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Lower bound for the diameter of planar Brownian motion

by
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Abstract

Let $\mathrm{B}(t)$ be a standard planar Brownian motion and $\mathrm{r}(\theta)$ be the diameter of the projection of $\mathrm{B}([0,1])$ on the line generated by the unit vector $\mathrm{e}_\theta = (\cos\theta, \sin\theta)$, where $0 \le \theta \le \pi$. In this short note, we find the common cumulative distribution function F of the random variables $\mathrm{r}(\theta)$. Namely, we prove that

$$F(x) = 8\sum_{n=1}^{\infty}\left(\frac{1}{x^2} + \frac{1}{(2n-1)^2\pi^2}\right)\exp\left(-\frac{(2n-1)^2\pi^2}{2x^2}\right),$$

for every $x > 0$. As immediate consequence, lower bound for the expected diameter of the set $\mathrm{B}([0,1])$, better than known, is obtained. Namely, it is known that $\mathbb{E}\mathrm{d} \ge 1.601$, where d is the diameter of the set $\mathrm{B}([0,1])$. In this note we show $\mathbb{E}\mathrm{d} \ge 1.856$.

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1 Introduction

1.1. Brownian motion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X is a random variable, then the expectation of X will be denoted by $\mathbb{E}X$ with respect to the given probability $\mathbb{P}$. Suppose that B_1 and B_2 are two independent standard linear Brownian motions. Then the stochastic process B given by

$$\mathrm{B} = (\mathrm{B}_1, \mathrm{B}_2),$$

is called a standard two-dimensional Brownian motion or equivalently a standard planar Brownian motion. For a complete information related to the theory of Brownian motion we refer to the monograph [3].

1.2. The main result. Let $\mathrm{B}(t)$, where $t \in [0,1]$, be a standard planar Brownian motion. Following [2] for $0 \le \theta \le \pi$ we introduce the parametrized range function r given by

$$\mathrm{r}(\theta) = \sup_{t\in[0,1]}(\mathrm{B}(t)\cdot \mathrm{e}_\theta) - \inf_{t\in[0,1]}(\mathrm{B}(t)\cdot \mathrm{e}_\theta),$$

with e_θ being the unit vector $(\cos\theta, \sin\theta)$. We are now ready to state the main result of this note.

Theorem 1. *Let F be the common cumulative distribution function of the random variables $\mathrm{r}(\theta)$, where $0 \le \theta \le \pi$. Then*

$$F(x) = 8\sum_{n=1}^{\infty}\left(\frac{1}{x^2} + \frac{1}{(2n-1)^2\pi^2}\right)\exp\left(-\frac{(2n-1)^2\pi^2}{2x^2}\right),$$

for $x > 0$.

2 Proof of the main result

This section is devoted to the proof of Theorem 1. First, we note that it is well known that density of $\mathrm{r}(\theta)$, where $0 \le \theta \le \pi$, is given explicitly as

$$f(x) = \frac{8}{\sqrt{2\pi}} \sum_{n=1}^{\infty} (-1)^{n-1} n^2 e^{-\frac{n^2 x^2}{2}}, \tag{2.1}$$

for $x > 0$ (see [1, 2]). Second, we will use some basic properties of the Theta Function. We recall that for $z \in \mathbb{C}$ and $\operatorname{Im} \tau > 0$, Jacobi Theta Function is defined by

$$\Theta(z;\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\mathrm{i}\pi\tau n^2 + 2\pi \mathrm{i} n z\right).$$

The following well known identity related to Jacobi Theta Function is a consequence of the Poisson Summation Formula

$$\Theta(z;\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\mathrm{i}\pi(z+n)^2}{\tau}\right). \tag{2.2}$$

As a special case of a previous identity (2.2), when $z = -\frac{1}{2}$ and $\tau = \frac{2\mathrm{i}x^2}{\pi}$, we find

$$\sum_{n\in\mathbb{Z}} (-1)^n e^{-2n^2x^2} = \frac{\sqrt{2\pi}}{2x} \sum_{n\in\mathbb{Z}} e^{-\frac{(2n-1)^2\pi^2}{8x^2}}, \quad x > 0,$$

or equivalently

$$1 - 2\sum_{n=1}^{\infty} (-1)^{n-1} e^{-2n^2x^2} = \frac{\sqrt{2\pi}}{x} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{8x^2}}, \quad x > 0. \tag{2.3}$$

Proof of Theorem 1. We have

$$F(x) = \mathbb{P}\{\mathrm{r}(\theta) \le x\} = \int_0^x f(t)dt.$$

Let

$$L(x) = 1 - 2\sum_{n=1}^{\infty} (-1)^{n-1} e^{-2n^2x^2}, \quad x > 0.$$

By straightforward calculation and formula (2.1), we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{2}{x} L'\left(\frac{x}{2}\right), \quad x > 0.$$

On the other hand

$$L(x) \overset{(2.3)}{=} \frac{\sqrt{2\pi}}{x} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{8x^2}}, \quad x > 0,$$

which implies

$$\frac{L(x)}{x^a} \to 0 \text{ when } x \to 0^+,$$

for every $a \in \mathbb{R}$. By using linear change of variable and integration by parts, we get

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^x \frac{1}{\sqrt{2\pi}} \frac{2}{t} L'\left(\frac{t}{2}\right) \mathrm{d}t \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\frac{x}{2}} \frac{1}{t} L'(t)\, \mathrm{d}t \\
&= \frac{2}{\sqrt{2\pi}} \left(\frac{2}{x} L\left(\frac{x}{2}\right) + \int_0^{\frac{x}{2}} \frac{1}{t^2} L(t)\, \mathrm{d}t \right) \\
&= \frac{2}{\sqrt{2\pi}} \left(\frac{4\sqrt{2\pi}}{x^2} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{2x^2}} + \sqrt{2\pi} \int_0^{\frac{x}{2}} \frac{1}{t^3} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{8t^2}} \mathrm{d}t \right) \\
&= 2 \left(\frac{4}{x^2} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{2x^2}} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \int_0^{\frac{x}{2}} \mathrm{d}\left(e^{-\frac{(2n-1)^2\pi^2}{8t^2}} \right) \right) \\
&= 8 \left(\frac{1}{x^2} \sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{2x^2}} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\pi^2} e^{-\frac{(2n-1)^2\pi^2}{2x^2}} \right) \\
&= 8 \sum_{n=1}^{\infty} \left(\frac{1}{x^2} + \frac{1}{(2n-1)^2\pi^2} \right) e^{-\frac{(2n-1)^2\pi^2}{2x^2}}.
\end{aligned}$$

This completes the proof. □

3 Lower bound for the diameter of planar Brownian motion

Let d be the diameter of the set $\mathrm{B}([0,1])$, that is

$$\mathrm{d} = \operatorname{diam} \mathrm{B}([0,1]) = \sup\{\|\mathrm{B}(t) - \mathrm{B}(s)\| : t, s \in [0,1]\},$$

where $\|\cdot\|$ denotes the two-dimensional Euclidean norm. In [2] McRedmond and Xu proved that

$$1.601 \le \mathbb{E}\mathrm{d} \le 2.355.$$

Among other things, they used the following formula (see [2, Lemma 6])

$$\mathrm{d} = \sup_{0 \le \theta \le \pi} \mathrm{r}(\theta),$$

and the fact that random variables $\mathrm{r}(0)$ and $\mathrm{r}(\pi/2)$ are independent. As immediate consequence of Theorem 1 we provide better lower bound for the expected diameter of the set $\mathrm{B}\,([0,1])$. Namely, we have the following result.

Corollary 1. *Let* d *be the diameter of the set* $\mathrm{B}\,([0,1])$*, where* B *is a standard planar Brownian motion. Then*

$$\mathbb{E}\mathrm{d} \geq 1.856.$$

Proof. The independent random variables $\mathrm{r}(0)$ and $\mathrm{r}(\pi/2)$ have the common cumulative distribution function F as in Theorem 1. Since the distribution function of $\max\{\mathrm{r}(0), \mathrm{r}(\pi/2)\}$ is F^2 we conclude

$$\mathbb{E}\mathrm{d} \geq \mathbb{E}\max\{\mathrm{r}(0), \mathrm{r}(\pi/2)\} = \int_0^{\infty} \left(1 - F(x)^2\right) \mathrm{d}x \geq \int_0^{6} \left(1 - F(x)^2\right) \mathrm{d}x.$$

By using *Mathematica* we find that $F(6) = 0.9999999921$ and since F is an increasing function we obtain a good approximation by replacing above mentioned improper integral with appropriate proper integral, thus providing greater accuracy when calculating using *Mathematica*. Hence, by using *Mathematica* again, we get

$$\int_0^{6} \left(1 - F(x)^2\right) \mathrm{d}x \approx 1.8562.$$

Finally, we obtain $\mathbb{E}\mathrm{d} \geq 1.856$. □

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References

[1] W. Feller, The asymptotic distribution of the range of sums of independent random variables, *Ann. Math. Statist.*, **22**, 427–432 (1951).

[2] J. McRedmond, C. Xu, On the expected diameter of planar Brownian motion, *Stat. Prob. Lett.*, **130**, 1–4 (2017).

[3] P. Mörters, Y. Peres, *Brownian Motion*, Cambridge University Press (2010).

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