Bull. Math. Soc. Sci. Math. Roumanie
Tome 64 (112), No. 4, 2021, 317–327

Hamiltonicity in directed Toeplitz graphs $T_n\langle 1,3,4;t\rangle$

by

Shabnam Malik

Abstract

An $(n\times n)$ matrix $A=(a_{ij})$ is called a Toeplitz matrix if it has constant values along all diagonals parallel to the main diagonal. A directed Toeplitz graph is a digraph with Toeplitz adjacency matrix. In this paper, we obtain new results and improve existing results on hamiltonicity of directed Toeplitz graph $T_n\langle 1,3,4;t\rangle$.

Key Words: Adjacency matrix, Toeplitz matrix, Toeplitz graph, Hamiltonian graph, increasing and decreasing edge, length of an edge.
2010 Mathematics Subject Classification: Primary 05C20; Secondary 05C45.

1 Introduction

We use [18] for basic terminology and notation not defined here. We consider finite, directed and simple graphs.

A *Toeplitz matrix* is a square matrix having constant values along all diagonals parallel to the main diagonal. A *directed Toeplitz graph* $T_n\langle s_1,\ldots,s_k;t_1,\ldots,t_l\rangle$ of order n is a digraph with Toeplitz adjacency matrix of order n. The edges of directed Toeplitz graph $T_n\langle s_1,s_2,\ldots,s_k;t_1,t_2,\ldots,t_l\rangle$ are of two types: *increasing edges* $(u,\ v)$, for which $u<v$, and *decreasing edges* $(u,\ v)$, where $u>v$. In the directed Toeplitz graph $T_n\langle s_1,\ldots,s_k;t_1,\ldots,t_l\rangle$, the edge $(i,\ j)$ occurs if and only if $j-i=s_p$ or $i-j=t_q$ for some integers p and q $(1\le p\le k$ and $1\le q\le l)$. Note that any increasing edge has length s_p for some p, and any decreasing edge has length t_q for some q, and that $T_n\langle s_1,\ldots,s_k;t_1,\ldots,t_l\rangle$ and $T_n\langle t_1,\ldots,t_l;s_1,\ldots,s_k\rangle$ are obtained from each other by reversing the orientation of all edges. We define the *length* of an edge (u,v) to be $|u-v|$.

Suppose that H is a hamiltonian cycle in $T_n\langle s_1,s_2,\ldots,s_k;t_1,t_2,\ldots,t_l\rangle$. The hamiltonian cycle H is determined by two paths $H_{1\to n}$ (from 1 to n) and $H_{n\to 1}$ (from n to 1), i.e., $H=H_{1\to n}\cup H_{n\to 1}$. Then for every vertex v in H, we have $d^-(v)=1=d^+(v)$. The vertices which are not covered by $H_{n\to 1}$ would be covered by $H_{1\to n}$.

Properties of Toeplitz graphs, such as colourability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension have been studied in [1-5, 7-15, 17]. Hamiltonian properties of Toeplitz graphs were first investigated by R. van Dal et al. in [6] and then studied in [12, 16, 24], while the hamiltonicity in directed Toeplitz graphs was first studied by S. Malik and A.M. Qureshi in [18], then by S. Malik and T. Zamfirescu in [21] and by S. Malik in [19, 20, 22, 23].

In [20], the hamiltonicity of the Toeplitz graphs $G=T_n\langle 1,3,4;t\rangle$ was investigated, where it was shown the following. For $t=2$, G is hamiltonian for $n\in\{5,7\}$ and all $n\cong 0,3,4$ (mod 6); for $t=3$, G is hamiltonian for $n\in\{5,\ 6,\ 7,\ 9\}$; for $t=4$, G is hamiltonian for

$n \in \{5, 7, 8,9, 11, 14, 15, 17, 18, 20, 21\}$ and all $n \geq 23$; for $t = 5$, G is hamiltonian for all n if and only if $n \neq 7$; for $t \in \{6, 7\}$, G is hamiltonian for all n; for $t = 8$, G is hamiltonian for all n if and only if $n \neq 14$; for $t = 9$, G is hamiltonian for all n different from 15; for all $t \geq 10$, G is hamiltonian for all n. It was also shown that $T_6\langle 1,3,4;4\rangle$ and $T_{10}\langle 1,3,4;4\rangle$ are non-hamiltonian.

Here in this paper, we improve upon [20] by adding some positive and negative results, and addressing some conjectures on hamiltonicity of $T_n\langle 1,3,4;t\rangle$. For $t = 2$, we show that $T_n\langle 1,3,4;t\rangle$ is hamiltonian for $n \cong 1,5 \pmod 6$, and is not hamiltonian for $n \cong 2 \pmod 6$. For $t = 4$, we show that $T_n\langle 1,3,4;t\rangle$ is hamiltonian for $n \in \{13, 16, 19, 22\}$, and is not hamiltonian for $n = 12$. For $t = 9$, we show that $T_{15}\langle 1,3,4;t\rangle$ is hamiltonian. The paper concludes with a conjecture which completes the hamiltonicity investigation in directed Toeplitz graphs $T_n\langle 1,3,4;t\rangle$.

2 Toeplitz graphs $T_n\langle 1,3,4;t\rangle$ with $t = 2$

Remark 1: Since $T_n\langle 1,3,4;2\rangle$ can use increasing edges of length 4 and decreasing edges of length 2, for any vertex a in $T_n\langle 1,3,4;2\rangle$, there exists a path, say $N(a)$, from a to $a+6$ containing all the vertices of the same parity as a, such that $N(a) = (a, a+4, a+2, a+6)$, see Fig. 1. We define two such *consecutive paths* $N(a)$ from a, namely $N(a) \cup N(a+6)$, such that $N(a) \cup N(a+6) = (a, a+4, a+2, a+6, a+10, a+8, a+12)$, which contains all vertices between a and $a+12$ of the same parity as a, see Fig. 2 for an illustration.

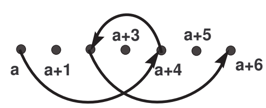

Figure 1: The path $N(a) = (a, a+4, a+2, a+6)$ in $T_n\langle 1,3,4;2\rangle$

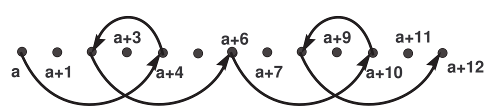

Figure 2: Two path $N(a) \cup N(a+6) = (a, a+4, a+2, a+6, a+10, a+8, a+12)$ in $T_n\langle 1,3,4;2\rangle$

The following lemma will be applied in the proof of Theorem 2.

Lemma 1. *Let* $a < b \leq n$. *In* $T_n\langle 1,3,4;2\rangle$*, the maximum number of consecutive paths* $N(a)$ *between* a *and* b *is* $\lfloor \frac{b-a}{6} \rfloor$*, and the last vertex of this path is* $a + \lfloor \frac{b-a}{6} \rfloor 6$

Proof. Since six successive vertices are required to construct one path $N(a)$, the maximum number of such consecutive paths between a and b in $T_n\langle 1,3,4;2\rangle$ is equal to $\lfloor \frac{b-a}{6} \rfloor$, that is, $N(a) \cup N(a+6) \cup \ldots \cup N(a + (\lfloor \frac{b-a}{6} \rfloor - 1)6)$. Clearly, the last vertex in this path is $a + (\lfloor \frac{b-a}{6} \rfloor - 1)6 + 6 = a + \lfloor \frac{b-a}{6} \rfloor 6$.

Theorem 1. *[20] $T_n\langle 1,3,4;2\rangle$ is hamiltonian for $n \in \{5,7\}$ and $n \cong 0,3,4 \pmod 6$.*

In Theorem 1, it was shown that $T_n\langle 1,3,4;2\rangle$ is hamiltonian for $n \cong 0, 3, 4 \pmod 6$ and $n \in \{5,7\}$. In [20], it was stated as conjecture that $T_n\langle 1,3,4;2\rangle$ is non-hamiltonian for all $n \cong 1, 2, 5 \pmod 6$ such that $n \notin \{5,7\}$. Here we show that $T_n\langle 1,3,4;2\rangle$ is hamiltonian for all $n \cong 1,5 \pmod 6$, and that $T_n\langle 1,3,4;2\rangle$ is non-hamiltonian for all $n \cong 2 \pmod 6$. Thus we can refine Theorem 1 as follows:

Theorem 2. *$T_n\langle 1,3,4;2\rangle$ is hamiltonian if and only if $n \ncong 2 \pmod 6$*

Proof. Claim. $T_n\langle 1,3,4;2\rangle$ is hamiltonian for all $n \cong 1,5 \pmod 6$.

For $n \in \{5,7\}$, $T_n\langle 1,3,4;2\rangle$ has a hamiltonian cycle containing the edge $(n-3, n)$. Indeed $T_5\langle 1,3,4;2\rangle$ has a hamiltonian cycle (1, 4, $\underline{2, 5}$, 3, 1) containing the edge (2, 5), see Fig. 3a, and $T_7\langle 1,3,4;2\rangle$ has a hamiltonian cycle (1, 2, 6, $\underline{4, 7}$, 5, 3, 1) containing the edge (4, 7), see Fig. 3b. Starting from $n \in \{5,7\}$, we can extend a hamiltonian cycle

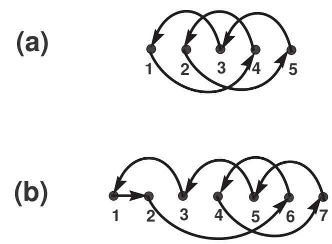

Figure 3: A hamiltonian cycle in (a) $T_5\langle 1,3,4;2\rangle$, and (b) $T_7\langle 1,3,4;2\rangle$

in $T_n\langle 1,3,4;2\rangle$ containing the edge $(n-3, n)$ to a hamiltonian cycle in $T_{n+6}\langle 1,3,4;2\rangle$ with the same property by replacing the edge $(n-3, n)$ with the path $(n-3,\ n+1,\ n+5,\ \underline{n+3,\ n+6},\ n+4,\ n+2,\ n)$. See Fig. 4 for an illustration. Since the vertices 7 and

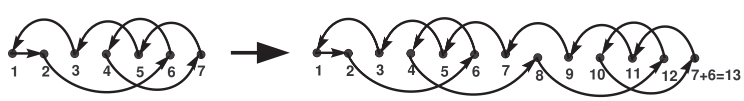

Figure 4: Transformation of a hamiltonian cycle in $T_7\langle 1,3,4;2\rangle$ to that in $T_{13}\langle 1,3,4;2\rangle$

5 are representative in class 1 and 5 modulo 6, respectively, it follows that $T_n\langle 1,3,4;2\rangle$ is hamiltonian for all $n \cong 1,5 \pmod 6$.

By Theorem 1, $T_n\langle 1,3,4;2\rangle$ is hamiltonian for all $n \cong 0,3,4 \pmod 6$. This together with the above claim shows that $T_n\langle 1,3,4;2\rangle$ is hamiltonian for all $n \ncong 2 \pmod 6$.

Conversely, we need to show that $T_n\langle 1,3,4;2\rangle$ is not hamiltonian for all $n \cong 2 \pmod 6$. Assume, to the contrary, that $T_n\langle 1,3,4;2\rangle$ is hamiltonian for $n \cong 2 \pmod 6$, and let H be a hamiltonian cycle in $T_n\langle 1,3,4;2\rangle$. Thus $H = H_{1\to n} \cup H_{n\to 1}$. The path $H_{n\to 1}$ can not cover more than 3 successive vertices (except $(1,2,3,4)$ or $(n-3, n-2, n-1, n)$), for otherwise

$H_{1\to n}$ would not be able to cover rest of the vertices as $H_{1\to n}$ can not use increasing edge of length greater than 4. Since $T_n\langle 1,3,4;2\rangle$ has decreasing edges of length 2 only and that n is even (as $n \cong 2 \pmod 6$), two possibilities exist for $H_{n\to 1}$.

Case 1: $H_{n\to 1} = (n, n-2, \ldots, 2, 3, 1)$

Figure 5: $H_{n\to 1} = (n, n-2, \ldots, 2, 3, 1)$ in $T_n\langle 1,3,4;2\rangle$

Since $H_{n\to 1}$ covers all the even vertices only, between n and 4, $H_{1\to n}$ has to cover all the odd vertices between 5 and n. Thus $H_{1\to n}$ has to use consecutive paths as described in Remark 1. By Lemma 1, the longest such path in $H_{1\to n}$ from vertex 5 is $N(5)\cup N(5+6)\cup \ldots \cup N(5+(\lfloor\frac{n-5}{6}\rfloor-1)6)$, and the last vertex of this path is $5+\lfloor\frac{n-5}{6}\rfloor 6$ which is $n-3$, because $5+\lfloor\frac{n-5}{6}\rfloor 6 = 5+\lfloor\frac{n-2-3}{6}\rfloor 6 = 5+(\lfloor\frac{n-2}{6}\rfloor+\lfloor\frac{-1}{2}\rfloor)6 = 5+(\lfloor\frac{n-2}{6}\rfloor-\lceil\frac{1}{2}\rceil)6 = 5+\lfloor\frac{n-2}{6}\rfloor 6-6 = n-3$ (since $n \cong 2 \pmod 6$, $\lfloor\frac{n-2}{6}\rfloor = \frac{n-2}{6}$). Since there is no path $P_{n-3\to n}$ in $H_{1\to n}$, $H_{1\to n}$ terminates at $n-3$. This is a contradiction.

Case 2: $H_{n\to 1} = (n, n-2, n-1, n-3, n-5, \ldots, 1)$.

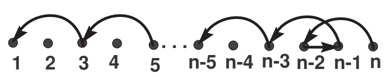

Figure 6: $H_{n\to 1} = (n, n-2, n-1, n-3, n-5, \ldots, 1)$ in $T_n\langle 1,3,4;2\rangle$

Consider $(1,2)\in E(H_{1\to n})$. Since last four vertices have already been visited by $H_{n\to 1}$, by Lemma 1, the longest path in $H_{1\to n}$ between 2 and $n-4$ is $N(2)\cup N(2+6)\cup \ldots \cup N(2+(\lfloor\frac{n-6}{6}\rfloor-1)6)$ and the last vertex of this path is $2+\lfloor\frac{n-6}{6}\rfloor 6$ which is $n-6$, because $2+\lfloor\frac{n-6}{6}\rfloor 6 = 2+\lfloor\frac{n-2-4}{6}\rfloor 6 = 2+\lfloor\frac{n-2}{6}\rfloor 6+\lfloor\frac{-4}{6}\rfloor 6 = 2+\lfloor\frac{n-2}{6}\rfloor 6-\lceil\frac{2}{3}\rceil 6 = 2+\lfloor\frac{n-2}{6}\rfloor 6-6 = n-6$ (since $n \cong 2 \pmod 6$, $\lfloor\frac{n-2}{6}\rfloor = \frac{n-2}{6}$). Since there is no path $P_{n-6\to n}$ in $H_{1\to n}$, $H_{1\to n}$ terminates at vertex $n-6$. This is a contradiction.

Consider $(1,4)\in E(H_{1\to n})$. Then $(4,2),(2,6)\in E(H_{1\to n})$, for otherwise vertex 2 would be missed. By Lemma 1, the longest path in $H_{1\to n}$ between 6 and $n-4$ is $N(6)\cup N(6+6)\cup \ldots \cup N(6+(\lfloor\frac{n-10}{6}\rfloor-1)6)$, and the last vertex of this path is $6+\lfloor\frac{n-10}{6}\rfloor 6$ which is $n-8$, because $6+\lfloor\frac{n-10}{6}\rfloor 6 = 6+\lfloor\frac{n-2-8}{6}\rfloor 6 = 6+\lfloor\frac{n-2}{6}\rfloor 6+\lfloor\frac{-8}{6}\rfloor 6 = 6+\lfloor\frac{n-2}{6}\rfloor 6-\lceil\frac{4}{3}\rceil 6 = 6+\lfloor\frac{n-2}{6}\rfloor 6-12 = n-8$ (since $n \cong 2$ mod 6, $\lfloor\frac{n-2}{6}\rfloor = \frac{n-2}{6}$). Since there is no path $P_{n-8\to n}$ in $H_{1\to n}$, $H_{1\to n}$ terminates at vertex $n-8$. This is a contradiction.

This completes the proof.

3 Toeplitz graphs $T_n\langle 1,3,4;t\rangle$ with $t=4$

Theorem 3. *[20] $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \in \{5, 7, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$ and all $n \geq 23$.*

Theorem 4. *[20] $T_6\langle 1,3,4;4\rangle$ is non-hamiltonian.*

Theorem 5. *[20] $T_{10}\langle 1,3,4;4\rangle$ is non-hamiltonian.*

In [20], it was shown that $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \in \{5, 7, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$ and all $n \geq 23$. Furthermore it was shown that $T_6\langle 1,3,4;4\rangle$ and $T_{10}\langle 1,3,4;4\rangle$ are non-hamiltonian, and a conjecture was stated, that is, $T_n\langle 1,3,4;4\rangle$ is non-hamiltonian for $n \in \{12, 13, 16, 19, 22\}$. Here we show that $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \in \{13, 16, 19, 22\}$. We also show that $T_{12}\langle 1,3,4;4\rangle$ is non-hamiltonian. Thus we can refine Theorem 3-5 as follows:

Theorem 6. *$T_n\langle 1,3,4;4\rangle$ is hamiltonian if and only if $n \notin \{6, 10, 12\}$*

Proof. Claim 1. For $n \in \{13, 16, 19, 22\}$, $T_n\langle 1,3,4;4\rangle$ is hamiltonian.

In Fig. 7, we display a hamiltonian cycle $(1, 2, 6, 10, 11, 7, 3, 4, 8, 12, 13, 9, 5, 1)$ in $T_{13}\langle 1,3,4;4\rangle$. In Fig. 8, we display a hamiltonian cycle $(1, 2, 3, 4, 7, 8, 11, 15, 16, 12, 13, 14, 10, 6, 9, 5, 1)$ in $T_{16}\langle 1,3,4;4\rangle$. In Fig. 9, we display a hamiltonian cycle $(1, 2, 3, 6, 7, 10, 11, 14, 18, 19, 15, 16, 17, 13, 9, 12, 8, 4, 5, 1)$ in $T_{19}\langle 1,3,4;4\rangle$. In Fig. 10, we display a hamiltonian cycle $(1, 2, 3, 6, 9, 10, 13, 14, 17, 21, 22, 18, 19, 20, 16, 12, 15, 11, 7, 8, 4, 5, 1)$ in $T_{22}\langle 1,3,4;4\rangle$.

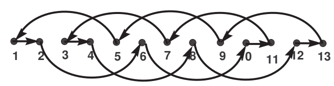

Figure 7: A hamiltonian cycle in $T_{13}\langle 1,3,4;4\rangle$

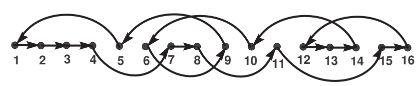

Figure 8: A hamiltonian cycle in $T_{16}\langle 1,3,4;4\rangle$

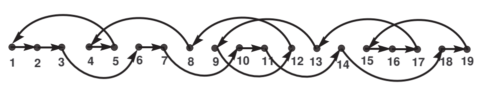

Figure 9: A hamiltonian cycle in $T_{19}\langle 1,3,4;4\rangle$

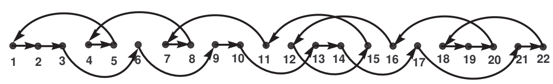

Figure 10: A hamiltonian cycle in $T_{22}\langle 1,3,4;4\rangle$

By Theorem 3, $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \in \{5, 7, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$ and all $n \geq 23$. This together with Claim 1 shows that $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \notin \{6, 10, 12\}$.

Conversely, we show that $T_n\langle 1,3,4;4\rangle$ is not hamiltonian for $n \in \{6, 10, 12\}$.

Claim 2. $T_{12}\langle 1,3,4;4\rangle$ is non-hamiltonian.

Assume, to the contrary, that $T_{12}\langle 1,3,4;4\rangle$ is hamiltonian. Let $H = H_{1\to 12} \cup H_{12\to 1}$ be a hamiltonian cycle in $T_{12}\langle 1,3,4;4\rangle$. Let $V(H_{12\to 1} \setminus \{1, 12\}) = V_1 \cup V_2 \cup \cdots \cup V_k$, where each

V_i, $i \in \{1, 2, \ldots, k\}$, is a disjoint set of successive vertices. But then order of each V_i should not be greater than 3 because $H_{1\to12}$ has no edge of length greater than 3. Thus $|V_i| \le 3$. Let A be the set of all decreasing edges in $T_{12}\langle 1,3,4;4\rangle$. Since the decreasing edges are of length 4 only, $A = \{(12,8), (11,7), (10,6), (9,5), (8,4), (7,3), (6,2), (5,1)\}$, and $|A| = 8$. Let B be the set of all decreasing edges in $H_{12\to1}$. Since $B \subseteq A$ and $d^-(1) = d^+(12) = 1$, $(12,8)$, $(5,1) \in B$. Since $H_{12\to1}$ can not have only these two edges as its decreasing edges, for otherwise $H_{12\to1}$ terminates at vertex 8, $3 \le |B| \le 8$. Six cases arise as per number of decreasing edges in $H_{12\to1}$ (other than $(12,8)$ and $(5,1)$).

Case 1. If $|B| = 3$. Since $(12,8), (5,1) \in B$, six subcases arise:
(*i*) Let $(11,7) \in B$. Then $H_{12\to1} = (12,8) \cup P_{8\to11} \cup (11,7) \cup P_{7\to5} \cup (5,1)$, and $P_{8\to11} = (8,11)$. Since there is no path $P_{7\to5}$ in $H_{12\to1}$, $H_{12\to1}$ terminates at vertex 7.
(*ii*) Let $(10,6) \in B$. Then $H_{12\to1} = (12,8) \cup P_{8\to10} \cup (10,6) \cup P_{6\to5} \cup (5,1)$, and $P_{8\to10} = (8,9,10)$. Since there is no path $P_{6\to5}$ in $H_{12\to1}$, $H_{12\to1}$ terminates at vertex 6.
(*iii*) Let $(7,3) \in B$. Then $H_{12\to1} = (12,8) \cup P_{8\to7} \cup (7,3) \cup P_{3\to5} \cup (5,1)$, and $P_{3\to5} = (3,4,5)$. Since there is no path $P_{8\to7}$ in $H_{12\to1}$, $H_{12\to1}$ terminates at vertex 8.
(*iv*) Let $(6,2) \in B$. Then $H_{12\to1} = (12,8) \cup P_{8\to6} \cup (6,2) \cup P_{2\to5} \cup (5,1)$. Since there is no path $P_{8\to6}$ in $H_{12\to1}$, $H_{12\to1}$ terminates at vertex 8.
(*v*) Let $(9,5) \in B$. Then $H_{12\to1} = (12,8) \cup P_{8\to9} \cup (9,5) \cup (5,1)$, and $P_{8\to9} = (8,9)$. Thus $H_{12\to1} = (12,8,9,5,1)$. But then, by considering all the following possible cases, we see that there is no path $H_{1\to12}$:

Consider $(1,2), (2,3), (3,4) \in E(H_{1\to12})$. If $(4,7), (7,10), (10,11) \in E(H_{1\to12})$, but then $H_{1\to12}$ terminates at vertex 11, for otherwise vertex 6 would be missed. If $(4,7), (7,10), (10,6) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 6. If $(4,7), (7,11) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 11, for otherwise vertices 6 and 10 would be missed.

Consider $(1,2), (2,3) \in E(H_{1\to12})$. If $(3,6), (6,10), (10,7) \in E(H_{1\to12})$ or $(3,6), (6,10), (10,11), (11,7) \in E(H_{1\to12})$, but then $H_{1\to12}$ terminates at vertex 7. If $(3,6), (6,7), (7,11) \in E(H_{1\to12})$, or $(3,7), (7,10), (10,11) \in E(H_{1\to12})$, or $(3,7), (7,11) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 11. If $(3,7), (7,10), (10,6) \in E(H_{1\to12})$ but then $H_{1\to12}$ terminates at vertex 6.

Consider $(1,2) \in E(H_{1\to12})$. If $(2,6), (6,10), (10,11), (11,7), (7,3), (3,4) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 4. If $(2,6), (6,7), (7,10), (10,11)) \in E(H_{1\to12})$, or $(2,6), (6,7), (7,11) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 11.

Consider $(1,4) \in E(H_{1\to12})$. If $(4,7), (7,10), (10,11) \in E(H_{1\to12})$, or $(4,7), (7,11) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 11, for otherwise some vertices would be missed. If $(4,7), (7,10), (10,6), (6,2), (2,3) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 3.
(*vi*) Let $(8,4) \in B$. Then $E(H_{12\to1}) = (12,8) \cup (8,4) \cup P_{4\to5} \cup (5,1)$. Since the only possibility for the path $P_{4\to5}$ in $H_{12\to1}$ is $P_{4\to5} = (4,5)$, $H_{12\to1} = (12,8,4,5,1)$. But then, by considering all the following possible cases, we see that there is no path $H_{1\to12}$:

Consider $(1,2), (2,3) \in E(H_{1\to12})$. If $(3,6), (6,7), (7,10) \in E(H_{1\to12})$, but then $H_{1\to12}$ terminates at vertex 10, for otherwise vertex 9 would be missed. If $(3,6), (6,7), (7,11) \in E(H_{1\to12})$, or $(3,7), (7,10), (10,11) \in E(H_{1\to12})$, or $(3,6), (7,11) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 11. If $(3,6), (6,10), (10,11), (11,7) \in E(H_{1\to12})$, or $(3,6), (6,9), (9,10), (10,11), (11,7) \in E(H_{1\to12})$, then $H_{1\to12}$ terminates at vertex 7. If $(3,7), (7,10), (10,6), (6,9) \in E(H_{1\to12})$ but then $H_{1\to12}$ terminates at vertex 9.

Consider $(1,2), (2,6) \in E(H_{1\to12})$. If $(6,10), (10,11), (11,7), (7,3) \in E(H_{1\to12})$, or

$(6,9), (9,10), (10,11), (11,7), (7,3) \in E(H_{1\to 12})$, then $H_{1\to 12}$ terminates at vertex 3.

Case 2. If $|B| = 4$. Since $(12,8), (5,1) \in B$, then ${C_2}^6 = 15$ subcases arise:
(i) $\{(6,2), (7,3)\}$, (ii) $\{(6,2), (11,7)\}$, (iii) $\{(10,6), (7,3)\}$, (iv) $\{(10,6), (11,7)\}$. Since, for subcases (i)-(iv), there exists $V_i \in V(H_{12\to 1} \setminus \{1,12\})$ such that $| V_i |> 3$, this is a contradiction.
(v) $\{(11,7),\ (9,5)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 11} \cup (11,7) \cup P_{7\to 9} \cup (9,5) \cup (5,1)$, and $P_{8\to 11} = (8,11)$. Since there is no path $P_{7\to 9}$ in $H_{12\to 1}$, $H_{12\to 1}$ terminates at vertex 7.
(vi) Let $\{(11,7),\ (8,4)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 11} \cup (11,7) \cup P_{7\to 5} \cup (5,1)$. Since there is no paths $P_{4\to 11}$ in $H_{12\to 1}$, $H_{12\to 1}$ terminates at vertex 4.
(vii) Let $\{(11,7),\ (7,3)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 11} \cup (11,7) \cup (7,3) \cup P_{3\to 5} \cup (5,1)$, $P_{8\to 11} = (8,11)$, and $P_{3\to 5} = (2,4,5)$. Thus $H_{1\to 12} = (1,2,6,9,10) \cup P_{10\to 12}$. Since there is no path $P_{10\to 12}$ in $H_{1\to 12}$, $H_{1\to 12}$ terminates at vertex 10.
(viii) Let $\{(10,6),\ (9,5)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 10} \cup (10,6) \cup P_{6\to 9} \cup (9,5) \cup (5,1)$. Since there is no path $P_{8\to 10}$ in $H_{12\to 1}$, $H_{12\to 1}$ terminates at vertex 8.
(ix) Let $\{(10,6),\ (8,4)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 10} \cup (10,6) \cup P_{6\to 5} \cup (5,1)$. Since there is no path $P_{4\to 10}$ in $H_{12\to 1}$ (for otherwise $| V_i |> 3$), $H_{12\to 1}$ terminates at vertex 4.
(x) Let $\{(10,6),\ (6,2)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 10} \cup (10,6) \cup (6,2) \cup P_{2\to 5} \cup (5,1)$, $P_{8\to 10} = (8,9,10)$, and $P_{2\to 5} = (2,5)$. But then $H_{1\to 12}$ terminates at vertex 1.
(xi) Let $\{(9,5),\ (8,4)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 9} \cup (9,5) \cup (5,1)$. Since there is no path $P_{4\to 9}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.
(xii) Let $\{(9,5),\ (7,3)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 7} \cup (7,3) \cup P_{3\to 9} \cup (9,5) \cup (5,1)$. Since there is no path $P_{8\to 7}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 8.
(xiii) Let $\{(9,5),\ (6,2)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 6} \cup (6,2) \cup P_{2\to 9} \cup (9,5) \cup (5,1)$. Since there is no path $P_{8\to 6}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 8.
(xiv) Let $\{(8,4),\ (7,3)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 7} \cup (7,3) \cup P_{3\to 5} \cup (5,1)$, and $P_{4\to 7} = (4,7)$. Since there is no path $P_{3\to 5}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at 3.
(xv) Let $\{(8,4),\ (6,2)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 6} \cup (6,2) \cup P_{2\to 5} \cup (5,1)$. Since there is no path $P_{4\to 6}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.

Case 3. If $|B| = 5$. Since $(12,8), (5,1) \in B$, then ${C_3}^6 = 20$ subcases arise:
(i) $\{(6,2), (7,3), (8,4)\}$, (ii) $\{(6,2), (7,3), (9,5)\}$, (iii) $\{(6,2), (8,4), (11,7)\}$,
(iv) $\{(6,2), (7,3), (11,7)\}$, (v) $\{(6,2), (9,5), (11,7)\}$, (vi) $\{(10,6), (7,3), (6,2)\}$,
(vii) $\{(10,6), (7,3), (8,4))\}$, (viii) $\{(10,6), (11,7), (6,2)\}$, (ix) $\{(10,6), (11,7), (7,3)\}$,
(x) $\{(10,6), (11,7), (8,4)\}$, (xi) $\{(10,6), (11,7), (9,5)\}$. Since, for (i)-(xi) subsets B, there exist $V_i \in V(H_{12\to 1} \setminus \{1,12\})$ such that $| V_i |> 3$, this is a contradiction.
(xii) Let $\{(6,2), (8,4), (9,5)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 11} \cup (11,7) \cup P_{7\to 9} \cup (9,5) \cup (5,1)$. Since there is no path $P_{4\to 11}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.
(xiii) Let $\{(6,2), (8,4), (10,6)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 11} \cup (11,7) \cup (7,3) \cup P_{3\to 9} \cup (9,5) \cup (5,1)$, and $P_{8\to 11} = (8,11)$. Since there is no path $P_{3\to 9}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 3.
(xiv) Let $\{(6,2), (9,5), (10,6)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup P_{8\to 11} \cup (11,7) \cup (7,3) \cup P_{3\to 8} \cup (8,4) \cup (4,5) \cup (5,1)$, and $P_{8\to 11} = (8,11)$. Since there is no path $P_{3\to 8}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 3.
(xv) Let $\{(6,2), (7,3), (10,6)\} \subset B$. Then $H_{12\to 1} = (12,8) \cup (8,4) \cup P_{4\to 10} \cup (10,6) \cup P_{6\to 9} \cup$

$(9,5)\cup(5,1)$. Since there is no path $P_{4\to 10}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.
(xvi) Let $\{(7,3),(8,4),(11,7)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup P_{8\to 10}\cup(10,6)\cup(6,2)\cup P_{2\to 9}\cup(9,5)\cup(5,1)$. Since there is no path $P_{8\to 10}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 8.
(xvii) Let $\{(7,3),(8,4),(9,5)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup(8,4)\cup P_{4\to 10}\cup(10,6)\cup(6,2)P_{2\to 5}\cup(5,1)$. Since there is no path $P_{4\to 10}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.
(xviii) Let $\{(7,3),(9,5),(11,7)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup P_{8\to 10}\cup(10,6)\cup(6,2)\cup P_{2\to 7}\cup(7,3)\cup P_{3\to 5}\cup(5,1)$. Since there is no path $P_{8\to 10}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 8.
(xix) Let $\{(8,4),(9,5),(10,6)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup(8,4)\cup P_{4\to 7}\cup(7,3)\cup P_{3\to 9}\cup(9,5)\cup(5,1)$, and $P_{4\to 7}=(4,7)$. Since there is no path $P_{3\to 9}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 3.
(xx) Let $\{(8,4),(9,5),(11,7)\}\subset B$. Thus $H_{12\to 1}=(12,8)\cup(8,4)\cup P_{4\to 6}\cup(6,2)\cup P_{2\to 9}\cup(9,5)\cup(5,1)$. Since there is no path $P_{4\to 6}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.

Case 4. If $|B|=6$. Since $(12,8),(5,1)\in B$, then ${C_4}^6=15$ subcases arise:
(i) $\{(11,7),(10,6),(9,5),(8,4)\}$, (ii) $\{(11,7),(10,6),(9,5),(7,3)\}$,
(iii) $\{(11,7),(10,6),(9,5),(6,2)\}$, (iv) $\{(6,2),(7,3),(8,4),(9,5)\}$,
(v) $\{(6,2),(7,3),(8,4),(10,6)\}$, (vi) $\{(6,2),(7,3),(8,4),(11,7)\}$,
(vii) $\{(6,2),(7,3),(11,7),(10,6)\}$, (viii) $\{(6,2),(7,3),(11,7),(9,5)\}$,
(ix) $\{(6,2),(8,4),(11,7),(10,6)\}$, (x) $\{(6,2),(8,4),(11,7),(9,5)\}$,
(xi) $\{(6,2),(7,3),(10,6),(9,5)\}$, (xii) $\{(7,3),(8,4),(11,7),(10,6)\}$,
(xiii) $\{(7,3),(8,4),(9,5),(10,6)\}$. Since for (i)-(xiii) subsets of B there exist $V_i\in V(H_{12\to 1}\setminus\{1,12\})$ such that $\mid V_i\mid>3$, this is a contradiction.
(xiv) $\{(6,2),(8,4),(10,6),(9,5)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup(8,4)\cup P_{4\to 10}\cup(10,6)\cup(6,2)\cup P_{2\to 9}\cup(9,5)\cup(5,1)$. Since there is no path $P_{4\to 7}$ in $H_{12\to 1}$ (as otherwise $\mid V_i\mid>3$), $H_{1\to 12}$ terminates at vertex 4.
(xv) $\{(7,3),(8,4),(11,7),(9,5)\}\subset B$. Then $H_{12\to 1}=(12,8)\cup(8,4)\cup P_{4\to 11}\cup(11,7)\cup(7,3)\cup P_{3\to 9}\cup(9,5)\cup(5,1)$. Since there is no path $P_{4\to 11}$ in $H_{12\to 1}$, $H_{1\to 12}$ terminates at vertex 4.

Case 5. If $|B|=7$. Since $(12,8),(5,1)\in B$, then the following ${C_5}^6=6$ subcases arise:
(i) $\{(11,7),(10,6),(9,5),(8,4),(7,3)\}\subset B$
(ii) $\{(11,7),(10,6),(9,5),(8,4),(6,2)\}\subset B$
(iii) $\{(11,7),(10,6),(9,5),(7,3),(6,2)\}\subset B$.
(iv) $\{(11,7),(10,6),(8,4),(7,3),(6,2)\}\subset B$.
(v) $\{(11,7),(9,5),(8,4),(7,3),(6,2)\}\subset B$.
(vi) $\{(10,6),(9,5),(8,4),(7,3),(6,2)\}\subset B$.

For all these six subcases, since there exist $V_i\in V(H_{12\to 1}\setminus\{1,12\})$ such that $\mid V_i\mid>3$, this is a contradiction because

Case 6. If $|B|=8$. Then $A=B$ and $V(H_{12\to 1}\setminus\{1,12\})=\{2,3,4,5,6,7,8,9,10,11\}=V_1$, where V_1 is a set of successive vertices. Since $\mid V_1\mid>3$, this is a contradiction.

In summary, $T_{12}\langle 1,3,4;4\rangle$ is non-hamiltonian.

By Theorem 4 and Theorem 5, $T_n\langle 1,3,4;4\rangle$ is not hamiltonian for $n\in\{6,10\}$. This together with Claim 1 and Claim 2 shows that $T_n\langle 1,3,4;4\rangle$ is hamiltonian if and only if $n\notin\{6,10,12\}$.

This completes the proof.

4 Toeplitz graphs $T_n\langle 1,3,4;t\rangle$ with $t=9$

Theorem 7. *[20] $T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n different from* 15.

In [20], it was shown that $T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n different from 15, further it was stated as conjecture that $T_{15}\langle 1,3,4;9\rangle$ is non-hamiltonian. But here we show that $T_{15}\langle 1,3,4;9\rangle$ is hamiltonian. Thus we refine Theorem 7 as follows:

Theorem 8. *$T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n.*

Proof. Claim. $T_{15}\langle 1,3,4;9\rangle$ is hamiltonian.
Indeed $T_{15}\langle 1,3,4;9\rangle$ contains the hamiltonian cycle

$$(1,2,3,7,11,14,5,9,13,4,8,12,15,6,10,1).$$

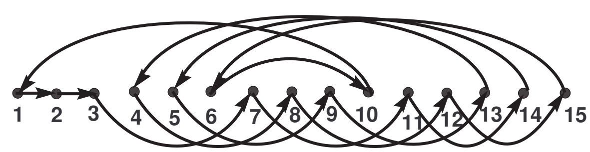

Figure 11: Hamiltonian cycle in $T_{15}\langle 1,3,4;9\rangle$

By Theorem 7, $T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n different from 15. This together with the above claim shows that $T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n.
This finishes the proof.

Conjecture: $T_n\langle 1,3,4;3\rangle$ is non-hamiltonian for $n\notin\{5,6,7,9\}$.

5 Concluding Remarks

In this paper we refine results of [20], and address to the stated conjectures in [20]. This completes the investigation of hamiltonnicity of the Toeplitz Graph $T_n\langle 1,3,4;t\rangle$ by proposing the above conjecture.

References

[1] A. Ahmad, F. Nadeem, A. Gupta, On super edge-magic deficiency of certain Toeplitz graphs, *Hacet. J. Math. Stat.*, **47 (3)**, 513-519 (2018).

[2] A. Ahmad, M. Baca, M. F. Nadeem, On edge irregularity strength of Toeplitz graphs, *U. P. B. Sci. Bull. ser. A.*, **78 (4)**, 155-162 (2016).

[3] S. Akbari, S. Hossein Ghorban, S. Malik, S. Qajar, Conditions for regularity and for 2-connectivity of Toeplitz graphs, *Util. Math.*, **110**, 305-314 (2019).

[4] L. Aslam, S. Sarwar, M. J. Yousaf, S. Waqar, Cycle discrepancy of cubic Toeplitz graphs, *Pakistan J. Eng. Appl. Sci.*, **22**, 14-19 (2018).

[5] S. Bau, A generalization of the concept of Toeplitz graphs, *Mong. Math. J.*, **15**, 54-61 (2011).

[6] R. van Dal, G. Tijssen, Z. Tuza, J. A. A. van der Veen, Ch. Zamfirescu, T. Zamfirescu, Hamiltonian properties of Toeplitz graphs, *Discrete Math.*, **159**, 69-81 (1996).

[7] R. Euler, H. LeVerge, T. Zamfirescu, A characterization of infinite, bipartite Toeplitz graphs, in: Ku Tung-Hsin (Ed.), *Combinatorics and Graph Theory.* **1**, Academia Sinica, World Scientific, Singapore, 119-130 (1995).

[8] R. Euler, Coloring infinite, planar Toeplitz graphs, *Tech. Report, LIBr*, November (1998).

[9] R. Euler, Coloring planar Toeplitz graphs and the stable setpolytope, *Discrete Math.*, **276**, 183-200 (2004).

[10] R. Euler, Characterizing bipartite Toeplitz graphs, *Theor. Comput. Sci.*, **263**, 47-58 (2001).

[11] R. Euler, T. Zamfirescu, On planar Toeplitz graphs, *Graphs and Combinatorics*, **29**, 1311-1327 (2013).

[12] C. Heuberger, On hamiltonian Toeplitz graphs, *Discrete Math.*, **245**, 107-125 (2002).

[13] S. Hossein Ghorban, Toeplitz graph decomposition, *Transactions on Combinatorics*, **1 (4)**, 35-41 (2012).

[14] J. B. Liu, M. F. Nadeem, H. M. A. Siddiqui, W. Nazir, Computing metric dimension of certain families of Toeplitz graphs, *IEEE Access*, **7**, 126734-126741 (2019).

[15] M. Baca, Y. Bashir, F. Nadeem, A. Shabbir, On super edge-antimagic total labeling of Toeplitz graphs, *Springer Proceedings in Mathematics and Statistics*, **98**, 1-10 (2015).

[16] M. F. Nadeem, A. Shabbir, T. Zamfirescu, Hamiltonian connectedness of Toeplitz graphs, *Mathematics in the 21st Century, Springer Proceedings in Mathematics and Statistics,* **98**, 135-149 (2015).

[17] S. Nicoloso, U. Pietropaoli, On the chromatic number of Toeplitz graphs, *Discrete Applied Mathematics*, **164 (1)**, 286-296 (2014).

[18] S. Malik, A. M. Qureshi, Hamiltonian cycles in directed Toeplitz graphs, *Ars Comb.*, **CIX** April, 511-526 (2013).

[19] S. Malik, Hamiltonian cycles in directed Toeplitz graphs-Part 2, *Ars Comb.*, **116**, 303-319 (2014).

[20] S. Malik, Hamiltonicity in directed Toeplitz graphs of maximum (out or in) degree 4, *Util. Math.*, **89**, 33-68 (2012).

[21] S. Malik, T. Zamfirescu, Hamiltonian connectedness in directed Toeplitz graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, **53 (101)**, 145-156 (2010).

[22] S. Malik, Hamiltonian cycles in directed Toeplitz graphs $T_n\langle 1,2;t_1 \leq 5,t_2\rangle$, *Util. Math.*, **99**, 3-17 (2016).

[23] S. Malik, Hamiltonicity in directed Toeplitz graphs $T_n\langle 1,2;t_1,t_2\rangle$, *Australas. J. Combin.*, **78 (3)**, 434-449 (2020).

[24] H. Zafar, N. Akhter, M. K. Jamil, F. Nadeem, Hamiltonian connectedness and Toeplitz graphs, *American Scientific Research Journal for Engineering, Technology, and Sciences*, **33 (1)**, 255-268 (2017).

Received: 13.10.2020
Revised: 04.05.2021
Accepted: 04.05.2021

Forman Christian College (A Chartered University)
Lahore, Pakistan
E-mail: shabnam.malik@gmail.com