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**A classifier for equidimensional contact unimodal map germs**

by

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**Abstract**

We found an overlapping in the classification of equidimensional unimodular map germs with respect to contact equivalence (type 3 and 5) given by Dimca and Gibson over $\mathbb{C}$. The reason is an error in the classification of nets of conics found by the anonymous referee. On this basis we correct the classification. Moreover, we characterize this classification in terms of certain invariants and on the basis of this characterization we present an algorithm to compute the type of the equidimensional contact unimodular map germs without computing the normal form and also give its implementation in the computer algebra system Singular.

**Key Words:** Equidimensional map germ, $\mathcal{K}$-equivalence, codimension.
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# 1 Introduction

Classification and Recognition of singularities of map germs upto some equivalence relation are two fundamental problems in local singularity theory. Classification for map germs under some equivalence relation, means finding a list of map germs and showing that all map germs are equivalent to a map germ in the list. Recognition means finding some criteria which describe, how to find an equivalent map germ in the list. There are many classification results for singularities of map germs, however studies for recognition problem are not so much. Therefore one important aim of the paper is to develop and implement algorithms to compute the type of a map germ by using certain invariants but not computing the normal form (which is usually much more difficult).

Let $\mathrm{A}(3,3) =< x,y,z > \mathbb{C}[[x,y,z]]^3$ be the set of map germs $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$. We denote by $\mathcal{K}=\mathcal{A}\times\mathcal{L}$, where $\mathcal{A}:=Aut_{\mathbb{C}}(\mathbb{C}[[x,y,z]])$ and $\mathcal{L}:=Gl_3(\mathbb{C}[[x,y,z]])$ . Then $\mathcal{K}$ acts on $\mathrm{A}(3,3)$ as follows:

$$\mathcal{K}\times \mathrm{A}(3,3)\to \mathrm{A}(3,3),$$

such that

$$((\psi,M),f)\mapsto \psi^{-1}\circ Mf.$$

The map germs $f,g\in \mathrm{A}(3,3)$ are called $\mathcal{K}$-equivalent say $f\sim_{\mathcal{K}} g$ if they lie in the same orbit under the action of $\mathcal{K}$. For any $f\in \mathrm{A}(3,3)$, we define the orbit map $\vartheta_f:\mathcal{K}\to \mathrm{A}(3,3)$ by $\vartheta_f(\psi,M)=\psi^{-1}oMf$. In particular, we have $\vartheta_f(id)=f$. Let $\mathcal{K}_f=img(\vartheta_f)$ is the orbit of $f$ under the action of $\mathcal{K}$. Then the corresponding tangent map to the orbit at $f = (f_1, f_2, f_3)$ has as image the tangent space defined as:

$$\mathrm{T}_{\mathcal{K}_f,id} =< x, y, z >< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >_{\mathbb{C}[[x,y,z]]} + < f_1, f_2, f_3 > \mathbb{C}[[x, y, z]]^3.$$

The classification problem for singularities of map germs from the plane into the plane with respect to $\mathcal{A}-$equivalence was studied by Rieger [11], [12]. He classified map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with corank one and $\mathcal{A}_e-$codimension $\leq 6$. He also classified $\mathcal{A}-$simple and $\mathcal{A}-$unimodal singularities of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Moreover Dimca and Gibson [6] gave the classification of $\mathcal{K}-$simple and $\mathcal{K}-$unimodal singularities of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. These classifications are characterized in [3], [2], [4], [5] and [10]. Also in [7], Dimca and Gibson gave the classification of equidimensional contact unimodal map germs. The aim of this article is to study the recognition problem for this classification.

Let us recall some basic definitions related to this article:

**Definition 1.1.** The $\mathcal{K}$-modality of a map germ $f$ is the smallest integer $m$ such that a sufficiently small neighbourhood of $f$ can be covered by a finite number of $m$-parameter families of orbits under the action of $\mathcal{K}$ on $f$.

A map germ $f$ is called $\mathcal{K}$-unimodular if the $\mathcal{K}$-modality of the germ is 1

**Definition 1.2.** Let $f \in \mathrm{A}(3, 3)$ be a map germ. Then codimension of the tangent space at $f = (f_1, f_2, f_3)$, the Milnor number of $f$ and a number closely connected with the order of $k-$determinancy of $f$ are denoted by $c_f$, $\mu_f$ and $\sigma_f$ respectively and defined as:

$$c_f = dim_{\mathbb{C}} \frac{< x, y, z > \mathbb{C}[[x, y, z]]}{\mathrm{T}_{\mathcal{K}_f,id}}.$$

$$\mu_f = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y, z]]}{f} - 1.$$

$$\sigma_f = min\{s :< x, y, z >^s \mathbb{C}[[x, y, z]]^3 \subset \mathrm{T}_{\mathcal{K}_f,id}\}.$$

# 2 Characterization of equidimensional map germs from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$

In this section we give the characterization of equidimensional unimodal map germs in terms of certain invariants.

**Proposition 2.1.** *Table* 1 *contains the classification of nets of conics.*

*Proof.* For a proof [1] see [7] and [14]. □

**Proposition 2.2.** *Let* $f(x, y, z)$ *be a map germ from* $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$. *Then*

[1] The anonymous referee gave us the hint that normal form in the tables of [7] and [14] corresponding to type 3 is wrong and had to be replaced by $(x^2 - yz, y^2 - xz, z^2)$.

| Type | Normal form | Conditions |
|---|---|---|
| 1 | $(x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$ | $\lambda^3 \neq -1, 0, 8$ |
| 2 | $(xy, x^2+yz, y^2+xz)$ | $-$ |
| 3 | $(x^2-yz, y^2-xz, z^2)$ | $-$ |
| 4 | $(x^2+yz, xy, xz)$ | $-$ |
| 5 | $(x^2+yz, xy, z^2)$ | $-$ |
| 6 | $(x^2+yz, y^2, z^2)$ | $-$ |
| 7 | $(xy, yz, xz)$ | $-$ |
| 8 | $(x^2, z^2+yz, xy)$ | $-$ |
| 9 | $(xz, x^2+z^2, yz)$ | $-$ |
| 10 | $(x^2, y^2, z^2)$ | $-$ |
| 11 | $(xy, xz, z^2)$ | $-$ |
| 12 | $(y^2, z^2, xy)$ | $-$ |
| 13 | $(x^2+yz, xy, y^2)$ | $-$ |
| 14 | $(x^2, xy, xz)$ | $-$ |
| 15 | $(x^2, y^2, xy)$ | $-$ |

Table 1:

1. *if $j^2(f)$ is of type $< 9$ or of type 10 then $f$ is always unimodular.*
2. *if $j^2(f)$ is of type 9, 11 or 12 then $f$ is either unimodular or of a higher modality.*
3. *if $j^2(f)$ is of type 13, 14 or 15 then $f$ is at least bimodular.*

*Proof.* For a proof see [7]. ∎

**Proposition 2.3.** *Table 2 contains the classification of equidimensional contact map germs.*

*Proof.* For a proof see [7] except for type 3. ∎

To obtain the classification for type 3 we prove the following lemma.

**Lemma 2.4.** *Let $A, B, C \in \langle x, y, z\rangle^3$ then*

$$(x^2-yz+A, y^2-xz+B, z^2+C) \sim_{\mathcal{K}} (x^2-yz, y^2-xz, z^2).$$

*Proof.* Note that $\mathcal{G} = \{x^2-yz+A, y^2-xz+B, z^2+C\}$ is a standard basis of the ideal $I := \langle x^2-yz+A, y^2-xz+B, z^2+C\rangle$ with respect to the local degree lexicographical ordering $D_s$ (cf. [9]). This implies that the normal form of $A$ (resp. $B$, resp. $C$) with respect to $\mathcal{G}$ is $\alpha xyz$ (resp. $\beta xyz$, resp. $\gamma xyz$) for suitable $\alpha, \beta, \gamma \in \mathbb{C}$. We obtain

$$I = \langle x^2-yz+\alpha xyz, y^2-xz+\beta xyz, z^2+\gamma xyz\rangle.$$

Especially

$$\langle x^2-yz+A, y^2-xz+B, z^2+C\rangle \sim_{\mathcal{K}} \langle x^2-yz+\alpha xyz, y^2-xz+\beta xyz, z^2+\gamma xyz\rangle.$$

| Type of $j^2(f)$ | Normal form | Conditions |
|---|---|---|
| 1 | $(x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$ | $\lambda^3 \neq -1, 0, 8$ |
| 2 | $(xy+z^p, x^2+yz, y^2+xz)$ | $p>2$ |
| 3 | $(x^2-yz, y^2-xz, z^2)$ | – |
| 4 | $(x^2+yz, xy+z^p, xz+y^q)$ | $p \geq q > 2$ |
| 5 | $(x^2+yz, xy, z^2+y^p)$ | $p>2$ |
| 6 | $(x^2+yz, y^2, z^2)$ | – |
| 7 | $(xy+z^p, yz+x^q, xz+y^r)$ | $p \geq q \geq r > 2$ |
| 8 | $(x^2+y^p+z^q, z^2+yz, xy)$ | $p \geq q > 2$ or $q=\infty$ |
| 9 | $(xz+y^3, x^2+z^2+\lambda y^3, yz)$ | $\lambda \in \mathbb{C}$ |
| | $(xz+xy^{p-1}+y^p, x^2+z^2+y^3, yz)$ | $p>3$ |
| | $(xz+y^p, x^2+z^2+y^3, yz)$ | $p>3$ |
| | $(xz+xy^p, x^2+z^2+y^3, yz)$ | $p>1$ |
| 10 | $(x^2, y^2, z^2)$ | – |
| 11 | $(xz+xy^2+y^3, yz, x^2+y^3+\lambda z^p)$ | $\lambda \in \mathbb{C}$, $p>2$ |
| | $(xz+xy^2+y^3, yz, x^2+z^p)$ | $p>2$ |
| | $(xz+y^3, yz, x^2+z^p)$ | $p>2$ |
| | $(xz+y^p, yz, x^2+y^3+z^q)$ | $p>2$, $q>2$ |
| | $(xz+xy^p, yz, x^2+y^3+z^q)$ | $p>1$, $q>2$ |
| 12 | $(y^2+x^3+x^2z, z^2, xy)$ | – |
| | $(y^2+x^3, z^2, xy)$ | – |
| | $(y^2+x^p, z^2+x^3, xy)$ | $p>2$ |
| | $(y^2+x^pz, z^2+x^3, xy)$ | $p>1$ |
| | $(y^2+x^p+x^{p-1}z, z^2+x^3, xy)$ | $p>2$ |
| | $(y^2+x^{p+2}+x^pz, z^2+x^3, xy)$ | $p>1$ |

Table 2:

Using the automorphism $x \rightsquigarrow x$, $y \rightsquigarrow y$, $z \rightsquigarrow z-\frac{1}{2}\gamma xy$ and the normal form argument as before we may assume that $\gamma=0$.
Now we consider the automorphism $\varphi$ defined by $x \rightsquigarrow ax$, $y \rightsquigarrow by$, $z \rightsquigarrow z$ with $a, b \in \mathbb{C}[[x,y,z]]^*$. We obtain that

$$\varphi(\langle x^2-yz+\alpha xyz, y^2-xz+\beta xyz, z^2\rangle) = \langle a^2x^2-byz+\alpha abxyz, b^2y^2-axz+\beta abxyz, z^2\rangle$$

$$= \langle x^2-\frac{b}{a^2}yz(1-\alpha ax), y^2-\frac{a}{b^2}xz(1-\beta by), z^2\rangle.$$

This implies that

$$\langle x^2-yz+\alpha xyz, y^2-xz+\beta xyz, z^2\rangle \sim_{\mathcal{K}} \langle x^2-\frac{b}{a^2}yz(1-\alpha ax), y^2-\frac{a}{b^2}xz(1-\beta by), z^2\rangle.$$

For $b=\frac{a^2}{1-\alpha ax}$ we obtain (using normal form as above) that

$$\langle x^2-yz+\alpha xyz, y^2-xz+\beta xyz, z^2\rangle \sim_{\mathcal{K}} \langle x^2-yz, y^2-xz+\delta xyz, z^2\rangle$$

for suitable $\delta \in \mathbb{C}$.
Again using the automorphism $\psi$ defined by $x \rightsquigarrow ax$, $y \rightsquigarrow a^2y$, $z \rightsquigarrow z$. We obtain

$$\psi(\langle x^2 - yz, y^2 - xz + \delta xyz, z^2\rangle) = \langle a^2x^2 - a^2yz, a^4y^2 - axz + \delta a^3xyz, z^2\rangle$$

$$= \langle x^2 - yz, y^2 - (\frac{1}{a^3} - \frac{\delta}{a}y)xz, z^2\rangle.$$

Using the implicit function theorem we find $a \in \mathbb{C}[[x,y,z]]^*$ with $a^3 + \delta a^2 y - 1 = 0$. This implies that

$$(x^2 - yz, y^2 - xz + \delta xyz, z^2) \sim_{\mathcal{K}} (x^2 - yz, y^2 - xz, z^2).$$

□

| Type | $c_f$ | $\mu_f$ | $\sigma_f$ |
|---|---|---|---|
| 1 | 10 | 7 | 3 |
| 2 | $p+7$ | $p+5$ | $p$ |
| 3 | 10 | 7 | 3 |
| 4 | $p+q+5$ | $p+q+3$ | $p$ |
| 5 | $p+8$ | $p+5$ | $p$ |
| 6 | 11 | 7 | 3 |
| 7 | $p+q+r+3$ | $p+q+r+1$ | $p$ |
| | $2p+7$ | $2p+3$, $q=\infty$ or $p=q$, $p,q$ even | $p+1$ |
| 8 | $p+q+6$ | $p+q+3$, $p>q$ | $p$ |
| | 13 | 9 | 4 |
| 9 | $2p+6$ | $2p+3$ | $p$ |
| | $2p+6$ | $2p+3$ | $p$ |
| | $2p+9$ | $2p+6$ | $p+2$ |
| 10 | 12 | 7 | 3 |
| 11 | $p+10$ | $p+7$ | $p$ |
| | $p+11$ | $p+7$ | 4 if $p=3$ |
| | | | $p$ if $p>3$ |
| | $p+11$ | $p+7$ | 4 if $p=3$ |
| | | | $p$ if $p>3$ |
| | $2p+q+4$ | $2p+q+1$ | $max\{p,q\}$ |
| | $2p+q+7$ | $2p+q+4$ | $p+2$ |
| 12 | 14 | 9 | 4 |
| | 15 | 9 | 4 |
| | $2p+8$ | $2p+3$ | $p+1$ |
| | $2p+11$ | $2p+6$ | $p+3$ |
| | $2p+7$ | $2p+3$ | $p$ |
| | $2p+10$ | $2p+6$ | $p+2$ |

Table 3: Invariants use for the characterization of equidimensional contact map germs:

In the following series of results $I_2$ denotes the ideal generated by the components of 2-jet of a map germ $f$ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ and $J(I_2)$ denotes the Jacobian matrix of $I_2$. The determinant of $J(I_2)$ and the Milnor number of the corresponding surface is an important ingredient in our recognition. The anonymous referee inspired us to consider the cubic plane curve defined by $det(J(I_2))$ in $\mathbb{P}^2_{\mathbb{C}}$ and compare it with the discriminant cubic curve which occurs in [14] and is drawn there in the list of nets. He claimed that this could be the same curve. This is indeed true for 9 curves in the list (Type 1, Type 2, Type 4, Type 7, Type 10, Type 11, Type 12, Type 13 and Type 15). See Table 4 and Table 5 for a comparison between the discriminant [2] cubic curves obtained by Emsalem and Iarrobino and the cubic curves defined by $det(J(I_2)) = 0$ (obtain by the authors). We give now a list of the equations defining these curves according to the different types in the classification of nets of conics.

| Normal form | $\det(J(I_2))$ | picture |
|---|---|---|
| $(x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$ | $x^3+y^3+z^3+txyz, t^3 \neq -1$ | smooth |
| $(xy, x^2+yz, y^2+xz)$ | $x^3+y^3-xyz$ | nodal |
| $(x^2-yz, y^2-xz, z^2)$ | $z^3-4xyz$ | conic+chord |
| $(x^2+yz, xy, xz)$ | $x^3-xyz$ | conic+chord |
| $(x^2+yz, xy, z^2)$ | $yz^2-2zx^2$ | conic+tangent |
| $(x^2+yz, y^2, z^2)$ | $xyz$ | triangle |
| $(xy, yz, xz)$ | $xyz$ | triangle |
| $(x^2, z^2+yz, xy)$ | $x^2y+2x^2z$ | line + double line |
| $(xz, x^2+z^2, yz)$ | $x^2z-z^3$ | 3 concurrent lines |
| $(x^2, y^2, z^2)$ | $xyz$ | triangle |
| $(xy, xz, z^2)$ | $xz^2$ | line + double line |
| $(y^2, z^2, xy)$ | $y^2z$ | line + double line |
| $(x^2+yz, xy, y^2)$ | $y^3$ | a triple line |
| $(x^2, xy, xz)$ | $x^3$ | a triple line |
| $(x^2, y^2, xy)$ | 0 | - |

Table 4:

**Proposition 2.5.** *Let $f(x,y,z)$ be a map germ from $\mathbb{C}^3 \to \mathbb{C}^3$ such that $\mu(\det J(I_2)) = 8$. If $c_f = 10$ and $\sigma_f = 3$, then $f \sim_{\mathcal{K}} (x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$.*

*Proof.* As $\mu(\det J(I_2)) = 8$, therefore we can assume $j^2(f) \sim_{\mathcal{K}} (x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$. Now if $c_f = 10$ and $\sigma_f = 3$, then the result follows immediately from the classification of Dimca and Gibson. □

[2] To obtain the discriminant cubic curves, we follow the procedure suggested by the referee: Let $f = (f_1, f_2, f_3)$ is a net of conic. Consider the general member of the net such that $G = uf_1+vf_2+wf_3$, where $u, v, w$ are new variables. Let $M(u,v,w)$ be the $3 \times 3$-symmetric matrix associated in the usual way to the quadratic form $G$. Then the equation for the corresponding discriminant cubic curve is $\det(M(u,v,w)) = 0$.

| Normal form | Discriminant | picture |
|---|---|---|
| $(x^2+\lambda yz, y^2+\lambda xz, z^2+\lambda xy)$ | $u^3+v^3+w^3+tuvw, t^3\neq -1$ | smooth |
| $(xy, x^2+yz, y^2+xz)$ | $u^3+v^3-uvw$ | nodal |
| $(x^2-yz, y^2-xz, z^2)$ | $u^3+v^3-4uvw$ | nodal |
| $(x^2+yz, xy, xz)$ | $u^3-uvw$ | conic+chord |
| $(x^2+yz, xy, z^2)$ | $u^3+v^2w$ | cuspidal |
| $(x^2+yz, y^2, z^2)$ | $u^3-4uvw$ | conic+chord |
| $(xy, yz, xz)$ | $uvw$ | triangle |
| $(x^2, z^2+yz, xy)$ | $uv^2+vw^2$ | conic+tangent |
| $(xz, x^2+z^2, yz)$ | $vw^2$ | line + double line |
| $(x^2, y^2, z^2)$ | $uvw$ | triangle |
| $(xy, xz, z^2)$ | $u^2w$ | line + double line |
| $(y^2, z^2, xy)$ | $vw^2$ | line + double line |
| $(x^2+yz, xy, y^2)$ | $u^3$ | a triple line |
| $(x^2, xy, xz)$ | 0 | - |
| $(x^2, y^2, xy)$ | 0 | - |

Table 5:

**Proposition 2.6.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that zero set, $V(det(J(I_2))$ of $det(J(I_2))$ is a nodal. If $c_f=\sigma_f+7$, then $f$ is unimodal of type $(xy+z^{\sigma_f}, x^2+yz, y^2+xz)$.*

*Proof.* Since the zero set, $V(det(J(I_2))$ of $det(J(I_2))$ is a nodal, therefore we can assume that $j^2f$ is $\mathcal{K}$-equivalent to $(xy, x^2+yz, y^2+zx)$. Now if $c_f=\sigma_f+7$, then from the classification of Dimca and Gibson, it follows that $f$ is unimodal of type $(xy+z^{\sigma_f}, x^2+yz, y^2+xz)$. □

**Proposition 2.7.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that the zero set, of $V(det(J(I_2))$ of $det(J(I_2))$ is a conic and a chord.*

1. *If $c_f=10$ and $\mu_f=7$, then $f$ is unimodal of type $(x^2-yz, y^2-xz, z^2)$.*
2. *If $c_f=\mu_f+2>10$, then $f$ is unimodal of type $(x^2+yz, xy+z^{\sigma_f}, xz+y^{c_f-\sigma_f-5})$.*

*Proof.* Since the zero set, $V(det(J(I_2))$ of $det(J(I_2))$ is a conic and a chord, therefore we can assume that $j^2f$ is either $\mathcal{K}$-equivalent to $(x^2-yz, y-xz, z^2)$ $(x^2+yz, xy, xz)$ . If $I_2$ has only one primary ideals, then $j^2f$ is $\mathcal{K}$-equivalent to $(x^2-yz, y^{-}xz, z^2)$ and if $c_f=10$ and $\mu_f=7$, then $f$ is unimodal of type $(x^2-yz, y^2-xz, z^2)$. If $I_2$ has three primary ideals, then $j^2f$ is $\mathcal{K}$-equivalent to $(x^2+yz, xy, xz)$ and if $c_f=\mu_f+2$, then $f$ is unimodal of type $(x^2+yz, xy+z^{\sigma_f}, xz+y^{c_f-\sigma_f-5})$. □

**Proposition 2.8.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line and $I_2$ has three primary ideals each of dimension one. If $c_f=\mu_f+4$, then $f$ is unimodal of type $(x^2+$*

*$y^{\sigma_f-1}+z^{\sigma_f-1}, yz+z^2, xy)$ and if $c_f=\mu_f+3$, then $f$ is unimodal of type $(x^2+y^{\sigma_f}+z^{c_f-\sigma_f-6}, yz+z^2, xy)$.*

*Proof.* Since the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line and $I_2$ has three primary ideals each of dimension one, therefore we can assume $j^2f$ is $\mathcal{K}$-equivalent to $(x^2, yz+z^2, xy)$. Then the result follows from the classification of Dimca and Gibson. □

**Proposition 2.9.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ is a triangle. If $I_2$ has only one primary ideal, then*

1. *$c_f=11$ and $\sigma_f=3$, give $f$ is unimodal of type $(x^2+yz, y^2, z^2)$.*
2. *$c_f=12$ and $\sigma_f=3$, give $f$ is unimodal of type $(x^2, y^2, z^2)$.*

**Proposition 2.10.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ is a triangle. If $I_2$ has three primary ideals, then $c_f=\mu_f+2$, give $f$ is unimodal of type $(xy+z^p, x^q+yz, xz+y^r)$ . The values of $p, q$ and $r$ can be computed by using Algorithm 1.*

**Proposition 2.11.** *Let $f$ be a map germ from $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ is a conic and a tangent line. If $I_2$ has two primary ideals, then $c_f=\mu_f+3$, give $f$ is unimodal of type $(x^2+yz, xy, z^2+y^{\sigma_f})$ .*

Proofs of Proposition 2.9, 2.10 and Proposition 2.11 are straight forward.

**Algorithm 1.** ***Input:*** *An ideal $f$ of type $(xy+z^p, yz+x^q, xz+y^r)$.*
***Output:*** *Integer vector $v$, the values of $p, q$ and $r$.*

1: *Compute a list $L =$: tangentcone$(f)$*
2: *Compute $n =$: size$(L[1])$;*
3: *return $(deg(L[1][n])-2, deg(L[1][n-1])-1, deg(L[1][n-2])-1)$.*

Let $g$ be a map germ of type $(xy+z^p, yz+x^q, xz+y^r)$. Then by using the following code, we can compute the values of $p, q$ and $r$ for $g$.

```
ring R=0,(x,y,z),ds;

ideal g=x2+2xy+y2+xz+yz+x5+5x4z+10x3z2+10x2z3+5xz4+z5,
        x2+xy+xz+yz+x4+4x3y+6x2y2+4xy3+y4+4x3z+12x2yz+
        12xy2z+4y3z+6x2z2+12xyz2+6y2z2+4xz3+4yz3+z4,x2+
        xy+2xz+yz+z2+x3+3x2y+3xy2+y3;
> expValue(g);
[1]:
   5
[2]:
   4
[3]:
   3
```

Above computations show that $g$ is contact equivalent to $(xy + z^5, yz + x^4, xz + y^3)$.

The following lemma help us to differentiate the germs $(xz+xy^{p-1}+y^p, x^2+z^2+y^3, yz)$ and $(xz + y^p, x^2 + z^2 + y^3, yz)$.

**Lemma 2.12.** *If $f = (xz+xy^{p-1}+y^p, x^2+z^2+y^3, yz)$ and $g = (xz+y^p, x^2+z^2+y^3, yz)$, where $p > 3$ then $y^{p+1} \in I_g$ but $y^{p+1} \notin I_f$.*

*Proof.* Compute the standard bases for $I_f$ and $I_g$, then the result follows immediately. □

*Remark* 2.13. Let $f = (xz+xy^{p-1}+y^p, x^2+z^2+y^3, yz)$ and $g = (xz+y^p, x^2+z^2+y^3, yz)$, where $p > 3$. If $I$ is an ideal such that $I \sim_{\mathcal{K}} I_f$ then there exist two independent elements $f_1, f_2$ with $ord(f_1) = ord(f_2) = 1$ such that $f_1^{p+1}, f_2^{p+1} \in I$. Similarly if $J$ is an ideal such that $J \sim_{\mathcal{K}} I_g$ then there exist three independent elements $f_1, f_2, f_3$ with $ord(f_1) = ord(f_2) = ord(f_3) = 1$ such that $f_1^{p+1}, f_2^{p+1}, f_3^{p+1} \in J$.

The number of independent elements of order 1, belongs to the ideals $I_f$ and $I_g$ can be computed by using the following algorithm.

**Algorithm 2.** ***Input:*** *Ideal $I = f, g, h$ and an integer $p > 0$.*
***Output:*** *Ideal $J = h_1, h_2, h_3$, the required normal form.*

1: *Assume $f = ax + by + cz$;*
2: *Compute $f_1, \dots, f_k \in \mathbb{K}[a, b, c]$, the coefficients of $f^{p+1}$ mod $I+ < x, y, z >^{p+2}$;*
3: *Compute $n = dim(\frac{\mathbb{K}[a,b,c]}{<f_1,\dots,f_k>})$;*
4: *return $(n + 1)$.*

Let $f$ and $g$ are two map germs of type $(xz + xy^3 + y^4, x^2 + z^2 + y^3, yz)$ and $(xz + y^4, x^2 + z^2 + y^3, yz)$ respectively. Then by using the following code, we can compute the number of independent elements of order 1 for $f$ and $g$.

```
ring R=0,(x,y,z),ds;

ideal f=x2+xy+2xz+yz+z2+2x4+8x3y+12x2y2+8xy3+2y4+x3z+
        3x2yz+3xy2z+y3z,2x2+2xy+y2+4xz+2yz+2z2+x3+3x2y
       +3xy2+y3,x2+xy+xz+yz;
> testp(f,4);
2

ideal g=x2+xy+2xz+yz+z2+x4+4x3y+6x2y2+4xy3+y4,2x2+2xy+
        y2+4xz+2yz+2z2+x3+3x2y+3xy2+y3,x2+xy+xz+yz;
> testp(g,4);
3
```

**Proposition 2.14.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $j^2 f$ is $\mathcal{K}$-equivalent to $(xz, x^2 + z^2, yz)$. Then $f$ is unimodal if and only if $f \sim_{\mathcal{K}} (xz + xy^p A(y) + y^q B(y), x^2 + z^2 + y^3 C(y), yz)$ with $A(0), B(0), C(0) \neq 0$ and $p \geq 2$, $q \geq 3$.*

*Proof.* This is a consequence of Proposition 2.1 (ii) [6] and the classification. ∎

**Proposition 2.15.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are three concurrent lines. If $I_2$ has two primary ideals and $f \sim_{\mathcal{K}} (xz + xy^p A(y) + y^q B(y), x^2 + z^2 + y^3 C(y), yz)$, then*

1. *$c_f = 13$ and $\sigma_f = 4$ give, either $f$ is unimodal of type $(xz + y^3, x^2 + z^2 + \lambda y^3, yz)$ or $(xz + xy^2, x^2 + z^2 + y^3, yz)$.*
2. *$c_f = 2\sigma_f + 6$ gives, either $f$ is unimodal of type $(xz + xy^{\sigma_f - 1} + y_f^{\sigma}, x^2 + z^2 + y^3, yz)$ or $(xz + y_f^{\sigma}, x^2 + z^2 + y^3, yz)$.*
3. *$c_f = 2\sigma_f + 5$ and $\sigma_f > 4$ give, $f$ is unimodal of type $(xz + xy^{\sigma_f - 2}, x^2 + z^2 + y^3, yz)$.*

*Proof.* Since the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are three concurrent lines, then $j^2 f$ is $\mathcal{K}$-equivalent to $(x^2 + yz, y^2, z^2)$ or $(xy, yz, xz)$ or $(xz, x^2 + z^2, yz)$ or $(x^2, y^2, z^2)$. As $I_2$ has two primary ideals, therefore we can assume $j^2 f$ is $\mathcal{K}$-equivalent to $(xz, x^2 + z^2, yz)$. If $c_f = 13$ and $\sigma_f = 4$ then either $f$ is unimodal of type $(xz + y^3, x^2 + z^2 + \lambda y^3, yz)$ or $(xz + xy^2, x^2 + z^2 + y^3, yz)$. These two types can be differentiated by computing the milnor number, $\mu_f$ of $f$, i.e., if $\mu_f = 9$, then $f$ is unimodal of type $(xz + y^3, x^2 + z^2 + \lambda y^3, yz)$ and if $\mu_f = 10$, then $f$ is unimodal of type $(xz + xy^2, x^2 + z^2 + y^3, yz)$.
Now if $c_f = 2\sigma_f + 6$, then $f$ is unimodal of type $(xz + xy^{\sigma_f - 1} + y_f^{\sigma}, x^2 + z^2 + y^3, yz)$ or $(xz + y_f^{\sigma}, x^2 + z^2 + y^3, yz)$. These two germs can be differentiated by using Lemma 2.12. Moreover if $c_f = 2\sigma_f + 5$ and $\sigma_f > 4$ , then $f$ is unimodal of type $(xz + xy^{\sigma_f - 2}, x^2 + z^2 + y^3, yz)$. ∎

To differentiate the germs $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^3)$ and $(xz + y^3, yz, x^2 + y^3 + z^3)$, we use the following lemma.

**Lemma 2.16.** *Let $f = (xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^3)$ and $g = (xz + y^3, yz, x^2 + y^3 + z^3)$ be map germs from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ . Then $y^4 \notin I_f$ but $y^4 \in I_g$.*

*Proof.* This can be easily seen by computing standard bases of $I_f$ and $I_g$. ∎

*Remark* 2.17. Let $f = (xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^3)$ and $g = (xz + y^3, yz, x^2 + y^3 + z^3)$. If $I$ is an ideal such that $I \sim_{\mathcal{K}} I_f$ then there exist only one independent element $f_1$ with $ord(f_1) = 1$ such that $f_1^4 \in I$. Similarly if $J$ is an ideal such that $J \sim_{\mathcal{K}} I_g$ then there are two independent element $f_1, f_2$ with $ord(f_1) = ord(f_2) = 1$ such that $f_1^4, f_2^4 \in J$.

The following lemma help us to compute the values of exponents $p$, $q$ and the type of map germs $(xz + y^p, yz, x^2 + y^3 + z^q)$, $(xz + xy^p, yz, x^2 + y^3 + z^q)$, $(xz + xy^2 + y^3, yz, x^3 + y^3, \lambda z^p)$.

**Lemma 2.18.** *Let $f = (xz + y^p, yz, x^2 + y^3 + z^q)$, $p > 2, q > 2$, $g = (xz + xy^p, yz, x^2 + y^3 + z^q)$, $p > 1, q > 2$ and $h = (xz + xy^2 + y^3, yz, x^3 + y^3, \lambda z^p)$ then $\mu(det(J(I_f))) = 2p + 2q + 4$, $\mu(det(J(I_g))) = 2p + 2q + 7$ and $\mu(det(J(I_h))) = 2p + 10$.*

*Proof.* We use standard bases ([9]), for the proof. We fix the local degree ordering $d_s$. Note that

$$det(J(I_f)) = qz^{q+1} - 3zy^3 + 2pxy^p - 2x^2y;$$

and therefore

$$J(det(J(I_f)) =< 2py^p - 4xz, -9zy^2 + 2p^2xy^{p-1}, q(q+1)z^q - 3y^3 - 2x^2 > .$$

If $3 = p \leq q$ then leading ideal of standard basis of $J(det(J(I_f))$ is

$$< x^2, xz, xy^2, y^3z, y^2z^2, y^5, z^{q+2} > .$$

This gives, the monomials do not belong to the leading ideal are

$$1, x, y, y^2, y^3, y^4, z, z^2, \ldots, z^{q+1}, xy, yz, yz^2, \ldots, yz^{q+1}, y^2z.$$

This implies $\mu(det(J(I_f))) = 2q + 10 = 2p + 2q + 4$.
If $3 < p \leq q$ then leading ideal of standard basis of $J(det(J(I_f))$ is

$$< x^2, xz, xy^p, y^2z, y^{p+2}, z^{q+2} > .$$

This gives, the monomials do not belong to the leading ideal are

$$1, x, y, y^2, \ldots, y^{p+1}, z, z^2, \ldots, z^{q+1}, xy, xy^2, \ldots, xy^{p-1}, yz, yz^2, \ldots, yz^{q+1}.$$

This implies $\mu(det(J(I_f))) = 2 + p + 1 + q + 1 + p - 1 + q + 1 = 2p + 2q + 4$.
If $3 \leq q < p$ then leading ideal of standard basis of $J(det(J(I_f))$ is

$$< x^2, xz, y^2z, y^{p+2}, z^{q+1} > .$$

This gives, the monomials do not belong to the leading ideal are

$$1, x, y, y^2, \ldots, y^{p+1}, z, z^2, \ldots, z^q, xy, xy^2, \ldots, xy^{p+1}, yz, yz^2, \ldots, yz^q.$$

This implies $\mu(det(J(I_f))) = 2 + p + 1 + q + p + 1 + q = 2p + 2q + 4$.
In a similar way, we can show that $\mu(det(J(I_g))) = 2p+2q+7$ and $\mu(det(J(I_h))) = 2p+10$.
∎

**Proposition 2.19.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $j^2f$ is $\mathcal{K}$-equivalent to $(xz, yz, x^2)$. Then $f$ is unimodal if and only if $f \sim_{\mathcal{K}} (xz + xy^pA(y) + y^3B(y), yz, x^2 + y^rC(y) + z^sD(z))$ with $A(0), B(0), C(0), D(0) \neq 0$ and $p \geq 2$, $r, s \geq 3$ or $f \sim_{\mathcal{K}} (xz+xy^pA_1(y)+y^qB_1(y), yz, x^2+y^3C_1(y)+D_1(z)z^s)$ with $A_1(0), B_1(0), C_1(0), D_1(0) \neq 0$ and $p \geq 2$, $q, s \geq 3$.*

*Proof.* This is a consequence of Proposition 2.2 (ii) [6] and the classification. ∎

**Proposition 2.20.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line. If $I_2$ has two primary ideals each of dimension one and $f \sim_{\mathcal{K}} (xz+xy^pA(y)+y^3B(y), yz, x^2+y^rC(y)+z^sD(z))$ or $f \sim_{\mathcal{K}} (xz + xy^pA_1(y) + y^qB_1(y), yz, x^2 + y^3C_1(y) + D_1(z)z^s)$. Then*

1. *$c_f = 13$ and $\sigma_f = 3$ give, $f$ is unimodal of type* $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^3)$ *or* $(xz + y^3, yz, x^2 + y^3 + z^3)$.
2. *$c_f = \mu_f + 4$ gives, either $f$ is unimodal of type* $(xz + xy^2 + y^3, yz, x^2 + z^{c_f - 11})$ *or* $(xz + y^3, yz, x^2 + z^{c_f - 11})$.
3. *$c_f \geq 14$ and $c_f = \mu_f + 3$ give, $f$ is unimodal of type* $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^{c_f - 10})$ *or* $(xz + y^{\sigma_f}, yz, x^2 + y^3 + z^{c_f - 2\sigma_f - 4})$ *or* $(xz + y^{\frac{c_f - \sigma_f - 4}{2}}, yz, x^2 + y^3 + z^{\sigma})$ *or* $(xz + xy^{\sigma_f - 2}, yz, x^2 + y^3 + z^{c_f - 2\sigma_f - 3})$.

*Proof.* Since the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line and $I_2$ has two primary ideals each of dimension one, then we can assume $j^2 f$ is $\mathcal{K}$-equivalent to $(xz, yz, x^2)$.
If $c_f = 13$ and $\sigma_f = 3$, then $f$ is unimodal of type $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^3)$ or $(xz + y^3, yz, x^2 + y^3 + z^3)$. These two germs can be differentiated by using Lemma 2.16.
If $c_f = \mu_f + 4$,, then either $f$ is unimodal of type $(xz + xy^2 + y^3, yz, x^2 + z^{c_f - 11})$ or $(xz + y^3, yz, x^2 + z^{c_f - 11})$. These two types can be differentiated by using Algorithm 3.
Now if $c_f > 13$ and $c_f = \mu_f + 3$, then $f$ is unimodal of type $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^p)$ or $(xz + y^p, yz, x^2 + y^3 + z^q)$ or $(xz + xy^p, yz, x^2 + y^3 + z^q)$. These germs can be differentiated by using Lemma 2.18, i.e., if $\mu(det(J(I_f))) = 2(c_f - \sigma_f) - 3$, then $f$ is unimodal of type $(xz + xy^p, yz, x^2 + y^3 + z^q)$ with $p = \sigma_f - 2$, $q = c_f - 2\sigma_f - 3$ and if $\mu(det(J(I_f))) = 2(c_f - \sigma_f - 2)$, then $f$ is of type $(xz + y^p, yz, x^2 + y^3 + z^q)$ with $p = \sigma_f$, $q = c_f - 2\sigma_f - 4$ and if $\mu(det(J(I_f))) = c + \sigma_f$, then $f$ is of type $(xz + xy^2 + y^3, yz, x^2 + y^3 + \lambda z^p)$ with $p = c_f - 10$ or $(xz + y^p, yz, x^2 + y^3 + z^q)$ with $p = \frac{c_f - \sigma_f - 4}{2}$, $q = \sigma_f$. These two types can be differentiated by using a similar algorithm as Algorithm 3. ∎

**Lemma 2.21.** *Let $f = (y^2 + x^3 + x^2 z, z^2, xy)$ and $g = (y^2 + x^3, z^2 + x^3, xy)$, then $z^2 \in I_f$ but $z^2 \notin I_g$.*

*Proof.* This can be easily seen by computing standard bases of $I_f$ and $I_g$. ∎

*Remark* 2.22. Let $f = (y^2 + x^3 + x^2 z, z^2, xy)$ and $g = (y^2 + x^3, z^2 + x^3, xy)$. If $I$ is an ideal such that $I \sim_{\mathcal{K}} I_f$ then there exist only one independent element $f_1$ with $ord(f_1) = 1$ such that $f_1^2 \in I$. Similarly if $J$ is an ideal such that $J \sim_{\mathcal{K}} I_g$ then there exist no independent element $f_1$ with $ord(f_1) = 1$ such that $f_1^2 \in J$.

**Algorithm 3.** ***Input:*** *Ideal $I = f, g, h$ and two integers $p, \sigma > 0$.*
***Output:*** *Ideal $I = h_1, h_2, h_3$, the required normal form.*

1: *Define a map $\varphi$, $\varphi(x) = \sum_{1 \leq i+j+k \leq p-2} a_{ijk} x^i y^j z^k$, $\varphi(y) = \sum_{1 \leq i+j+k \leq p-2} b_{ijk} x^i y^j z^k$, $\varphi(z) = \sum_{1 \leq i+j+k \leq p-2} c_{ijk} x^i y^j z^k$ with parameters up to the degree $p - 2$.*
2: *Define a $3 \times 3$-matrix $T$ with polynomial entries of degree $p - 3$ and parameters as coefficients, i.e., $T = (t_{ij})$, $t_{ij} = \sum_{0 \leq l+m+n \leq p-3} a_{ij,l,m,n} x^l y^m z^n$, $a_{ij,l,m,n}$ are parameters.*
3: *Define the expected normal form $(xz + y^3, yz, x^2 + z^{\sigma})$.*
4: *Define* $J := \varphi(I) - T \begin{pmatrix} xz + y^3 \\ yz \\ x^2 + z^{\sigma} \end{pmatrix}$.

*Let $K$ be the ideal generated by all the coefficients of $J$ with respect to $x, y, z$.*
*Compute $S$, the standard basis of $K$.*
***if** $S =< 1 >$ **then***
    *return*$((xz + xy^2 + y^3, yz, x^2 + z^\sigma))$;
***if** $S \neq< 1 >$ **then***
    *return* $((xz + y^3, yz, x^2 + z^\sigma))$.

**Proposition 2.23.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $j^2f$ is $\mathcal{K}$-equivalent to $(y^2, z^2, xy)$. Then $f$ is unimodal if and only if $f \sim_{\mathcal{K}} (y^2 + x^3A(x) + x^q zB(x), z^2 + x^r C(x), xy)$ with $A(0), B(0), C(0) \neq 0$ and $q \geq 2$, $r \geq 3$ or $f \sim_{\mathcal{K}} (y^2 + x^p A_1(x) + x^q zB_1(x), z^2 + x^3C_1(x), xy)$ with $A_1(0), B_1(0), C_1(0) \neq 0$ and $p, q \geq 2$.*

*Proof.* This is a consequence of Proposition 2.3 (ii) [6] and the classification. □

**Proposition 2.24.** *Let $f$ be a map germ from $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line. If $I_2$ has two primary ideals, one of dimension 0 and one of dimension 1 and either $f \sim_{\mathcal{K}} (y^2 + x^3A(x)+x^q zB(x), z^2+x^r C(x), xy)$ or $f \sim_{\mathcal{K}} (y^2+x^p A_1(x)+x^q zB_1(x), z^2+x^3C_1(x), xy)$. Then*

1. *$c_f = 13$ and $\sigma_f = 3$ give, $f$ is unimodal of type $(y^2 + x^3 + x^2z, z^2 + x^3, xy)$.*
2. *$c_f = 14$ and $\sigma_f = 4$ give, $f$ is unimodal of type $(y^2 + x^3 + x^2z, z^2, xy)$ or $(y^2 + x^3, z^2 + x^3, xy)$ or $(y^2 + x^4 + x^2z, z^2 + x^3, xy)$.*
3. *$c_f = 15$ and $\sigma_f \geq 4$ give, $f$ is unimodal of type $(y^2 + x^3, z^2, xy)$ or $(y^2 + x^2z, z^2 + x^3, xy)$ or $(y^2 + x^4 + x^3z, z^2 + x^3, xy)$.*
4. *$c_f > 15$ and $c_f = 2\sigma_f + 5$ give, $f$ is unimodal of type $(y^2 + x^{\sigma_f -3}z, z^2 + x^3, xy)$ .*
5. *$c_f > 15$ and $c_f = 2\sigma_f + 6$ give, either $f$ is unimodal of type $(y^2 + x^{\sigma_f -1}, z^2 + x^3, xy)$ or $(y^2 + x^{\sigma_f} + x^{\sigma_f -2}z, z^2 + x^3, xy)$.*
6. *$c_f > 15$ and $c_f = 2\sigma_f + 7$ give, $f$ is unimodal of type $(y^2 + x^{\sigma_f} + x^{\sigma_f -1}z, z^2 + x^3, xy)$.*

*Proof.* Since the zero set, $V(det(J(I_2)))$ of $det(J(I_2))$ are two lines one of them is a double line and $I_2$ has two primary ideals, one of dimension 0 and one of dimension 1, therefore we can assume $j^2f$ is $\mathcal{K}$-equivalent to $(y^2, z^2, xy)$.
If $c_f = 14$ and $\sigma_f = 4$, then $f$ is unimodal of type $(y^2 + x^3 + x^2z, z^2, xy)$ or $(y^2 + x^3, z^2 + x^3, xy)$ or $(y^2 + x^4 + x^2z, z^2 + x^3, xy)$. Now if $\mu_f = 10$, then $f$ is of type $(y^2+x^4+x^2z, z^2+x^3, xy)$ and if $\mu_f = 9$, then the remaining two types can be differentiated by using Lemma 2.21.
If $c_f = 13$ and $\sigma_f = 3$ then $f$ is unimodal of type $(y^2 + x^3 + x^2z, z^2 + x^3, xy)$. Now if $c_f = 15$ and $\sigma_f = 5$, then $f$ is unimodal of type $(y^2 + x^2z, z^2 + x^3, xy)$ and if $c_f = 15$ and $\sigma_f = 4$, then $f$ is unimodal of type $(y^2 + x^3, z^2, xy)$ or $(y^2 + x^4 + x^3z, z^2 + x^3, xy)$. These two types can be differentiated by computing the Milnor number, i.e., if $\mu_f = 9$, then $f$ is unimodal of type $(y^2 + x^3, z^2, xy)$ and if $\mu_f = 11$, then $f$ is unimodal of type $(y^2 + x^4 + x^3z, z^2 + x^3, xy)$.
If $c_f > 15$ and $c_f = 2\sigma_f + 5$, then $f$ is unimodal of type $(y^2 + x^{\sigma_f -3}z, z^2 + x^3, xy)$ and if

$c_f > 15$ and $c_f = 2\sigma_f + 7$, then $f$ is unimodal of type $(y^2 + x^{\sigma_f} + x^{\sigma_f - 1}z, z^2 + x^3, xy)$ . Now if $c_f > 15$ and $c_f = 2\sigma_f + 6$, then either $f$ is unimodal of type $(y^2 + x^{\sigma_f - 1}, z^2 + x^3, xy)$ or $(y^2 + x^{\sigma_f} + x^{\sigma_f - 2}z, z^2 + x^3, xy)$. These two types can be differentiated by computing the Milnor number, i.e., if $\mu_f = 2\sigma_f + 1$, then $f$ is unimodal of type $(y^2 + x^{\sigma_f - 1}, z^2 + x^3, xy)$ and if $\mu_f = 2\sigma_f + 2$, then $f$ is unimodal of type $(y^2 + x^{\sigma_f} + x^{\sigma_f - 2}z, z^2 + x^3, xy)$. □

## 3 Singular examples

We have implemented the Algorithm in the computer algebra system SINGULAR [8]. Code can be downloaded from https://www.mathcity.org/files/ahsan/Procedure-Eqdim.txt. We give some examples.

```
ring R=0,(x,y,z),ds;

ideal f=16x2+32xy+16y2+16xz+16yz+4z2+16x3+68x2y
       +96xy2+45y3+28x2z+80xyz+57y2z+16xz2+23yz2+3z3,
       4x2+8xy+4y2+8xz+8yz+4z2, 8x2+20xy+12y2+8xz+10yz+2z2;
eqdimMapgerms(f);
f is of type (y2+x3+x2z,z2,xy)


ideal f=1536x2+3024xy+1489y2+2128xz+2094yz+737z2,
        960x2+1880xy+920y2+1352xz+1324yz+476z2,
        576x2+1152xy+576y2+768xz+768yz+256z2+64000x3
        +192000x2y+192000xy2+64000y3+134400x2z+268800xyz
        +134400y2z+94080xz2+94080yz2+21952z3;
eqdimMapgerms(f);
f is of type (x2+yz,xy, z2+y3)


ideal f=xy+y2+xz+2yz+z2+y4, x2+2xy+2y2+2xz+4yz+2z2+y3, y2+yz;
eqdimMapgerms(f);
f is of type (xz+y4, x2+z2+y3, yz)
```

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