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Saddle point formulations for a class of nonlinear boundary value problems

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Abstract

A class of boundary value problems governed by two subdifferential inclusions is considered. In the present paper we study the weak solvability by means of the saddle point theory. The boundary value problems under consideration are nonlinear problems arising from electricity and elasticity.

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1 Introduction

In the present work we address the following class of boundary value problems.

Problem 1. *Find* $u : \bar{\Omega} \to \mathbb{R}$ *and* $\boldsymbol{D} : \bar{\Omega} \to \mathbb{R}^N$ *such that*

$$\begin{aligned}
div\, \boldsymbol{D}(\boldsymbol{x}) &= f_0(\boldsymbol{x}) && in\ \ \Omega,\\
-\boldsymbol{D}(\boldsymbol{x}) &\in \partial\varphi(\nabla u(\boldsymbol{x})) + \beta\, \nabla\, u(\boldsymbol{x}) && in\ \ \Omega,\\
u(\boldsymbol{x}) &= 0 && on\ \ \Gamma_1,\\
\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && on\ \ \Gamma_2,\\
\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) &\in \partial\psi(u(\boldsymbol{x})) && on\ \ \Gamma_3.
\end{aligned}$$

Herein $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with smooth boundary Γ, partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas(\Gamma_i) > 0$, $i \in \{1, 2, 3\}$. As usual, $\boldsymbol{\nu}$ is the outward unit normal vector defined a.e. on the boundary Γ.

Given the functionals $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$, and β a positive constant, Problem 1 is a boundary value problem governed by the subdifferential inclusions

$$-\boldsymbol{D}(\boldsymbol{x}) \in \partial\varphi(\nabla u(\boldsymbol{x})) + \beta\, \nabla\, u(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega, \tag{1.1}$$

and

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \in \partial\psi(u(\boldsymbol{x})) \quad \boldsymbol{x} \in \Gamma_3, \tag{1.2}$$

where, according to the definition of the subdifferential,

$$\partial\varphi(\nabla u(\boldsymbol{x})) = \{\boldsymbol{\xi} \in \mathbb{R}^N \,|\, \varphi(\boldsymbol{v}) - \varphi(\nabla u(\boldsymbol{x})) \geq \boldsymbol{\xi} \cdot (\boldsymbol{v} - \nabla u(\boldsymbol{x})) \ \text{ for all } \boldsymbol{v} \in \mathbb{R}^N\} \quad (\boldsymbol{x} \in \Omega),$$

and

$$\partial\psi(u(\boldsymbol{x})) = \{\xi \in \mathbb{R} \,|\, \psi(r) - \psi(u(\boldsymbol{x})) \geq \xi(r - u(\boldsymbol{x})) \quad \text{for all } r \in \mathbb{R}\} \quad (\boldsymbol{x} \in \Gamma_3),$$

" $\cdot$ " denoting the inner product in $\mathbb{R}^N$.

In the present paper φ is a bounded seminorm, i.e., $\varphi : \mathbb{R}^N \to [0, \infty)$ is a seminorm such that

$$\exists\, M_\varphi > 0 : \quad \varphi(\boldsymbol{v}) \leq M_\varphi \, \|\boldsymbol{v}\| \ \text{ for all } \ \boldsymbol{v} \in \mathbb{R}^N.$$

Here and everywhere below, by $\|\cdot\|$ we denote the Euclidean norm on $\mathbb{R}^N$. Concerning the functional ψ, we will consider two particular cases of proper, convex and lower semicontinuous functionals: firstly, ψ will be a bounded seminorm; secondly, ψ will be the indicator function of a nonempty, convex, closed subset $K \subseteq X$.

Problem 1 has a physical significance in the electricity theory. The scalar function $u : \bar{\Omega} \to \mathbb{R}$ denotes the electrostatic potential and the vectorial function $\boldsymbol{D} : \bar{\Omega} \to \mathbb{R}$ denotes the electric field. In this context, (1.1) is a generalized electric constitutive law and (1.2) is a generalized electrically contact condition. If $\varphi \equiv 0$ and $\Gamma_3 \equiv 0$ we are led to the classical boundary value problem in the electrostatic theory,

$$\begin{array}{rcll} div\, \boldsymbol{D}(\boldsymbol{x}) & = & f_0(\boldsymbol{x}) & \text{in } \ \Omega, \\ \boldsymbol{D}(\boldsymbol{x}) & = & -\beta\, \nabla\, u(\boldsymbol{x}) & \text{in } \ \Omega, \\ u(\boldsymbol{x}) & = & 0 & \text{on } \ \Gamma_1, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) & = & f_2(\boldsymbol{x}) & \text{on } \ \Gamma_2; \end{array}$$

see, e.g., Section 26, Chapter 8 in [9].

Problem 1 has a physical significance in the elasticity theory, too. To give an example, let us consider $N = 2$, $\varphi \equiv 0$ and $\psi : \mathbb{R} \to [0, \infty)$, $\psi(r) = g|r|$, where g is a positive constant. Here and everywhere below $|\cdot|$ denotes the absolute value of a real number. In this case, it can be proved that (1.2) is equivalent to

$$\left|\beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x})\right| \leq g, \quad \beta \frac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) = -g \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \ \text{ if } u(\boldsymbol{x}) \neq 0 \ \text{ on } \Gamma_3$$

which is the Tresca's friction law. Thus, we are led to the following boundary value problem.

$$\begin{array}{rcll} -\beta \triangle\, u(\boldsymbol{x}) & = & f_0(\boldsymbol{x}) & \text{in } \ \Omega, \\ u(\boldsymbol{x}) & = & 0 & \text{on } \ \Gamma_1, \\ -\beta \dfrac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) & = & f_2(\boldsymbol{x}) & \text{on } \ \Gamma_2, \\ \left|\beta \dfrac{\partial\, u}{\partial\, \nu}(\boldsymbol{x})\right| & \leq & g, & \\ \beta \dfrac{\partial\, u}{\partial\, \nu}(\boldsymbol{x}) & = & -g \dfrac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \quad \text{if } u(\boldsymbol{x}) \neq 0, & \text{on } \ \Gamma_3. \end{array}$$

This is an antiplane model, the unknown $u : \bar{\Omega} \to \mathbb{R}$ being the third component of the displacement field; see, e.g., Chapter 9 in [16] for a mathematical study in terms of variational inequalities of second kind. Recall that $\frac{\partial u}{\partial \nu} = \nabla u \cdot \boldsymbol{\nu}$.

In the present work we deliver a weak formulation of Problem 1 as a saddle point problem on Hilbert spaces. Then, we study the existence, the uniqueness and the stability of the weak solution.

The present paper brings a contribution to the qualitative analysis of Problem 1 providing a convenient variational formulation for the numerical analysis. The approach via saddle points allows to efficiently approximate the weak solutions of contact models; see, e.g., [8] for a numerical study based on a variational formulation as a saddle point problem. Basic elements in the saddle point theory can be found in, e.g., [5, 7].

Below we mention an auxiliary abstract result.

Let $(\mathcal{X}, (\ ,\ )_{\mathcal{X}}, \|\ \|_{\mathcal{X}})$, $(\mathcal{Y}, (\ ,\ )_{\mathcal{Y}}, \|\ \|_{\mathcal{Y}})$ be two Hilbert spaces. We consider the following mixed variational problem.

Problem 2. *Given $f \in \mathcal{X}$, find $u \in \mathcal{X}$ and $\lambda \in \Lambda \subseteq \mathcal{Y}$ such that*

$$\hat{a}(u,v) + \hat{b}(v,\lambda) = (f,v)_{\mathcal{X}} \qquad \text{for all } v \in \mathcal{X} \tag{1.3}$$

$$\hat{b}(u,\mu-\lambda) \le 0 \qquad \text{for all } \mu \in \Lambda. \tag{1.4}$$

Assumption 1. *Let $\hat{a} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a symmetric, bilinear form. Moreover,*
(i) there exists $M_{\hat{a}} > 0 : |\hat{a}(u,v)| \le M_{\hat{a}}\, \|u\|_{\mathcal{X}}\, \|v\|_{\mathcal{X}}$ for all $u, v \in \mathcal{X}$;
(ii) there exists $m_{\hat{a}} > 0 : \hat{a}(v,v) \ge m_{\hat{a}}\, \|v\|^2_{\mathcal{X}}$ for all $v \in \mathcal{X}$.

Assumption 2. *Let $\hat{b} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a bilinear form. Moreover,*
(i) there exists $M_{\hat{b}} > 0 : |\hat{b}(v,\mu)| \le M_{\hat{b}}\, \|v\|_{\mathcal{X}}\, \|\mu\|_{\mathcal{Y}}$ for all $v \in \mathcal{X}$, $\mu \in \mathcal{Y}$;
(ii) there exists $\alpha > 0 : \inf_{\mu \in \mathcal{Y}, \mu \ne 0_{\mathcal{Y}}} \sup_{v \in \mathcal{X}, v \ne 0_{\mathcal{X}}} \dfrac{\hat{b}(v,\mu)}{\|v\|_{\mathcal{X}}\, \|\mu\|_{\mathcal{Y}}} \ge \alpha.$

Assumption 3. *Let Λ be a closed, convex subset of $\mathcal{Y}$ that contains $0_{\mathcal{Y}}$.*

Theorem 1. *Assumptions 1-3 hold true. Then Problem 2 has a unique solution $(u,\lambda) \in \mathcal{X} \times \Lambda$. Moreover, there exists $c > 0$ such that*

$$\|u_1 - u_2\|_{\mathcal{X}} + \|\lambda_1 - \lambda_2\|_{\mathcal{Y}} \le c\, \|f_1 - f_2\|_{\mathcal{X}},$$

where $(u_1,\lambda_1), (u_2,\lambda_2) \in \mathcal{X} \times \Lambda$ are the solutions of Problem 2 corresponding to the data $f_1 \in \mathcal{X}$ and $f_2 \in \mathcal{X}$, respectively.

The proof of this theorem is based on the saddle point theory, see, e.g., [5, 7] or [2, 10, 11] for even more general results.

Problem 2 is a saddle point problem since it is equivalent to the following problem

$$\mathcal{L}(u,\mu) \le \mathcal{L}(u,\lambda) \le \mathcal{L}(v,\lambda) \text{ for all } v \in \mathcal{X},\ \mu \in \Lambda,$$

where $\mathcal{L} : \mathcal{X} \times \Lambda \to \mathbb{R}$ is the associated functional

$$\mathcal{L}(v,\mu) = \frac{1}{2}\, \hat{a}(v,v) + \hat{b}(v,\mu) - (f,v)_{\mathcal{X}} \quad \text{for all } v \in \mathcal{X},\ \mu \in \Lambda.$$

We recall here the definition of the saddle point for the convenience of the readers.

Definition 1. *Let A and B be two nonempty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if*

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda) \text{ for all } v \in A,\ \mu \in B.$$

Problem 1 will be studied in a functional framework governed by Hilbert spaces. For reference, see, e.g., [1, 3, 4, 13].

The present paper is organized as follows. In Section 2 and Section 3 we describe the functional setting and we deliver weak formulations of Problem 1 as saddle point problems. In Section 4 we discuss the existence, the uniqueness and the stability of the solutions of the weak formulations.

2 Some notation and preliminaries

Herein, the main goal is to describe the functional setting we need. The content of this section is rather standard. However, we find it useful to easily follow the present paper. Let us consider the space

$$X = \{v \,:\, v \in H^1(\Omega), \gamma\, v = 0 \text{ a.e. on } \Gamma_1\},$$

where $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the trace operator. As it is known, the trace operator γ is a linear, continuous and compact operator; see, e.g., [12] for a trace theorem.

The space X is a closed subspace of the Hilbert space $H^1(\Omega)$. Indeed, let $(v_n)_n$ be a sequence of X such that $v_n \to v$ in $H^1(\Omega)$ as $n \to \infty$. In order to prove that $v \in X$, it is enough to prove that $\|\gamma\, v\|_{L^2(\Gamma_1)} = 0$. Indeed,

$$0 \leq \|\gamma\, v\|_{L^2(\Gamma_1)} = \|\gamma\, v - \gamma\, v_n + \gamma\, v_n\|_{L^2(\Gamma_1)} \leq \|\gamma\, v - \gamma\, v_n\|_{L^2(\Gamma_1)} + \|\gamma\, v_n\|_{L^2(\Gamma_1)}.$$

Therefore,

$$0 \leq \|\gamma\, v\|_{L^2(\Gamma_1)} \leq \|\gamma\, v - \gamma\, v_n\|_{L^2(\Gamma)} \leq c_{tr}\, \|v - v_n\|_{H^1(\Omega)},$$

where $c_{tr} > 0$ is a constant in the trace theorem. To get the conclusion, we have to pass to the limit as $n \to \infty$.

Since each closed subspace of a Hilbert space is a Hilbert space too, see, e.g., [4], page 127, then $(X, (\ ,\ )_{H^1(\Omega)}, \|\ \|_{H^1(\Omega)})$ is a Hilbert space.

But, as it is known, the space X can be endowed with the following particular inner product,

$$(u, v)_X = \int_\Omega \nabla\, u(\boldsymbol{x}) \cdot \nabla\, v(\boldsymbol{x})\, dx \quad \text{for all } u, v \in X,$$

the associated norm being

$$\|v\|_X = \|\nabla\, v\|_{L^2(\Omega)^N} \quad \text{for all } \ v \in X.$$

Recall that $(X, (\ ,\ )_X, \|\ \|_X)$ is a Hilbert space too, due to the inequality of Poincaré's type

$$\|u\|_{L^2(\Omega)} \leq c_p\, \|\nabla\, u\|_{L^2(\Omega)^N} \quad \text{for all } \ u \in X,$$

where $c_p = c_p(\Omega, \Gamma_1) > 0$, see, e.g., [14].

We note that, $\|\ \|_X$ and $\|\ \|_{H^1(\Omega)}$ are equivalent norms.

In the present paper, the dual of the space X will be denoted by X' and $(\ ,\ )_{X',X}$ will denote the duality pairing.

3 A weak formulation

In this section we deliver a weak formulation consisting of a variational system of type (1.3)-(1.4).

Let us make the following assumption.

Assumption 4. *Let* $f_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$ *and* $\beta > 0$.

Let u and $\boldsymbol{D}$ be smooth enough functions which verify Problem 1. By a Green formula for Sobolev spaces, see, e.g., [15], page 90, we get, for all $v \in X$,

$$\int_\Omega \beta \, \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \, \gamma \, v(\boldsymbol{x}) \, d\Gamma + \int_{\Gamma_2} f_2(\boldsymbol{x}) \, \gamma \, v(\boldsymbol{x}) \, d\Gamma$$
$$- \int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \, dx = \int_\Omega f_0(\boldsymbol{x}) \, v(\boldsymbol{x}) \, dx.$$

Let $a : X \times X \to \mathbb{R}$ be the bilinear form

$$a(u, v) = \int_\Omega \beta \, \nabla \, u(\boldsymbol{x}) \cdot \nabla \, v(\boldsymbol{x}) \, dx.$$

By using Riesz representation theorem, we define $f \in X$ by means of the following relation

$$(f, v)_X = \int_\Omega f_0 \, (\boldsymbol{x}) \, v(\boldsymbol{x}) \, dx - \int_{\Gamma_2} f_2 \, (\boldsymbol{x}) \gamma v \, (\boldsymbol{x}) d\Gamma \ \text{ for all } \ v \in X.$$

Let us introduce another bilinear form $b : X \times X' \to \mathbb{R}$ as follows:

$$b(v, \mu) = (\mu, v)_{X', X}.$$

We also consider $\lambda \in X'$ such that for all $z \in X$ we have,

$$(\lambda, z)_{X', X} = - \int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla \, z(\boldsymbol{x}) \, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \, \gamma z(\boldsymbol{x}) \, d\Gamma. \tag{3.1}$$

Therefore, we can write,

$$a(u, v) + b(v, \lambda) = (f, \, v)_X \ \text{ for all } \ v \in X. \tag{3.2}$$

Below we admit the following assumption.

Assumption 5. $\varphi : \mathbb{R}^N \to [0, \infty)$ *is a seminorm such that*

$$\textit{there exists } M_\varphi > 0 \, : \, \varphi(\boldsymbol{w}) \leq M_\varphi \, \|\boldsymbol{w}\| \ \textit{ for all } \ \boldsymbol{w} \in \mathbb{R}^N.$$

The next lemma will be helpful.

Lemma 1. *The following relations hold true.*

$$- \int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x}) dx = \int_\Omega \varphi(\nabla u(\boldsymbol{x})) dx, \tag{3.3}$$

$$- \int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \, \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \, dx \leq \int_\Omega \varphi(\nabla v(\boldsymbol{x})) \, dx \textit{ for all } v \in X. \tag{3.4}$$

Proof. Let $\boldsymbol{x} \in \Omega$. Due to relation (1.1),

$$\varphi(\boldsymbol{w}) - \varphi(\nabla u(\boldsymbol{x})) \geq -(\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot (\boldsymbol{w} - \nabla u(\boldsymbol{x})) \quad \text{for all } \boldsymbol{w} \in \mathbb{R}^N. \tag{3.5}$$

Setting $\boldsymbol{w} = \boldsymbol{0}$ in (3.5),

$$-\varphi(\nabla u(\boldsymbol{x})) \geq (\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x}).$$

Hence, after integration, it results

$$-\int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x}))\, dx \geq \int_\Omega \varphi(\nabla u(\boldsymbol{x}))\, dx. \tag{3.6}$$

Setting now $\boldsymbol{w} = 2\nabla u(\boldsymbol{x})$ in (3.5), since φ is a seminorm, we obtain

$$\varphi(\nabla u(\boldsymbol{x})) \geq -(\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x}).$$

Thus,

$$-\int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x}) dx \leq \int_\Omega \varphi(\nabla u(\boldsymbol{x})) dx. \tag{3.7}$$

Therefore, by relations (3.6) and (3.7) we obtain (3.3).

Let $v \in X$ and $\boldsymbol{x} \in \Omega$. We set $\boldsymbol{w} = \nabla v(\boldsymbol{x}) + \nabla u(\boldsymbol{x})$ in (3.5). Since φ is a seminorm, we have

$$\varphi(\nabla v(\boldsymbol{x})) \geq -(\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}).$$

After integration, we get (3.4). □

In order to complete the variational formulation, we consider two cases.

3.1 First case: ψ is a bounded seminorm

In this subsection we make the following assumption on ψ.

Assumption 6. $\psi : \mathbb{R} \to [0, \infty)$ *is a seminorm such that*

$$\text{there exists } M_\psi > 0 \,:\, \psi(s) \leq M_\psi \,|s| \ \text{ for all } \ s \in \mathbb{R}.$$

Let us introduce the following set:

$$\Lambda_1 = \{\mu \in X' : (\mu,\, v)_{X',X} \leq \int_\Omega \varphi(\nabla v(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(\gamma v(\boldsymbol{x}))\, d\Gamma \ v \in X\}. \tag{3.8}$$

Remark 1. *The set Λ_1 defined in (3.8) is bounded. Indeed, let $\mu \in \Lambda_1$.*

$$\begin{aligned} \frac{(\mu, v)_{X',X}}{\|v\|_X} &\leq \frac{\int_\Omega \varphi(\nabla\, v(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(\gamma v(\boldsymbol{x}))\, d\Gamma}{\|v\|_X} \\ &\leq \frac{M_\varphi \sqrt{meas(\Omega)}\, \|\nabla\, v\|_{L^2(\Omega)^N} + M_\psi \sqrt{meas(\Gamma_3)}\, \|\gamma v\|_{L^2(\Gamma)}}{\|v\|_X}. \end{aligned}$$

Thus,

$$\|\mu\|_{X'} \leq M_\varphi \sqrt{meas(\Omega)} + M_\psi \sqrt{meas(\Gamma_3)}\, c_{tr} \sqrt{c_p^2 + 1},$$

where c_p and c_{tr} are the Poincaré's constant and a trace constant, respectively; see Section 2.

We can prove that λ defined in (3.1) is an element of Λ_1. Indeed, by taking into account (1.2), we have on Γ_3,

$$\psi(r) - \psi(u(\boldsymbol{x})) \geq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})(r - u(\boldsymbol{x})) \quad \text{for all } r \in \mathbb{R}. \tag{3.9}$$

Let $v \in X$ and $\boldsymbol{x} \in \Gamma_3$. We set $r = \gamma v(\boldsymbol{x}) + u(\boldsymbol{x})$ in (3.9) to obtain,

$$\psi(\gamma v(\boldsymbol{x})) \geq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, \gamma v(\boldsymbol{x}).$$

Thus, by integration on Γ_3, we get

$$\int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \nu(\boldsymbol{x}) \gamma v(\boldsymbol{x})\, d\Gamma \leq \int_{\Gamma_3} \psi(\gamma v(\boldsymbol{x}))\, d\Gamma. \tag{3.10}$$

By (3.1), (3.4) and (3.10), we conclude that $\lambda \in \Lambda_1$.

Let us proceed by proving that

$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda_1. \tag{3.11}$$

To start, we check if

$$b(u, \lambda) = \int_\Omega \varphi(\nabla\, u(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(\gamma u(\boldsymbol{x}))\, d\Gamma. \tag{3.12}$$

Indeed,

$$b(u, \lambda) = -\int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta \nabla u(\boldsymbol{x})) \cdot \nabla u(\boldsymbol{x})\, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \nu(\boldsymbol{x}) \gamma u(\boldsymbol{x})\, d\Gamma.$$

On the other hand, by taking $r = 0$ in (3.9), we obtain on Γ_3

$$-\psi(u(\boldsymbol{x})) \geq -\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, u(\boldsymbol{x}),$$

and from this,

$$\int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \gamma\, u(\boldsymbol{x})\, d\Gamma \geq \int_{\Gamma_3} \psi(\gamma\, u(\boldsymbol{x}))\, d\Gamma. \tag{3.13}$$

Let $\boldsymbol{x} \in \Gamma_3$. Setting now $r = 2u(\boldsymbol{x})$ in (3.9), we have

$$\psi(u(\boldsymbol{x})) \geq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, u(\boldsymbol{x}).$$

Therefore,

$$\int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, \gamma u(\boldsymbol{x})\, d\Gamma \leq \int_{\Gamma_3} \psi(\gamma\, u(\boldsymbol{x})) d\Gamma. \tag{3.14}$$

By (3.13) and (3.14) we obtain

$$\int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \gamma u(\boldsymbol{x})\, d\Gamma = \int_{\Gamma_3} \psi(\gamma u(\boldsymbol{x}))\, d\Gamma. \tag{3.15}$$

Therefore, by (3.3) and (3.15), we get (3.12).

Due to the definition of Λ_1, we deduce that, for all $\mu \in \Lambda_1$,

$$b(u, \mu) \leq \int_\Omega \varphi(\nabla u(\boldsymbol{x}))\, dx + \int_{\Gamma_3} \psi(\gamma u(\boldsymbol{x}))\, d\Gamma. \tag{3.16}$$

Combining now (3.12) and (3.16), we get (3.11).

We obtain the following weak formulation for Problem 1.

Problem 3. *Find $u \in X$ and $\lambda \in \Lambda_1 \subseteq X'$ such that*

$$a(u,v) + b(v,\lambda) = (f,\, v)_X \quad \textit{for all } \; v \in X,$$

$$b(u, \mu - \lambda) \le 0 \qquad \textit{for all } \mu \in \Lambda_1.$$

3.2 Second case: ψ is an indicator function

Let us consider in this subsection the following nonempty, closed, convex subset of $\mathbb{R}$:

$$K = \{r \in \mathbb{R},\; r \le 0\}.$$

Therefore, $I_K : \mathbb{R} \to \overline{\mathbb{R}}$,

$$I_K(r) = \begin{cases} 0, & r \in K; \\ \infty, & r \notin K \end{cases}$$

is a proper, convex and lower semicontinuous function.

We make the following assumption.

Assumption 7. $\psi = I_K$.

To proceed, we introduce the following set:

$$\Lambda_2 = \{\mu \in X' : \; (\mu, v)_{X',X} \le \int_\Omega \varphi(\nabla v(\boldsymbol{x}))\, dx \; \text{ for all } \; v \in \mathcal{K}\}, \tag{3.17}$$

where

$$\mathcal{K} = \{v \in X,\; \gamma\, v \le 0 \; \text{ a.e. on } \; \Gamma_3\}. \tag{3.18}$$

Recall that

$$(\lambda, z)_{X',X} = -\int_\Omega (\boldsymbol{D}(\boldsymbol{x}) + \beta\, \nabla u(\boldsymbol{x})) \cdot \nabla\, z(\boldsymbol{x})\, dx + \int_{\Gamma_3} \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, \gamma z(\boldsymbol{x})\, d\Gamma$$

and

$$a(u,v) + b(v,\lambda) = (f,\, v)_X \quad \text{for all } \; v \in X.$$

The next targets are:

- $\lambda \in \Lambda_2$;
- $b(u, \mu - \lambda) \le 0$ for all $\mu \in \Lambda_2$.

To do this, we observe that, in the case $\psi = I_K$, the boundary condition (1.2) has a very convenient equivalent writing. For the convenience of the reader, even the result is rather standard, we describe this equivalent writing in the next lemma.

Lemma 2. *The following equivalence holds true a.e. on Γ_3:*

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \in \partial\, I_K(u(\boldsymbol{x})) \Leftrightarrow \begin{cases} u(\boldsymbol{x}) \le 0, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \ge 0, \\ \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, u(\boldsymbol{x}) = 0. \end{cases}$$

Proof. Let $\boldsymbol{x} \in \Gamma_3$. According to the definition of the subdifferential of a convex functional,

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \in \partial I_K(u(\boldsymbol{x})) \Leftrightarrow \tag{3.19}$$
$$I_K(s) - I_K(u(\boldsymbol{x})) \geq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})(s - u(\boldsymbol{x})) \text{ for all } s \in \mathbb{R}.$$

Let us prove the first implication in Lemma 2. To do this, we take $s = 0 \in K$ in (3.19). It results that,

$$I_K(u(\boldsymbol{x})) \leq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, u(\boldsymbol{x}). \tag{3.20}$$

If $u(\boldsymbol{x}) \notin K$ we get

$$\infty \leq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) u(\boldsymbol{x})$$

which is impossible. Thus,

$$u(\boldsymbol{x}) \in K. \tag{3.21}$$

Hence,

$$u(\boldsymbol{x}) \leq 0. \tag{3.22}$$

Keeping in mind (3.20), since $u(\boldsymbol{x}) \in K$, it results that

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) u(\boldsymbol{x}) \geq 0. \tag{3.23}$$

Setting now $s = 2\, u(\boldsymbol{x})$ in (3.19) we obtain

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) u(\boldsymbol{x}) \leq 0. \tag{3.24}$$

By (3.23) and (3.24) we get

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) u(\boldsymbol{x}) = 0. \tag{3.25}$$

Taking $s = r + u(\boldsymbol{x})$ with $r < 0$ in (3.19) and keeping in mind (3.22), we obtain

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \geq 0. \tag{3.26}$$

Conversely, we have to check if (3.19) takes place. Indeed, if $s \notin K$, then

$$\infty - I_K(u(\boldsymbol{x})) \geq \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, (s - u(\boldsymbol{x}))$$

which is clearly true. Otherwise, if $s \in K$,

$$\boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, (s - u(\boldsymbol{x})) = \boldsymbol{D}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})\, s \leq 0 = I_K(s) - I_K(u(\boldsymbol{x})).$$

□

Let us prove that $\lambda \in \Lambda_2$. Indeed, let $v \in \mathcal{K}$. Taking into account the definition of λ in (3.1) and using (3.4), (3.26), we can conclude that $\lambda \in \Lambda_2$.

It remains to verify that

$$b(u, \mu - \lambda) \leq 0 \text{ for all } \mu \in \Lambda_2. \tag{3.27}$$

Indeed, keeping in mind (3.1), due to (3.25) and (3.3), we get

$$b(u,\lambda) = (\lambda, u)_{X',X} = \int_\Omega \varphi(\nabla u(\boldsymbol{x}))\, dx.$$

On the other hand,

$$b(u,\mu) \leq \int_\Omega \varphi(\nabla u(\boldsymbol{x}))\, dx \quad \text{for all } \mu \in \Lambda_2.$$

Hence, we immediately get (3.27).

Thus, in this second case, the weak formulation for Problem 1 is given as follows.

Problem 4. *Find $u \in X$ and $\lambda \in \Lambda_2 \subseteq X'$*

$$a(u,v) + b(v,\lambda) = (f,\, v)_X \quad \textit{for all } v \in X,$$

$$b(u,\mu-\lambda) \leq 0 \quad \textit{for all } \mu \in \Lambda_2.$$

Remark 2. *The set Λ_2 defined in (3.17) is unbounded. Indeed, there exists a sequence $(\mu_n)_n \subset \Lambda_2$ such that $\|\mu_n\|_X \to \infty$ as $n \to \infty$. To give an example, let $\mu_n = n\,\mu_0$ where $\mu_0 \in \Lambda_2$ is defined as follows*

$$(\mu_0, v)_{X',X} = \int_{\Gamma_3} \gamma v(\boldsymbol{x})\, d\Gamma \;\; \textit{for all } v \in X.$$

Since, for all $v \in \mathcal{K}$,

$$\int_{\Gamma_3} \gamma v(\boldsymbol{x})\, d\Gamma \leq 0 \leq \int_\Omega \varphi(\nabla v(\boldsymbol{x}))\, dx$$

we deduce that $\mu_0 \in \Lambda_2$ and $\mu_n = n\mu_0 \in \Lambda_2$ for all $n \in \mathbb{N}$. Moreover, we observe that

$$\|\mu_n\|_{X'} = n\|\,\mu_0\|_{X'}$$

which goes to ∞ when $n \to \infty$.

Remark 3. *By means of a standard approach, we can give a variational formulation of Problem 1, like a variational inequality of the second kind, as follows: find $u_0 \in \mathcal{K}$ such that*

$$(Au_0, v-u_0)_{X',X} + J(v) - J(u_0) \geq (f, v-u_0)_{X',X} \;\; \textit{for all } v \in \mathcal{K}, \tag{3.28}$$

where

$$A : X \to X \;\; (Au,v)_{X',X} = a(u,v) = \int_\Omega \beta\, \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx,$$

$$J : X \to \mathbb{R}, \;\; J(v) = \int_\Omega \varphi(\nabla v(\boldsymbol{x}))\, dx,$$

$$(f,v)_{X',X} = \int_\Omega f_0\,(\boldsymbol{x})\, v(\boldsymbol{x})\, dx - \int_{\Gamma_2} f_2\,(\boldsymbol{x})\gamma v\,(\boldsymbol{x}) d\Gamma \;\; \textit{for all } v \in X.$$

It can be easily proved that the operator A is linear, strongly monotone and Lipschitz continuous, and the functional J is proper, convex and lower semicontinuous.

By applying the theory of the variational inequalities of second kind (see, e.g., Theorem 2.8 in [15]) it can be proved that there exists a unique $u_0 \in \mathcal{K}$ such that (3.28) holds true.

4 Well-posedness results

In this section we discuss the existence, the uniqueness and the stability of the solution of Problem 3 and Problem 4, respectively, based on the abstract result, Theorem 1.

Theorem 2. *Assumptions 4-6 hold true. Then, Problem 3 has a unique solution*

$$(u, \lambda) \in X \times \Lambda_1.$$

Moreover, there exists $k_1 > 0$ such that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_X \leq k_1 \left(\|f_0^1 - f_0^2\|_{L^2(\Omega)} + \|f_2^1 - f_2^2\|_{L^2(\Gamma_2)} \right), \tag{4.1}$$

where $(u_1, \lambda_1), (u_2, \lambda_2) \in X \times \Lambda_1$ are the solutions of Problem 3 corresponding to the data $(f_0^1, f_2^1), (f_0^2, f_2^2) \in L^2(\Omega) \times L^2(\Gamma_2)$.

Proof. We are going to apply Theorem 1 with $\mathcal{X} = X$, $\mathcal{Y} = X'$, $\hat{a} = a$, $\hat{b} = b$ and $\Lambda = \Lambda_1$. Obviously, a is a bilinear, continuous, coercive and symmetric form and so Assumption 1 takes place with, e.g., $M_{\hat{a}} = m_{\hat{a}} = \beta$. Also, b is a bilinear form and (i) in Assumption 2 holds true with, e.g., $M_{\hat{b}} = 1$. Moreover, b verifies the inf-sup property. Indeed,

$$\|\mu\|_{X'} = \sup_{v \in X,\ v \neq 0_X} \frac{(\mu, v)_{X',X}}{\|v\|_X} = \sup_{v \in X,\ v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X}.$$

Therefore, Assumption 2 (ii) is fulfilled with, e.g., $\alpha = 1$.

By a standard calculus, it can be proved that Λ_1 defined in (3.8) verifies Assumption 3.

Therefore, due to Theorem 1, Problem 3 has a unique solution.

Let us prove (4.1). We introduce $f_i \in X$ as follows,

$$(f_i, v)_X = \int_\Omega f_0^i \, v \, dx - \int_{\Gamma_2} f_2^i \, \gamma \, v \, d\Gamma \quad \text{for } i \in \{1, 2\}.$$

Then,

$$\|f_1 - f_2\|_X \leq c^* \left(\|f_0^1 - f_0^2\|_{L^2(\Omega)} + \|f_2^1 - f_2^2\|_{L^2(\Gamma_2)} \right)$$

with $c^* > 0$. Applying now Theorem 1 we get

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_X \leq c\|f_1 - f_2\|_X.$$

We can take $k_1 = c\, c^*$ to immediately get (4.1). □

With similar arguments we can prove the following theorem.

Theorem 3. *Assumptions 4-5 and 7 hold true. Then, Problem 4 has a unique solution*

$$(u, \lambda) \in X \times \Lambda_2.$$

In addition, there exists $k_2 > 0$ such that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_X \leq k_2 \left(\|f_0^1 - f_0^2\|_{L^2(\Omega)} + \|f_2^1 - f_2^2\|_{L^2(\Gamma_2)} \right),$$

where $(u_1, \lambda_1), (u_2, \lambda_2) \in X \times \Lambda_2$ are the solutions of Problem 4 corresponding to the data $(f_0^1, f_2^1), (f_0^2, f_2^2) \in L^2(\Omega) \times L^2(\Gamma_2)$.

Theorem 3 ensures the existence of a unique solution $(u, \lambda) \in X \times \Lambda_2$. But, keeping in mind (3.22), it is natural to hope that the first component of the pair solution of Problem 4 is in fact an element of $\mathcal{K} \subset X$, where $\mathcal{K}$ is defined in (3.18). This is the last objective of the present paper.

Theorem 4. *Let u be the first component of the pair solution $(u, \lambda) \in X \times \Lambda_2$ of Problem 4. Then,*

$$u \in \mathcal{K}.$$

Proof. Let $u_0 \in \mathcal{K}$ be the unique solution of the variational inequality (3.28). By means of u_0 we can define $\lambda_0 \in X'$ as follows

$$(\lambda_0, v)_{X',X} = (f, v)_{X',X} - (Au_0, v)_{X',X} \quad \text{for all } v \in X. \tag{4.2}$$

We prove that $\lambda_0 \in \Lambda_2$ and $b(u_0, \mu - \lambda_0) \leq 0$ for all $\mu \in \Lambda_2$.
Indeed, setting successively in (3.28) $v = 0_X \in \mathcal{K}$ and $v = 2\,u_0 \in \mathcal{K}$ we get,

$$(Au_0, u_0)_{X',X} + J(u_0) = (f, u_0)_{X',X}. \tag{4.3}$$

By (3.28) and (4.3) we obtain,

$$(Au_0, v)_{X',X} + J(v) \geq (f, v)_{X',X} \quad \text{for all } v \in \mathcal{K}. \tag{4.4}$$

Keeping in mind (4.2) and (4.4) we observe that

$$(\lambda_0, v)_{X',X} \leq J(v) \quad \text{for all } v \in \mathcal{K}. \tag{4.5}$$

Thus,

$$\lambda_0 \in \Lambda_2. \tag{4.6}$$

Let us prove now that $b(u_0, \mu - \lambda_0) \leq 0$ for all $\mu \in \Lambda_2$.
Let $\mu \in \Lambda_2$. Then,

$$(\mu, u_0)_{X',X} \leq J(u_0). \tag{4.7}$$

On the other hand, by (4.3) and (4.2) we have

$$(\lambda_0, u_0)_{X',X} = J(u_0). \tag{4.8}$$

By (4.7) and (4.8) we obtain

$$(\mu - \lambda_0, u_0)_{X',X} \leq 0 \quad \text{for all } \mu \in \Lambda_2. \tag{4.9}$$

We observe that

$$a(u_0, v) + b(v, \lambda_0) = (f, v)_{X',X} \quad \text{for all } v \in X,$$

and

$$b(u_0, \mu - \lambda_0) \leq 0 \quad \text{for all } \mu \in \Lambda_2.$$

Thus, the pair $(u_0, \lambda_0) \in \mathcal{K} \times \Lambda_2$ is a solution of Problem 4. But, Problem 4 has a unique solution in $X \times \Lambda_2$. Therefore, $u = u_0$. Consequently, $u \in \mathcal{K}$. $\square$

Remark 4. *Theorem 4 indicates us a connection between the mixed variational formulation and the variational formulation like a variational inequality of the second kind.*

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