have*

$$\int_0^\pi\varphi_t\left(t\right)\int_0^t g\left(t-s\right)\varphi_{xx}\left(s\right)dsdx$$

$$=\frac{1}{2}\frac{d}{dt}\left[\left(g\circ\varphi_x\right)\left(t\right)-\int_0^t g\left(s\right)ds\int_0^\pi\varphi_x^2\left(t\right)dx\right]+g\left(t\right)\int_0^\pi\varphi_x^2\left(t\right)dx-\frac{1}{2}\left(g'\circ\varphi_x\right).$$

Proof. We have

$$\int_0^\pi\varphi_t\left(t\right)\int_0^t g\left(t-s\right)\varphi_{xx}\left(s\right)dsdx=-\int_0^\pi\varphi_{xt}\left(t\right)\int_0^t g\left(t-s\right)\varphi_x\left(s\right)dsdx$$

$$\begin{aligned}&=\int_0^\pi\varphi_{xt}\left(t\right)\int_0^t g\left(t-s\right)\left(\varphi_x\left(t\right)-\varphi_x\left(s\right)\right)dsdx-\int_0^\pi\varphi_{xt}\left(t\right)\varphi_x\left(t\right)\int_0^t g\left(t-s\right)dsdx,\\&=\frac{1}{2}\int_0^\pi\int_0^t g\left(t-s\right)\frac{d}{dt}\left(\varphi_x\left(t\right)-\varphi_x\left(s\right)\right)^2dsdx-\frac{1}{2}\left(\int_0^t g\left(s\right)ds\right)\frac{d}{dt}\int_0^\pi\varphi_x^2\left(t\right)dx,\\&=\frac{1}{2}\frac{d}{dt}\int_0^\pi\int_0^t g\left(t-s\right)\left(\varphi_x\left(t\right)-\varphi_x\left(s\right)\right)^2dsdx-\frac{1}{2}\int_0^\pi\int_0^t g'\left(t-s\right)\left(\varphi_x\left(t\right)-\varphi_x\left(s\right)\right)^2dsdx\\&\quad-\frac{1}{2}\frac{d}{dt}\int_0^t g\left(s\right)ds\int_0^\pi\varphi_x^2\left(t\right)dx+g\left(t\right)\int_0^\pi\varphi_x^2\left(t\right)dx,\\&=\frac{1}{2}\frac{d}{dt}\left[\left(g\circ\varphi_x\right)\left(t\right)-\int_0^t g\left(s\right)ds\int_0^\pi\varphi_x^2\left(t\right)dx\right]+g\left(t\right)\int_0^\pi\varphi_x^2\left(t\right)dx-\frac{1}{2}\left(g'\circ\varphi_x\right).\end{aligned}$$

□

Lemma 3. *Suppose that (H1) holds, then along the solution of system (1.1)–(1.3) the energy $E\left(t\right)$ satisfies the property*

$$E'\left(t\right)\leq\frac{1}{2}\left(g'\circ\varphi_x\right)+\frac{1}{2}\left(h'\circ\psi_x\right)\leq 0. \tag{2.4}$$

Proof. Multiplying the equations of (1.1) by u_t, φ_t, ψ_t respectively and integrating with respect to x over $(0,\pi)$ we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^\pi u_t^2 dx+\frac{\mu}{2}\frac{d}{dt}\int_0^\pi u_x^2 dx+b\int_0^\pi \varphi u_{tx} dx+d\int_0^\pi \psi u_{tx} dx=0, \tag{2.5}$$

$$\frac{\kappa_1}{2}\frac{d}{dt}\int_0^\pi \varphi_t^2 dx+\frac{\alpha}{2}\frac{d}{dt}\int_0^\pi \varphi_x^2 dx+\frac{\alpha_1}{2}\frac{d}{dt}\int_0^\pi \varphi^2 dx+\beta\int_0^\pi \psi_x\varphi_{xt} dx+b\int_0^\pi u_x\varphi_t dx+\alpha_3\int_0^\pi \psi\varphi_t dx$$
$$+\frac{1}{2}\frac{d}{dt}\left[(g\circ\varphi_x)(t)-\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2(t)\,dx\right]=-g(t)\int_0^\pi \varphi_x^2(t)\,dx+\frac{1}{2}(g'\circ\varphi_x) \tag{2.6}$$

and

$$\frac{\kappa_2}{2}\frac{d}{dt}\int_0^\pi \psi_t^2 dx+\frac{\gamma}{2}\frac{d}{dt}\int_0^\pi \psi_x^2 dx+\frac{\alpha_2}{2}\frac{d}{dt}\int_0^\pi \psi^2 dx+\beta\int_0^\pi \psi_{xt}\varphi_x dx+d\int_0^\pi u_x\psi_t dx+\alpha_3\int_0^\pi \psi_t\varphi dx$$
$$+\frac{1}{2}\frac{d}{dt}\left[(h\circ\psi_x)(t)-\int_0^t h(s)\,ds\int_0^\pi \psi_x^2(t)\,dx\right]=-h(t)\int_0^\pi \psi_x^2(t)\,dx+\frac{1}{2}(h'\circ\psi_x). \tag{2.7}$$

The addition of (2.5)-(2.7) gives

$$\begin{aligned}E'(t)&=-g(t)\int_0^\pi \varphi_x^2(t)\,dx+\frac{1}{2}(g'\circ\varphi_x)-h(t)\int_0^\pi \psi_x^2(t)\,dx+\frac{1}{2}(h'\circ\psi_x),\\&\le\frac{1}{2}(g'\circ\varphi_x)+\frac{1}{2}(h'\circ\psi_x)\le 0.\end{aligned}$$

□

Lemma 4. *Let $v\in L^2(0,\pi)$ and $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ an integrable function, then for any $\epsilon>0$, we have*

$$\int_0^\pi v(x,t)\int_0^t f(t-s)\,v(x,s)\,dsdx\le \begin{array}{l}\left(\epsilon+\int_0^t f(s)\,ds\right)\int_0^\pi v^2(x,t)\,dx\\+\frac{1}{4\epsilon}\left(\int_0^t f(s)\,ds\right)(g\circ v).\end{array} \tag{2.8}$$

Proof. We have

$$\int_0^\pi v(x,t)\int_0^t f(t-s)\,v(x,s)\,dsdx$$
$$=-\int_0^\pi v(x,t)\int_0^t f(t-s)\,(v(x,t)-v(x,s))\,dsdx+\left(\int_0^t f(s)\,ds\right)\int_0^\pi v^2 dx. \tag{2.9}$$

Using Young's and Cauchy Schwarz' inequalities we obtain

$$-\int_0^\pi v(x,t)\int_0^t f(t-s)\,(v(x,t)-v(x,s))\,dsdx$$
$$\le\epsilon\int_0^\pi v^2(x,t)\,dx+\frac{1}{4\epsilon}\int_0^\pi\left(\int_0^t f(t-s)\,(v(x,t)-v(x,s))\,ds\right)^2 dx,$$

$$\leq \epsilon \int_0^\pi v^2(t)\,dx + \frac{1}{4\epsilon}\left(\int_0^t f(s)\,ds\right)\int_0^\pi \int_0^t f(t-s)\left(v(t)-v(s)\right)^2 ds dx. \tag{2.10}$$

Substituting (2.10) into (2.9), inequality (2.8) follows immediately. ∎

Remark 3. *Applying Lemma 4 and the fact that* $\int_0^t g(s)\,ds \leq \alpha - l$, $\int_0^t h(s)\,ds \leq \gamma - k$, *we have,*

$$\begin{aligned}\int_0^\pi \varphi_x(t)\int_0^t g(t-s)\varphi_x(s)\,dsdx &\leq \left(\epsilon + \int_0^t g(s)\,ds\right)\int_0^\pi \varphi_x^2 dx + \frac{1}{4\epsilon}\left(\int_0^t g(s)\,ds\right)(g\circ\varphi_x)\\ &\leq (\alpha+\epsilon)\int_0^\pi \varphi_x^2 dx + \frac{\alpha - l}{4\epsilon}(g\circ\varphi_x),\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi \psi_x(t)\int_0^t h(t-s)\psi_x(s)\,dsdx &\leq \left(\epsilon + \int_0^t h(s)\,ds\right)\int_0^\pi \psi_x^2 dx + \frac{1}{4\epsilon}\left(\int_0^t h(s)\,ds\right)(h\circ\psi_x)\\ &\leq (\gamma+\epsilon)\int_0^\pi \psi_x^2 dx + \frac{\gamma - k}{4\epsilon}(h\circ\psi_x).\end{aligned}$$

On the other hand, Cauchy Schwarz inequality yields

$$\left(\int_0^x u_t(y)\,dy\right)^2 \leq \left(\int_0^\pi dx\right)\int_0^\pi u_t^2(x)\,dx = \pi\int_0^\pi u_t^2(x)\,dx.$$

For the sake of completeness, we end this section by stating the following theorem, where the proof can by done by the use of Faedo-Galerkin method.

Theorem 1. *Assume that the assumptions (H1), (H2) and (2.3) hold, then for every* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in H_0^1(0,\pi)\times L^2(0,\pi)$ *and* $(u_0,u_1) \in H_*^1(0,\pi)\times L_*^2(0,\pi)$, *the problem (1.1)–(2.5) has a unique weak solution* (u,φ,ψ) *satisfies*

$$\begin{gathered}u \in C\left((0,+\infty),H_*^1(0,\pi)\right)\cap C^1\left((0,+\infty),L_*^2(0,\pi)\right),\\ \varphi,\psi \in C\left((0,+\infty),H_0^1(0,\pi)\right)\cap C^1\left((0,+\infty),L^2(0,\pi)\right),\end{gathered}$$

where

$$L_*^2(0,\pi) := \left\{v \in L^2(0,\pi);\int_0^\pi v(x)\,dx = 0\right\} \text{ and } H_*^1(0,\pi) = H^1(0,\pi)\cap L_*^2(0,\pi).$$

3 General decay

Now we are able to state and prove our main result, which reads as follows

Theorem 2. *Let* $(u_0,u_1)\in H_*^1(0,\pi)\times L_*^2(0,\pi)$ *and* $(\varphi_0,\varphi_1),(\psi_0,\psi_1)\in H_0^1(0,\pi)\times L^2(0,\pi)$ *and assume that (H1)–(H2) hold and*

$$\frac{\mu}{\rho} = \frac{\alpha}{\kappa_1} + \frac{d\beta}{b\kappa_2} = \frac{\gamma}{\kappa_2} + \frac{b\beta}{d\kappa_1}. \tag{3.1}$$

Then for any $t_0 > 0$, there exist two positive constants σ, ω such that the energy $E(t)$ satisfies

$$E(t) \leq \sigma e^{-\omega \int_{t_0}^{t} \chi(s)ds}, \text{ for a.e. } t \geq t_0, \tag{3.2}$$

where $\chi(t) = \min\{\xi(t), \eta(t)\}$.

Remark 4. *Our result is optimal if:*

1. *for all $t \geq 0$, $\xi(t) \neq \eta(t)$, or,*
2. *there exists $t_1 > 0$; $\xi(t_1) = \eta(t_1)$ and $\lim_{t \longrightarrow t_1} \frac{\xi(t) - \eta(t)}{t - t_1} = 0.$*

In both cases χ is differentiable for all $t \geq 0$.

The proof of Theorem 2 will be established through several lemmas.

Lemma 5. *For any $\varepsilon > 0$, the functional*

$$F_1(t) = \kappa_1 \int_0^{\pi} \varphi \varphi_t dx + \kappa_2 \int_0^{\pi} \psi_t \psi dx + \frac{\rho}{\mu} \int_0^{\pi} (b\varphi + d\psi) \int_0^{x} u_t(y)\, dydx$$

satisfies along the solution of (1.1)–(1.3), the estimate

$$\begin{aligned} F_1'(t) \leq & (\kappa_1 + C_\varepsilon) \int_0^{\pi} \varphi_t^2 dx + (\kappa_2 + C_\varepsilon) \int_0^{\pi} \psi_t^2 dx - \frac{\widehat{l}}{2} \int_0^{\pi} \varphi_x^2 dx - \frac{\widehat{k}}{2} \int_0^{\pi} \psi_x^2 dx \\ & - \widehat{\alpha}_1 \int_0^{\pi} \varphi^2 dx - \widehat{\alpha}_2 \int_0^{\pi} \psi^2 dx + c(g \circ \varphi_x) + c(h \circ \psi_x) + \varepsilon \int_0^{\pi} u_t^2 dx, \end{aligned} \tag{3.3}$$

where, $\widehat{l}$ and $\widehat{k}$ are positive constants depending on l, k, β and ϵ, and $\widehat{\alpha}_1, \widehat{\alpha}_2$ are positive constants depending on $b, d, \mu, \alpha_1, \alpha_2$ and α_3, and c is a positive constant.

Proof. The differentiation of F_1 gives

$$\begin{aligned} F_1'(t) = & \kappa_1 \int_0^{\pi} \varphi_t^2 dx + \kappa_2 \int_0^{\pi} \psi_t^2 dx + \frac{\rho}{\mu} \int_0^{\pi} (b\varphi_t + d\psi_t) \int_0^{x} u_t(y)\, dydx \\ & + \int_0^{\pi} \varphi \left(\alpha \varphi_{xx} + \beta \psi_{xx} - bu_x - \alpha_1 \varphi - \alpha_3 \psi - \int_0^{t} g(t-s)\, \varphi_{xx}(s)\, ds \right) dx \\ & + \int_0^{\pi} \psi \left(\beta \varphi_{xx} + \gamma \psi_{xx} - du_x - \alpha_3 \varphi - \alpha_2 \psi - \int_0^{t} h(t-s)\, \psi_{xx}(s)\, ds \right) dx \\ & + \frac{1}{\mu} \int_0^{\pi} (b\varphi + d\psi)(\mu u_x + b\varphi + d\psi)\, dx, \\ = & \kappa_1 \int_0^{\pi} \varphi_t^2 dx + \kappa_2 \int_0^{\pi} \psi_t^2 dx - \alpha \int_0^{\pi} \varphi_x^2 dx - \gamma \int_0^{\pi} \psi_x^2 dx - 2\beta \int_0^{\pi} \varphi_x \psi_x dx \\ & + \frac{\rho}{\mu} \int_0^{\pi} (b\varphi_t + d\psi_t) \int_0^{x} u_t(y)\, dydx - \alpha_1 \int_0^{\pi} \varphi^2 dx - 2\alpha_3 \int_0^{\pi} \varphi \psi dx - \alpha_2 \int_0^{\pi} \psi^2 dx \end{aligned}$$

$$+\frac{b^2}{\mu}\int_0^\pi \varphi^2 dx + \frac{d^2}{\mu}\int_0^\pi \psi^2 dx + \frac{2bd}{\mu}\int_0^\pi \varphi\psi dx$$
$$+\int_0^\pi \varphi_x(t)\int_0^t g(t-s)\varphi_x(s)\,dsdx + \int_0^\pi \psi_x(t)\int_0^t h(t-s)\psi_x(s)\,dsdx.$$

Using Young inequality and Remark 3, we infer that for any $\epsilon, \varepsilon, \varepsilon_1, \varepsilon_2 > 0$ we have

$$F_1'(t) \le \kappa_1\int_0^\pi \varphi_t^2 dx + \kappa_2\int_0^\pi \psi_t^2 dx - \left(\alpha_1 - \frac{b^2}{\mu}\right)\int_0^\pi \varphi^2 dx - \left(\alpha_2 - \frac{d^2}{\mu}\right)\int_0^\pi \psi^2 dx$$
$$+\frac{2(bd-\mu\alpha_3)}{\mu}\int_0^\pi \varphi\psi dx - 2\beta\int_0^\pi \varphi_x\psi_x dx - \left(\alpha - \int_0^t g(s)\,ds - \epsilon\right)\int_0^\pi \varphi_x^2 dx + \frac{\alpha - l}{4\epsilon} g\circ\varphi_x$$
$$-\left(\gamma - \int_0^t h(s)\,ds - \epsilon\right)\int_0^\pi \psi_x^2 dx + \frac{\gamma - k}{4\epsilon} h\circ\psi_x + \frac{\rho}{\mu}\int_0^\pi (b\varphi_t + d\psi_t)\int_0^x u_t(y)\,dydx,$$
$$F_1'(t) \le (\kappa_1 + C_\varepsilon)\int_0^\pi \varphi_t^2 dx + (\kappa_2 + C_\varepsilon)\int_0^\pi \psi_t^2 dx - (l - \beta\varepsilon_1 - \epsilon)\int_0^\pi \varphi_x^2 dx$$
$$-\left(k - \frac{\beta}{\varepsilon_1} - \epsilon\right)\int_0^\pi \psi_x^2 dx + \frac{\alpha - l}{4\epsilon} g\circ\varphi_x + \frac{\gamma - k}{4\epsilon} h\circ\psi_x + \varepsilon\int_0^\pi u_t^2 dx$$
$$-\left(\alpha_1 - \frac{b^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\varepsilon_2\right)\int_0^\pi \varphi^2 dx - \left(\alpha_2 - \frac{d^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\frac{1}{\varepsilon_2}\right)\int_0^\pi \psi^2 dx.$$

Inequality (2.1) and (2.3) allow us to choose $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\widehat{l} = l - \beta\varepsilon_1 > 0 \text{ and } \widehat{k} = k - \frac{\beta}{\varepsilon_1} > 0$$

and

$$\widehat{\alpha}_1 = \alpha_1 - \frac{b^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\varepsilon_2 > 0 \text{ and } \widehat{\alpha}_2 = \alpha_2 - \frac{d^2}{\mu} - \left(\frac{bd}{\mu} - \alpha_3\right)\frac{1}{\varepsilon_2} > 0.$$

Next we choose $\epsilon = \min\left\{\frac{\widehat{l}}{2}, \frac{\widehat{k}}{2}\right\}$ to get (3.3). ∎

Lemma 6. *For any $t_0 > 0$ and any $\varepsilon_1, \delta > 0$, the functional*

$$F_2(t) := -\kappa_1\int_0^\pi \varphi_t(t)\int_0^t g(t-s)(\varphi(t) - \varphi(s))\,dsdx$$

satisfies along the solution of (1.1)–(1.3) the estimate

$$F_2'(t) \le -\kappa_1\left(\int_0^{t_0} g(s)\,ds - \delta\right)\int_0^\pi \varphi_t^2(t)\,dx + 3\varepsilon_1\int_0^\pi \varphi_x^2 dx + \varepsilon_1\int_0^\pi \psi_x^2 dx$$
$$+\frac{c}{\varepsilon_1} g\circ\varphi_x + \varepsilon_1\int_0^\pi u_x^2 dx - \frac{c}{\delta} g'\circ\varphi_x. \tag{3.4}$$

Proof. The differentiation of F_2 and integration by parts give

$$F_2'(t) = \alpha \int_0^\pi \varphi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx$$

$$+\beta \int_0^\pi \psi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx$$

$$+b \int_0^\pi u_x \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx + \alpha_1 \int_0^\pi \varphi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx$$

$$+\alpha_3 \int_0^\pi \psi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx$$

$$-\int_0^\pi \left(\int_0^t g(t-s)\varphi_x(s)\, ds\right)\left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, ds\right) dx$$

$$-\kappa_1 \int_0^\pi \varphi_t(t) \int_0^t g'(t-s)(\varphi(t) - \varphi(s))\, dsdx - \kappa_1 \left(\int_0^t g(s)\, ds\right) \int_0^\pi \varphi_t^2(t)\, dx.$$

Now we estimate the terms in the right hand side of $F_2'(t)$ term by term recalling that

$$\int_0^t g(s)\, ds < \alpha \text{ and } \int_0^t h(s)\, ds < \gamma.$$

First, using Young's and Cauchy Schwarz' inequalities, we have for any $\varepsilon_1 > 0$,

$$I_1 := \alpha \int_0^\pi \varphi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx,$$

$$I_1 \le \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{\alpha^2}{4\varepsilon_1} \int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, ds\right)^2 dx,$$

$$I_1 \le \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{\alpha^3}{4\varepsilon_1} g \circ \varphi_x. \tag{3.5}$$

Similarly, for any $\varepsilon_2 > 0$, we have

$$I_2 := \beta \int_0^\pi \psi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s))\, dsdx \le \varepsilon_1 \int_0^\pi \psi_x^2 dx + \frac{\beta^2\alpha}{4\varepsilon_1} g \circ \varphi_x. \tag{3.6}$$

Next, we use Young's, Poincaré's and Cauchy Schwarz' inequalities to obtain for any $\varepsilon_1 > 0$ and $\delta > 0$, where C is a generic positive constant

$$I_3 := b \int_0^\pi u_x \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx \le \varepsilon_1 \int_0^\pi u_x^2 dx + \frac{Cb^2\alpha}{4\varepsilon_1} g \circ \varphi_x, \tag{3.7}$$

$$I_4 := \int_0^\pi \varphi \int_0^t g(t-s)(\varphi(t) - \varphi(s))\, dsdx \le \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{C\alpha}{4\varepsilon_1} g \circ \varphi_x, \tag{3.8}$$

$$I_5 := -\kappa_1 \int_0^\pi \varphi_t(t) \int_0^t g'(t-s)(\varphi(t) - \varphi(s))\, dsdx,$$

$$I_5 \leq \kappa_1 \delta \int_0^\pi \varphi_t^2 dx + \frac{\kappa_1}{4\delta} \int_0^\pi \left(\int_0^t g'(t-s)(\varphi(t) - \varphi(s)) ds \right)^2 dx,$$

$$I_5 \leq \kappa_1 \delta \int_0^\pi \varphi_t^2 dx + \frac{\kappa_1}{4\delta} \left(-\int_0^t g'(s) ds \right) \int_0^\pi \int_0^t -g'(t-s)(\varphi(t) - \varphi(s))^2 dsdx,$$

$$I_5 \leq \kappa_1 \delta \int_0^\pi \varphi_t^2 dx - \frac{\kappa_1 C g(0)}{4\delta} \int_0^\pi \int_0^t g'(t-s)(\varphi_x(t) - \varphi_x(s))^2 dsdx,$$

then

$$I_5 \leq \kappa_1 \delta \int_0^\pi \varphi_t^2 dx - \frac{c}{\delta} g' \circ \varphi_x, \tag{3.9}$$

$$\begin{aligned} I_6 \quad &:= -\int_0^\pi \left(\int_0^t g(t-s)\varphi_x(s) ds \right) \left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds \right) dx, \\ &= \int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) ds \right)^2 dx \\ &\quad - \left(\int_0^t g(s) ds \right) \int_0^\pi \varphi_x(t) \int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) dsdx, \end{aligned}$$

$$I_6 \leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \left[1 + \frac{\left(\int_0^t g(s) ds \right)^2}{4\varepsilon_1} \right] \int_0^\pi \left(\int_0^t g(t-s)(\varphi_x(t) - \varphi_x(s)) \right)^2 dsdx,$$

then

$$I_6 \leq \varepsilon_1 \int_0^\pi \varphi_x^2 dx + \frac{c}{\varepsilon_1} g \circ \varphi_x. \tag{3.10}$$

Substituting $I_1 - I_6$, in the expression of $F_2'(t)$ we arrive at

$$\begin{aligned} F_2'(t) \leq -\kappa_1 \left(\int_0^t g(s) ds \right) \int_0^\pi \varphi_t^2(t) dx + 3\varepsilon_1 \int_0^\pi \varphi_x^2 dx + \varepsilon_1 \int_0^\pi \psi_x^2 dx \\ + \frac{c}{\varepsilon_1} g \circ \varphi_x + \varepsilon_2 \int_0^\pi u_x^2 dx + \delta \kappa_1 \int_0^\pi \varphi_t^2 dx - \frac{c}{\delta} g' \circ \varphi_x \end{aligned}$$

then for any $t \geq t_0 > 0$, we have

$$\begin{aligned} F_2'(t) \leq -\kappa_1 \left(\int_0^{t_0} g(s) ds - \delta \right) \int_0^\pi \varphi_t^2(t) dx + 3\varepsilon_1 \int_0^\pi \varphi_x^2 dx + \varepsilon_1 \int_0^\pi \psi_x^2 dx \\ + \frac{c}{\varepsilon_1} g \circ \varphi_x + \varepsilon_1 \int_0^\pi u_x^2 dx - \frac{c}{\delta} g' \circ \varphi_x. \end{aligned}$$

□

Remark 5. *Similarly, the functional*

$$F_3(t) := -\kappa_2 \int_0^\pi \psi_t(t) \int_0^t h(t-s)(\psi(t)-\psi(s))\, dsdx,$$

satisfies for any $\varepsilon_1, \delta > 0$, *the estimate*

$$\begin{aligned} F_3'(t) \le -\kappa_2 \left(\int_0^{t_0} h(s)\, ds - \delta \right) \int_0^\pi \psi_t^2(t)\, dx + 3\varepsilon_1 \int_0^\pi \psi_x^2 dx + \varepsilon_1 \int_0^\pi \varphi_x^2 dx \\ + \frac{c}{\varepsilon_1} h \circ \psi_x + \varepsilon_1 \int_0^\pi u_x^2 dx - \frac{c}{\delta} h' \circ \psi_x. \end{aligned} \tag{3.11}$$

Lemma 7. *Suppose that (3.1) holds, then for any* $\varepsilon_2 > 0$, *the functional*

$$F_4(t) := b \int_0^\pi \varphi_x u_t dx + b \int_0^\pi u_x \varphi_t dx - \frac{b\rho}{\mu\kappa_1} \int_0^\pi u_t \int_0^t g(t-s)\varphi_x(s)\, dsdx$$

$$+ d \int_0^\pi \psi_x u_t dx + d \int_0^\pi u_x \psi_t dx - \frac{d\rho}{\mu\kappa_2} \int_0^\pi u_t \int_0^t h(t-s)\psi_x(s)\, dsdx,$$

satisfies along the solution of (1.1)–(1.3) the estimate

$$\begin{aligned} F_4'(t) \le -\frac{1}{2}\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2} \right) \int_0^\pi u_x^2 dx + \frac{c}{\varepsilon_2} \int_0^\pi \varphi_x^2 dx + \left(c + \frac{c}{\varepsilon_2} \right) \int_0^\pi \psi_x^2 dx \\ + \frac{bd\alpha}{2\mu\kappa_1} g \circ \varphi_x + \frac{b^2}{4\mu\kappa_1} g \circ \varphi_x + c\varepsilon_2 \int_0^\pi u_t^2 dx \\ + \frac{bd\gamma}{2\mu\kappa_2} h \circ \psi_x + \frac{d^2}{4\mu\kappa_2} h \circ \psi_x - \frac{c}{\varepsilon_2} g' \circ \varphi_x - \frac{c}{\varepsilon_2} h' \circ \psi_x. \end{aligned} \tag{3.12}$$

Proof. Differentiating $F_4(t)$ we obtain

$$F_4'(t) = \frac{b}{\rho} \int_0^\pi \varphi_x (\mu u_{xx} + b\varphi_x + d\psi_x)\, dx$$

$$+ \frac{b}{\kappa_1} \int_0^\pi u_x \left(\alpha\varphi_{xx} + \beta\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi - \int_0^t g(t-s)\varphi_{xx}(s)\, ds \right) dx$$

$$- \frac{b}{\mu\kappa_1} \int_0^\pi (\mu u_{xx} + b\varphi_x + d\psi_x) \int_0^t g(t-s)\varphi_x(s)\, dsdx$$

$$- \frac{b\rho g(0)}{\mu\kappa_1} \int_0^\pi u_t \varphi_x(t)\, dsdx - \frac{b\rho}{\mu\kappa_1} \int_0^\pi u_t \int_0^t g'(t-s)\varphi_x(s)\, dsdx$$

$$+ \frac{d}{\rho} \int_0^\pi \psi_x (\mu u_{xx} + b\varphi_x + d\psi_x)\, dx$$

$$+ \frac{d}{\kappa_2} \int_0^\pi u_x \left(\beta\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi - \int_0^t h(t-s)\psi_{xx}(s)\, ds \right) dx$$

$$-\frac{d}{\mu\kappa_2}\int_0^\pi(\mu u_{xx}+b\varphi_x+d\psi_x)\int_0^t h(t-s)\psi_x(s)\,dsdx$$
$$-\frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x(t)\,dsdx-\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\int_0^t h'(t-s)\psi_x(s)\,dsdx,$$

then

$$F_4'(t)=\left(\frac{b\mu}{\rho}-\frac{b\alpha}{\kappa_1}-\frac{d\beta}{\kappa_2}\right)\int_0^\pi\varphi_x u_{xx}dx+\left(\frac{d\mu}{\rho}-\frac{b\beta}{\kappa_1}-\frac{d\gamma}{\kappa_2}\right)\int_0^\pi\psi_x u_{xx}dx$$
$$-\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)\int_0^\pi u_x^2dx+\frac{b^2}{\rho}\int_0^\pi\varphi_x^2dx+\frac{d^2}{\rho}\int_0^\pi\psi_x^2dx+2\frac{bd}{\rho}\int_0^\pi\varphi_x\psi_xdx$$
$$-\left(\frac{b\alpha_1}{\kappa_1}+\frac{d\alpha_3}{\kappa_2}\right)\int_0^\pi u_x\varphi dx-\left(\frac{b\alpha_3}{\kappa_1}+\frac{d\alpha_2}{\kappa_2}\right)\int_0^\pi u_x\psi dx$$
$$-\frac{b^2}{\mu\kappa_1}\int_0^\pi\varphi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx-\frac{bd}{\mu\kappa_1}\int_0^\pi\psi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx$$
$$-\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x(t)\,dsdx-\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)\varphi_x(s)\,dsdx$$
$$-\frac{bd}{\mu\kappa_2}\int_0^\pi\varphi_x\int_0^t h(t-s)\psi_x(s)\,dsdx-\frac{d^2}{\mu\kappa_2}\int_0^\pi\psi_x\int_0^t h(t-s)\psi_x(s)\,dsdx$$
$$-\frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x(t)\,dsdx-\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\int_0^t h'(t-s)\psi_x(s)\,dsdx.$$

At this point we estimate the terms of the right hand side of $F_4'(t)$ term by term, taking into account (3.1). First we have

$$\begin{aligned}J_1&:=\frac{b^2}{\rho}\int_0^\pi\varphi_x^2dx+\frac{d^2}{\rho}\int_0^\pi\psi_x^2dx+\frac{2bd}{\rho}\int_0^\pi\varphi_x\psi_xdx,\\&=\frac{1}{\rho}\int_0^\pi(b\varphi_x+d\psi_x)^2dx,\\&\le\frac{2b^2}{\rho}\int_0^\pi\varphi_x^2dx+\frac{2d^2}{\rho}\int_0^\pi\psi_x^2dx.\end{aligned}$$

Secondly, for any $\xi>0$, we have

$$J_2:=-\left(\frac{b\alpha_1}{\kappa_1}+\frac{d\alpha_3}{\kappa_2}\right)\int_0^\pi u_x\varphi dx-\left(\frac{b\alpha_3}{\kappa_1}+\frac{d\alpha_2}{\kappa_2}\right)\int_0^\pi u_x\psi dx$$
$$\le\xi\int_0^\pi u_x^2dx+\frac{c}{\xi}\int_0^\pi\varphi_x^2dx+\frac{c}{\xi}\int_0^\pi\psi_x^2dx.$$

Thirdly,

$$J_3:=-\frac{b^2}{\mu\kappa_1}\int_0^\pi\varphi_x\int_0^t g(t-s)\varphi_x(s)\,dsdx=-\frac{b^2}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi\varphi_x^2dx$$

$$+\frac{b^2}{\mu\kappa_1}\int_0^\pi \varphi_x \int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx,$$

$$J_3 \le (\delta-1)\frac{b^2}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \varphi_x^2 dx + \frac{b^2}{4\delta\mu\kappa_1}\int_0^\pi\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx.$$

Taking $\delta = 1$, we get

$$J_3 \le \frac{b^2}{4\mu\kappa_1} g\circ\varphi_x.$$

Fourthly,

$$J_4 := -\frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x \int_0^t g(t-s)\varphi_x(s)\,dsdx = \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx$$
$$-\frac{bd}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \psi_x\varphi_x dx,$$

$$J_4 \le \frac{bd}{2\mu\kappa_1}\int_0^\pi \psi_x^2 dx + \frac{bd}{2\mu\kappa_1}\int_0^\pi\left(\int_0^t g(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)^2 dx$$
$$-\frac{bd}{\mu\kappa_1}\int_0^t g(s)\,ds\int_0^\pi \psi_x\varphi_x dx,$$

$$J_4 \le \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x^2 dx + \frac{bd}{2\mu\kappa_1}\left(\int_0^t g(s)\,ds\right)^2\int_0^\pi \varphi_x^2 dx + \frac{bd}{2\mu\kappa_1}\left(\int_0^t g(s)\,ds\right)(g\circ\varphi_x).$$

Therefore,

$$J_4 \le \frac{bd}{\mu\kappa_1}\int_0^\pi \psi_x^2 dx + \frac{bd\alpha^2}{2\mu\kappa_1}\int_0^\pi \varphi_x^2 dx + \frac{bd\alpha}{2\mu\kappa_1}(g\circ\varphi_x).$$

Fifthly, for any $\varepsilon_2 > 0$, we have

$$J_5 := -\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)\varphi_x(s)\,dsdx = -\frac{b\rho}{\mu\kappa_1}\int_0^t g'(s)\,ds\int_0^\pi u_t\varphi_x dx$$
$$+\frac{b\rho}{\mu\kappa_1}\int_0^\pi u_t\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))\,dsdx,$$

$$J_5 \le \frac{b\rho\varepsilon_2}{2\mu\kappa_1}\int_0^\pi u_t^2 dx + \frac{b\rho}{2\varepsilon_2\mu\kappa_1}\int_0^\pi\left(\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))\,ds\right)^2 dx$$
$$+\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx - \frac{b\rho g(t)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx,$$

$$J_5 \le \frac{b\rho\varepsilon_2}{\mu\kappa_1}\int_0^\pi u_t^2 dx + \frac{b\rho}{2\varepsilon_2\mu\kappa_1}\int_0^t g'(s)\,ds\int_0^\pi\int_0^t g'(t-s)(\varphi_x(t)-\varphi_x(s))^2\,dsdx$$
$$+\frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx + \frac{b\rho(g(t))^2}{2\varepsilon_2\mu\kappa_1}\int_0^\pi \varphi_x^2 dx,$$

then,

$$J_5 \le \frac{b\rho\varepsilon_2}{\mu\kappa_1}\int_0^\pi u_t^2dx + \frac{b\rho g(0)}{\mu\kappa_1}\int_0^\pi u_t\varphi_x dx + \frac{b\rho\left(g(0)\right)^2}{2\varepsilon_2\mu\kappa_1}\int_0^\pi \varphi_x^2dx - \frac{c}{\varepsilon_2}g'\circ\varphi_x.$$

Similarly, for the rest terms

$$J_6 := -\frac{bd}{\mu\kappa_2}\int_0^\pi \varphi_x\int_0^t h(t-s)\psi_x(s)\,dsdx$$

$$J_6 \le \frac{bd}{\mu\kappa_2}\int_0^\pi \varphi_x^2dx + \frac{bd\gamma^2}{2\mu\kappa_2}\int_0^\pi \psi_x^2dx + \frac{bd\gamma}{2\mu\kappa_2}h\circ\psi_x,$$

$$J_7 := -\frac{d^2}{\mu\kappa_2}\int_0^\pi \psi_x\int_0^t h(t-s)\psi_x(s)\,dsdx \le \frac{d^2}{4\mu\kappa_2}h\circ\psi_x,$$

and

$$J_8 := -\frac{d\rho}{\mu\kappa_2}\int_0^\pi u_t\int_0^t h'(t-s)\psi_x(s)\,dsdx,$$

then

$$J_8 \le \frac{d\rho\varepsilon_2}{\mu\kappa_2}\int_0^\pi u_t^2dx + \frac{d\rho h(0)}{\mu\kappa_2}\int_0^\pi u_t\psi_x dx + \frac{d\rho\left(h(0)\right)^2}{2\varepsilon_2\mu\kappa_2}\int_0^\pi \psi_x^2dx - \frac{c}{\varepsilon_2}h'\circ\psi_x.$$

Therefore,

$$\begin{aligned}
F_4'(t) \le& -\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}-\xi\right)\int_0^\pi u_x^2dx \\
&+\left(\frac{2b^2}{\rho}+\frac{c}{\xi}+\frac{bd\alpha^2}{2\mu\kappa_1}+\frac{c}{\varepsilon_2}+\frac{bd}{\mu\kappa_2}\right)\int_0^\pi \varphi_x^2dx \\
&+\left(\frac{2d^2}{\rho}+\frac{c}{\xi}+\frac{bd}{\mu\kappa_1}+\frac{bd\gamma^2}{2\mu\kappa_2}+\frac{c}{\varepsilon_2}\right)\int_0^\pi \psi_x^2dx \\
&+\frac{bd\alpha}{2\mu\kappa_1}g\circ\varphi_x+\frac{b^2}{4\mu\kappa_1}g\circ\varphi_x+\left(\frac{b\rho\varepsilon_2}{\mu\kappa_1}+\frac{d\rho\varepsilon_2}{\mu\kappa_2}\right)\int_0^\pi u_t^2dx \\
&+\frac{bd\gamma}{2\mu\kappa_2}h\circ\psi_x+\frac{d^2}{4\mu\kappa_2}h\circ\psi_x-\frac{c}{\varepsilon_2}g'\circ\varphi_x-\frac{c}{\varepsilon_2}h'\circ\psi_x.
\end{aligned}$$

If we choose $\xi = \frac{1}{2}\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)$ we obtain

$$\begin{aligned}
F_4'(t) \le& -\frac{1}{2}\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)\int_0^\pi u_x^2dx+\left(c+\frac{c}{\varepsilon_2}\right)\int_0^\pi \varphi_x^2dx+\left(c+\frac{c}{\varepsilon_2}\right)\int_0^\pi \psi_x^2dx \\
&+\frac{bd\alpha}{2\mu\kappa_1}g\circ\varphi_x+\frac{b^2}{4\mu\kappa_1}g\circ\varphi_x+c\varepsilon_2\int_0^\pi u_t^2dx \\
&+\frac{bd\gamma}{2\mu\kappa_2}h\circ\psi_x+\frac{d^2}{4\mu\kappa_2}h\circ\psi_x-\frac{c}{\varepsilon_2}g'\circ\varphi_x-\frac{c}{\varepsilon_2}h'\circ\psi_x.
\end{aligned}$$

□

Lemma 8. *The functional*

$$F_5(t) := -\rho \int_0^\pi u_t u dx$$

satisfies along the solution of (1.1)–(1.3) the estimate

$$F_5'(t) \leq -\rho \int_0^\pi u_t^2 dx + 2\mu \int_0^\pi u_x^2 dx + c \int_0^\pi \varphi_x^2 dx + c \int_0^\pi \psi_x^2 dx. \tag{3.13}$$

Proof. The Differentiation of $F_5'(t)$ and Young inequality lead to (3.13). □

3.1 Proof of the main result

Now we define the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2 (F_2(t) + F_3(t)) + N_4 F_4(t) + F_5(t).$$

Substituting (2.4), (3.3), (3.4), (2.5), (3.12) and (3.13) in the expression of $\mathcal{L}'(t)$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[N_2\kappa_1\left(\int_0^{t_0} g(s)\,ds - \delta\right) - N_1(\kappa_1 + C_\varepsilon)\right]\int_0^\pi \varphi_t^2 dx \\
& -\left[N_2\kappa_2\left(\int_0^{t_0} h(s)\,ds - \delta\right) - N_1(\kappa_2 + C_\varepsilon)\right]\int_0^\pi \psi_t^2 dx \\
& -[\rho - \varepsilon N_1 - c\varepsilon_2 N_4]\int_0^\pi u_t^2 dx - N_1\widehat{\alpha}_1\int_0^\pi \varphi^2 dx - N_1\widehat{\alpha}_2\int_0^\pi \psi^2 dx \\
& -\left[\frac{N_4}{2}\left(\frac{b^2}{\kappa_1} + \frac{d^2}{\kappa_2}\right) - 2\mu - 2N_2\varepsilon_1\right]\int_0^\pi u_x^2 dx \\
& -\left[N_1\frac{\widehat{l}}{2} - 4\varepsilon_1 N_2 - N_4\left(c + \frac{c}{\varepsilon_2}\right)\right]\int_0^\pi \varphi_x^2 dx - \left[N_1\frac{\widehat{k}}{2} - 4\varepsilon_1 N_2 - N_4\left(c + \frac{c}{\varepsilon_2}\right)\right]\int_0^\pi \psi_x^2 dx \\
& +\left[\frac{N}{2} - \frac{cN_4}{\varepsilon_2} - \frac{cN_2}{\delta}\right](g' \circ \varphi_x) + \left[\frac{N}{2} - \frac{cN_4}{\varepsilon_2} - \frac{cN_2}{\delta}\right](h' \circ \psi_x) \\
& +\left[N_4\frac{2bd\alpha + b^2}{4\mu\kappa_1} + cN_1 + \frac{c}{\varepsilon_1}N_2\right] g \circ \varphi_x \\
& +\left[cN_1 + N_4\frac{2bd\gamma + d^2}{4\mu\kappa_2} + \frac{c}{\varepsilon_1}N_2\right] h \circ \psi_x.
\end{aligned}$$

At this point, we choose the constants $N, N_1, N_2, N_3, N_4, \varepsilon, \varepsilon_1, \varepsilon_2$ and δ carefully.

First we choose $\delta > 0$ small such that

$$\int_0^{t_0} g(s)\,ds - \delta > 0, \text{ and } \int_0^{t_0} h(s)\,ds - \delta > 0,$$

then we choose N_4 such that $N_4\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)=6\mu$ we get

$$\frac{N_4}{2}\left(\frac{b^2}{\kappa_1}+\frac{d^2}{\kappa_2}\right)-2\mu=\mu.$$

Next we choose $\varepsilon_2>0$ such that

$$\rho-c\varepsilon_2 N_4\geq\frac{\rho}{2}.$$

After that we choose N_1 large enough such that

$$\left[N_1\frac{\widehat{l}}{2}-N_4\left(c+\frac{c}{\varepsilon_2}\right)-c\right]\geq\frac{N_1\widehat{l}}{3}$$

and

$$\left[N_1\frac{\widehat{k}}{2}-N_4\left(c+\frac{c}{\varepsilon_2}\right)-c\right]\geq\frac{N_1\widehat{k}}{3}.$$

Now we choose $\varepsilon>0$ such that

$$\frac{\rho}{2}-\varepsilon N_1>0,$$

the next step is to choose N_2 large enough such that

$$N_2\kappa_1\left(\int_0^{t_0}g\left(s\right)ds-\delta\right)-N_1\left(\kappa_1+C_\varepsilon\right)>0$$

and

$$N_2\kappa_2\left(\int_0^{t_0}h\left(s\right)ds-\delta\right)-N_1\left(\kappa_2+C_\varepsilon\right)>0$$

After that we choose $\varepsilon_1>0$ such that

$$\mu-2N_2\varepsilon_1>0,\ \frac{N_1\widehat{l}}{3}-4\varepsilon_1N_2>0\ and\ \frac{N_1\widehat{k}}{3}-4\varepsilon_1N_2>0.$$

Thus, there exists, $\varpi>0$ and $c_0>0$, such that

$$\mathcal{L}'\left(t\right)\leq-\varpi\int_0^\pi\left(\varphi_t^2+\psi_t^2+u_t^2+\varphi^2+\psi^2+u_x^2+\varphi_x^2+\psi_x^2\right)dx+c\left(g\circ\varphi_x+h\circ\psi_x\right)$$

$$+\left[\frac{N}{2}-c_0\right]\left(g'\circ\varphi_x+h'\circ\psi_x\right).$$

Now we let $\mathcal{L}\left(t\right)=N_1F_1\left(t\right)+N_2\left(F_2\left(t\right)+F_3\left(t\right)\right)+N_4F_4\left(t\right)+F_5\left(t\right)$, then

$$\left|\mathcal{L}\left(t\right)\right|\leq N_1\left(\kappa_1\int_0^\pi\left|\varphi\varphi_t\right|dx+\kappa_2\int_0^\pi\left|\psi_t\psi\right|dx+\frac{\rho}{\mu}\int_0^\pi\left|b\varphi+d\psi\right|\int_0^x\left|u_t\left(y\right)\right|dydx\right)$$

$$+N_2\left(\kappa_1\int_0^\pi\left|\varphi_t\left(t\right)\int_0^t g\left(t-s\right)\left(\varphi\left(t\right)-\varphi\left(s\right)\right)ds\right|dx\right)$$

$$+N_2\left(\kappa_2\int_0^\pi\left|\psi_t(t)\int_0^t h(t-s)(\psi(t)-\psi(s))\,ds\right|dx\right)$$
$$+N_4\left(|b|\int_0^\pi|\varphi_x u_t|\,dx+|b|\int_0^\pi|u_x\varphi_t|\,dx+\frac{|b|\rho}{\mu\kappa_1}\int_0^\pi\left|u_t\int_0^t g(t-s)\varphi_x(s)\,ds\right|dx\right)$$
$$+N_4\left(|d|\int_0^\pi|\psi_x u_t|\,dx+|d|\int_0^\pi|u_x\psi_t|\,dx+\frac{|d|\rho}{\mu\kappa_2}\int_0^\pi\left|u_t\int_0^t h(t-s)\psi_x(s)\,ds\right|dx\right)$$
$$+\rho\int_0^\pi|u_t u|\,dx.$$

Exploiting Young's, Cauchy–Schwarz' and Poincaré's inequalities and using Remark 3, we arrive at

$$|\mathcal{L}(t)|\le c\int_0^\pi\left[u_t^2+\varphi_t^2+\psi_t^2+u_x^2+\varphi^2+\psi^2+\varphi_x^2+\psi_x^2\right]dx+c\left[g\circ\varphi_x+h\circ\psi_x\right]\le cE(t).$$

Therefore,

$$|\mathcal{L}(t)-NE(t)|\le cE(t).$$

Consequently,

$$(N-c)E(t)\le\mathcal{L}(t)\le(N+c)E(t).$$

Now, we choose N large enough such that

$$N>c,\ \text{and}\ \frac{N}{2}-c_0>0.$$

Thus, there exists $\lambda,c,c_1,c_2>0$, such that

$$\mathcal{L}'(t)\le-\lambda E(t)+cg\circ\varphi_x+ch\circ\psi_x \tag{3.14}$$

and

$$c_1E(t)\le\mathcal{L}(t)\le c_2E(t).$$

Let $\chi(t)=\min\{\xi(t),\eta(t)\}$,then, multiplying (3.14) by $\chi(t)$ we obtain

$$\begin{aligned}\chi(t)\mathcal{L}'(t)&\le-\lambda\chi(t)E(t)+c\chi(t)g\circ\varphi_x+c\chi(t)h\circ\psi_x,\\&\le-\lambda\chi(t)E(t)+c\xi(t)g\circ\varphi_x+c\eta(t)h\circ\psi_x,\\&\le-\lambda\chi(t)E(t)-cg'\circ\varphi_x-ch'\circ\psi_x,\\&\le-\lambda\chi(t)E(t)-cE'(t).\end{aligned}$$

That is

$$\chi(t)\mathcal{L}'(t)+cE'(t)\le-\lambda\chi(t)E(t).$$

Recalling that χ is non increasing and $\mathcal{L}(t)\ge0$, we conclude that

$$\frac{d}{dt}\left(\chi(t)\mathcal{L}(t)+cE(t)\right)\le-\lambda\chi(t)E(t),\ a.e.\ t\ge t_0.$$

Let $\mathcal{F}(t)=\chi(t)\mathcal{L}(t)+cE(t)$, then $\mathcal{F}(t)\sim E(t)$ and there exists a positive constant ω such that

$$\mathcal{F}'(t)\le-\omega\chi(t)\mathcal{F}(t)\quad\forall t\ge t_0.$$

An integration over (t_0, t) gives

$$\mathcal{F}(t) \leq \mathcal{F}(t_0)\, e^{-\omega \displaystyle\int_{t_0}^{t} \chi(s)\, ds}, \ \forall\, t \geq t_0.$$

Using again the fact that $\mathcal{F}(t) \sim E(t)$, we deduce that

$$E(t) \leq \sigma e^{-\omega \displaystyle\int_{t_0}^{t} \chi(s)\, ds}, \ \forall t \geq t_0,$$

for a positive constant σ, which completes the proof of Theorem 2.

Remark 6. *Using the continuity and the boundedness of* E *and* χ, *the estimate (3.2) remains valid for* $t \in [0, t_0]$.

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