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**Krein-Rutman operators and a variant of Banach contraction principle in ordered Banach spaces**

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**Abstract**

Let $E$ be a real Banach space, $K$ be a cone in $E$ and $L$ be a linear positive and compact self mapping defined on $E$. The operator $L$ is said to be a Krein-Rutman operaor if it has a positive characteristic value $\lambda_L$ such that for all $h \in K \setminus \{0_E\}$, the nonhomogeneous equation $u - \theta Lu = h$ has no positive solution if $\theta \geq \lambda_L$ and a unique positive solution if $\theta \in (0, \lambda_L)$. M. G. Krein and M. A. Rutman have proved that if $L$ is strongly positive then $L$ is a Krein-Rutman operator with $\lambda_L = 1/r(L)$. Here $r(L)$ refers to the spectral radius of $L$.

The main goal of this article is to provide sufficient conditions making of $L$ a Krein-Rutman operator. The particular case where $E$ is a Hilbert space and $L$ is a self-adjoint operator is examined.

We also present in this article a version of the Banach contraction principle adapted to the case where the cone $K$ is normal and minihedral, making of the Banach space $E$ a Riesz space.

**Key Words**: Cones, positive operators, Krein-Rutman theory, Banach contraction principle.
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# 1 Introduction

Throughout this article, $E$ is a real Banach space. The standard notations $E^*$ and $\mathcal{L}(E)$ refer respectively to the topological dual of $E$ and the set of all linear bounded self-mapping defined on $E$. Let $K$ be a cone in $E$ inducing the order $\preceq_K$ on $E$ and let $L$ be a positive compact operator in $\mathcal{L}(E)$. We are mainly concerned in this work with the solvability in $K \setminus \{0_E\}$ of the abstract equation

$$u - \mu Lu = h \tag{1.1}$$

where $\mu$ is a positive real parameter and $h \in K$.

The set of positive characteristic value (pcv for short) of the operator $L$ consists of the set of positive real numbers $\mu$ for which Equation (1.1) for $h = 0_E$ is solvable in $K \setminus \{0_E\}$. The Krein-Rutman theory concerns the case of Equation (1.1) where the operator $L$ is strongly positive and it states that:

**a)** The operator $L$ has a unique pcv $\mu_L = 1/r(L)$, where $r(L)$ denotes the spectral radius of $L$.

**b)** Equation (1.1) within $h \in K \setminus \{0_E\}$, is uniquely solvable in $K$ for $\mu < \mu_L$ and has no solution in $K$ for $\mu \geq \mu_L$.

**c)** As a pcv of $L$, $\mu_L$ is algebraically simple.

**d)** $\mu_L$ is a simple pcv of the adjoint operator $L^*$ of $L$.

For a detailed presentation on the Krein-Rutman theory we refer the reader to [7] and [12].

It is proved in [3] (Proposition 3.16) that if the operator $L$ is strongly positive, then its unique pcv has the property:

**e)**

$$\begin{aligned}\mu_L = \lambda^-_{L,K} &= \sup\{\lambda \geq 0 : \ \exists u \succ_K 0_E \ \text{ such that } \ \lambda L u \preceq_K u\} \\ = \lambda^+_{L,K} &= \inf\{\lambda \geq 0 : \ \exists u \succ_K 0_E \ \text{ such that } \ \lambda L u \succeq_K u\}.\end{aligned}$$

The constants $\lambda^-_{L,K}$ and $\lambda^+_{L,K}$ have been introduced in [1] and [3] for positive maps in $\mathcal{L}(E)$ and for more general positive maps in [4] and [5]. Define for a positive mapping $N : E \to E$ the nonnegative real number $\lambda_{N,K} = \sup \Lambda_{N,K}$ where

$$\Lambda_{N,K} = \{\lambda \geq 0 : \ \exists u \succ_K 0_E \ \text{ such that } \ \lambda N u \preceq_K u\}.$$

We have from the above cited works, that if $N, N_1 : E \to E$ are two positive mappings such that $\lambda_{N_1,K} > 0$ and $N(u) \preceq_K N_1(u)$ for all $u \succ_K 0_E$, then $\lambda_{N,K} \geq \lambda_{N_1,K} > 0$. This property becomes more interesting when the mapping $N$ is completely continuous. In such a case, we have that the set $I_N$ defined by

$$I_N = \{\delta > 0 : \ i(\delta N, B(0_E, \rho) \cap K, K) = 0 \text{ for all } \rho > 0\}$$

is nonempty, where $i(\cdot,\cdot,\cdot)$ refers to the fixed point index. Roughly speaking, this property still holds in the case where the condition $N(u) \succeq_K N_1(u)$ for all $u \succ_K 0_E$,, is replaced by $N(u) \succeq_K N_1(u - g(u))$ for all $u \succ_K 0_E$, where $g : E \to E$ is $\circ(\|u\|)$ at $0_E$ or at $\infty$. In such a situation, we have that the set $J_N$ defined by

$$J_N = \{(\delta, \rho) \in (0, +\infty) : \ i(\delta N, B(0_E, \rho) \cap K, K) = 0\}$$

is nonempty. Such operators $N_1$ are said to have the strongly index jump property, see [1, 3, 4, 5] where this topological property was extensively used to obtain fixed point theorems for positive maps.

Unfortunately, there are functional spaces whose natural cone is not solid and the Krein-Rutman theorem can not be used. This is the case of $L^p(I)$ where $I$ is an interval of $\mathbb{R}$. Its natural cone $\{u \in L^p(I) : u \geq 0 \text{ a.e. in } I\}$ is not solid as it is mentioned in Example 1.1.3 in [9] and on page 219 in [7]. Thus, the main goal of this work is to adapt the Krein-Rutman theory to such spaces. In this article, a positive compact operator $L \in \mathcal{L}(E)$ having a pcv $\lambda_L$ such that Equation (1.1), within $h \in K \setminus \{0_E\}$, is uniquely solvable in $K$ for $\mu < \mu_L$ and has no solution in $K$ for $\mu \geq \mu_L$, is said to be a Krein-Rutman operator (KRO for short). Theorems 3, 6 and Corollaries 2, 3 provide sufficient conditions for a positive compact operator to be a KRO.

We also present in this article a version of Banach contraction principle in the case where the cone $K$ is normal and minihedral, making of the Banach space $E$ a Riesz space. We end the paper with two applications of our main results. The first application concerns the generalized Fisher equation and in the second one, we apply our version of the Banach contraction principle to obtain an existence and uniqueness result for a third-order boundary value problem (bvp for short) posed on the half-line.

## 2 Abstract background

**Definition 1.** *A nonempty closed and convex set $K$ is said to be a cone in $E$ if*
***i)*** $(tK) \subset K$ *for all* $t \geq 0$ *and*
***ii)*** $K \cap (-K) = \{0_E\}$.

It is well known that if $K$ is a cone in $E$, then $K$ induces a partial order in the Banach space $E$. We write for all $x, y \in X$ : $x \preceq_K y$ (or $y \succeq_K x$) if $y - x \in K$ and $x \prec_K y$ (or $y \succ_K x$) if $y - x \in K \setminus \{0_E\}$. Thus, vectors lying in $K \setminus \{0_E\}$ are said to be positive.

**Definition 2.** *Let $\Omega$ be a nonempty set in $E$. Then*

**a)** *$u \in E$ is said to be an upper bound of $\Omega$ if $v \preceq_K u$ for all $v \in \Omega$;*

**b)** *$u \in E$ is said to be a lower bound of $\Omega$ if $v \succeq_K u$ for all $v \in \Omega$;*

**c)** *$u \in E$ is said to be the least upper bound of $\Omega$ and we write $u = \sup \Omega$, if $u$ is an upper bound of $\Omega$ and $v \preceq_K w$ for all $v \in \Omega$ implies $u \preceq_K w$;*

**d)** *$u \in E$ is said to be the greatest lower bound of $\Omega$ and we write $u = \inf \Omega$, if $u$ is a lower bound of $\Omega$ and $v \succeq_K w$ for all $v \in \Omega$ implies $u \succeq_K w$.*

**Definition 3.** *Let $K$ be a cone in $E$. Then $K$ is said to be*

**a)** *total, if $\overline{K - K} = E$,*

**b)** *normal if there is a positive constant $n_K$ such that for all $u, v \in E$, $0_E \preceq_K u \preceq_K v$ implies $\|u\| \leq n_K \|v\|$,*

**c)** *minihedral if $\sup(x, y)$ exists for all $x, y \in E$.*

**Remark 1.** *Notice that if a cone $K$ is minihedral then $\inf(x, y)$ exists for all $x, y \in E$. Moreover, we have $\inf(x, y) = -\sup(-x, -y)$.*

**Remark 2.** *It is well known that if $K$ is a minihedral cone inducing the order $\preceq_K$ on $E$, then $(E, \preceq_K)$ is a Riesz space or a Banach lattice in the sence given in [10]..*

**Definition 4.** *Let $K$ be a minihedral cone in $E$ inducing the order $\preceq_K$ on $E$. For $x \in E$, we define the positive part, the negative part and the absolute value of the vector $x$ respectively by*

$$x^+ = \sup(x, 0), \quad x^- = \sup(-x, 0) \quad \text{and} \quad |x| = x^+ + x^-.$$

**Proposition 1.** *([10]) Let $K$ be a minihedral cone in $E$ inducing the order $\preceq_K$ on $E$. Then the absolute value define then a self-mapping on $E$ and it has the following properties:*

**i)** $|x| \succeq_K 0_E$ *for all* $x \in E$,

**ii)** $|x| = 0_E \Rightarrow x = 0_E$,

**iii)** $|tx| = |t|\,|x|$ *for all* $x \in E$ *and* $t \in \mathbb{R}$,

**iv)** $|x+y| \preceq_K |x| + |y|$ *for all* $x, y \in E$,

**v)** $||x| - |y|| \preceq_K |x-y|$ *for all* $x, y \in E$.

**Proposition 2.** *Let $K$ be a minihedral cone in $E$, then the following assertions are equivalents.*

**i)** *The mapping* $|\cdot| : E \to K$ *is continuous.*

**ii)** *The mapping* $|\cdot| : E \to K$ *is continuous at* $0_E$.

**iii)** *There exists* $\eta > 0$ *such that* $\||u|\| \leq \eta \|u\|$ *for all* $u \in E$.

*Proof.* The equivalence between i) and ii) is due to the inequality in v) of Proposition 1. It is easy to see that iii) implies ii) and , hence let us prove that ii) implies iii). Let $\epsilon_0 > 0$, there is $\delta_0 > 0$ such that for all $u \in E$, $\|u\| \leq \delta_0$ implies $\||u|\| \leq \epsilon_0$. Therefore, for all $u \in E$ with $u \neq 0_E$, we have

$$\frac{\delta_0}{\|u\|} \||u|\| = \left\| \left| \frac{\delta_0 u}{\|u\|} \right| \right\| \leq \epsilon_0,$$

leading to

$$\||u|\| \leq \eta \|u\| \text{ for all } u \in E$$

with $\eta = \epsilon_0 / \delta_0$. □

**Remark 3.** *It follows from Proposition 2 that the mapping $|\cdot| : E \to K$ is continuous if and only if* $\sup_{\|u\|=1} \||u|\| < \infty$.

**Definition 5.** *Let $K$ be a cone in $E$, a mapping $L \in \mathcal{L}(E)$ is said to be:*
***i)*** *positive, if* $L(K) \subset K$,
***ii)*** *strongly positive, if $K$ is solid and* $L(K \setminus \{0_E\}) \subset int(K)$.

For a cone $K$ in $E$, $\mathcal{L}_K(E)$ will denote the subset in $\mathcal{L}(E)$ of all positive mapping. The dual cone associated with $K$ is defined by

$$K^* = \{\varphi \in E^* : \ (\varphi, u) \geq 0 \text{ for all } u \in K\}.$$

In general, $K^*$ is a not a cone with respect of the Definition 1, it happens that $K^* \cap (-K^*) \neq \{0_{E^*}\}$. However, $K^*$ is a cone if and only if $K$ is total. With this definition of the dual cone, for all operators $L$ in $\mathcal{L}_K(E)$, $L^*$ the adjoint operator associated with $L$ belongs to $\mathcal{L}_{K^*}(E^*)$.

**Definition 6.** *Let $K$ be a cone in $E$. A vector $u \in K$ is said to be strictly positive if $(\varphi, u) > 0$ for all $\varphi \in K^* \setminus \{0_{E^*}\}$ and a functional $\varphi \in K^*$ is said to be strictly positive, if $(\varphi, u) > 0$ for all $u \succ_K 0_E$.*

**Definition 7.** *Let $K$ be a cone in $E$ and $L \in \mathcal{L}_K(E)$, a positive real number $\mu$ is said to be a pcv of $L$ if there exist $u \succ_K 0_E$ such that $\mu L u = u$. The vector $u$ is then a positive eigenvector associated with the pcv $\mu$.*

For detailled presentations on cones and positivity we refer the reader to [7] and [9]. The reader will observe that the definition of the minihedrality given here is that of [7]. In [9], a cone $C$ is said to be minihedral if $\sup(x, y)$ exists for all pair $(x, y) \in E^2$ having an upper bounded. To ensure the existence of $\sup(x, y)$ for all $x, y \in E$ when such is the definition of the minihedrality, one may assume that the cone $C$ is generating (i.e. $E = K - K$). Indeed, for all $x, y \in E$ there exist $x_1, x_2, y_1, y_2 \in K$ such that $x = x_1 - x_2$ and $y = y_1 - y_2$. Therefore, we have $x \preceq_K x_1 + y_1$ and $y \preceq_K x_1 + y_1$.

In all this work, we use the following notations: for $L \in \mathcal{L}(E)$, $CV(L)$ denotes the set of all characteristic values of $L$. The spectral radius of $L$, is defined to be

$$r(L) = \begin{cases} \inf\left\{ |\mu|^{-1} : \quad \mu \in CV(L) \right\} & \text{if } CV(L) \neq \emptyset, \\ 0 & \text{if } CV(L) \neq \emptyset \end{cases}$$

and we have by the Gelfand formula

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

For $\mu \notin CV(L)$, $R(\mu, L) = (I - \mu L)^{-1}$ is the resolvent mapping associated with $L$ and we have for all $\mu \in \mathbb{C}$ with $|\mu| < 1/r(L)$,

$$R(\mu, L) = (I - \mu L)^{-1} = \sum_{n=0}^{\infty} \mu^n L^n. \tag{2.1}$$

Notice that if $K$ is a cone in $E$ we have from (2.1) that $R(\mu, L) \in \mathcal{L}_K(E)$ for all $L \in \mathcal{L}_K(E)$ and all $0 < \mu < 1/r(L)$.

We recall now briefly what is known as the Riesz-Schauder theory. Let $L \in \mathcal{L}(E)$ be compact, we have:

**A)** $CV(L)$ is empty or finite or consists of a sequence $(\mu_k)$ with $\lim |\mu_k| = +\infty$,

**B)** $L^*$ is compact,

**C)** if $\mu \in CV(L)$ then the geometric multiplicity of $\mu$, $m(\mu) = \dim N(I - \mu L) < \infty$ and $R(I - \mu L)$ is closed,

**D)** We have that $CV(L) \subset CV(L^*)$. Moreover, for all $\mu \in CV(L)$,

$$\begin{aligned} &N(I - \mu L)^{\perp} = R(I^* - \mu L^*), \\ &N(I^* - \mu L^*)^{\perp} = R(I - \mu L), \\ &\dim N(I - \mu L) = \dim N(I^* - \mu L^*). \end{aligned}$$

**E)** For all $\mu \in CV(L)$, there is a smallest $n(\mu)$ such that for all $k \geq n(\mu)$

$$N_\mu = N(I-\mu L)^{n(\mu)} \supset N(I-\mu L)^k, \qquad R_\mu = R(I-\mu L)^{n(\mu)} \subset R(I-\mu L)^k,$$

$$L(N_\mu) \subset N_\mu, \qquad L(R_\mu) \subset R_\mu \quad \text{and} \quad N_\mu \oplus R_\mu = E.$$

**F)** If $\mu \in CV(L)$ and $P_\mu$ and $Q_\mu$ are respectively the projections of $E$ on $N_\mu$ and $R_\mu$, then

$$LP_\mu = P_\mu L, \quad \text{and} \quad LQ_\mu = Q_\mu L.$$

The integer $\varkappa(\mu) = \dim N(I-\mu L)^{n(\mu)}$ is the algebraic multiplicity of the characteristic value $\mu$. For a detailed presentation on the Riesz-Schauder theory, we refer the reader to [11].

The following theorem is due to M. G. Krein and M. A. Rutman and plays a key role in this paper.

**Theorem 1** ([7, 12])**.** *Let $K$ be a cone in $E$ and let $L$ be a compact operator in $\mathcal{L}_K(E)$. Assume that the cone $K$ is total in $E$ and $r(L) > 0$, then $1/r(L)$ is a pcv of $L$ and $L^*$.*

# 3 Main results

## 3.1 Notations and preliminaries

The statements of the main results in this article as well as their proofs need to introduce some notations and preliminary lemmas. For any cone $K$ in $E$ and any operator $L$ in $\mathcal{L}_K(E)$, we let

$$\begin{aligned}
&F_K = \overline{K-K} \\
&CV_K(L) = \{\mu > 0 : \ \exists u \succ_K 0_E \text{ such that } \mu L u = u\}, \\
&\mu^-_{L,K} = \begin{cases} \inf CV_K(L) \text{ if } CV_K(L) \neq \emptyset \\ +\infty \qquad\qquad \text{if } CV_K(L) = \emptyset \end{cases} \\
&\mu^+_{L,K} = \begin{cases} \sup CV_K(L) \text{ if } CV_K(L) \neq \emptyset \\ +\infty \qquad\qquad \text{if } CV_K(L) = \emptyset. \end{cases}
\end{aligned}$$

Notice that $K$ is a total cone in $F_K$, $L(F_K) \subset F_K$ and $L_F$, the restriction of $L$ to $F_K$, belongs to $\mathcal{L}_K(F_K)$ and is compact if $L$ is. Moreover, we have $CV_K(L) = CV_K(L_F)$.

With any cone $K$ in $E$ and any operator $L$ in $\mathcal{L}_K(E)$ is associated the abstract equation:

$$u - \mu L u = h \qquad\qquad (\mathcal{E}_{\mu,h,L})$$

where $\mu$ is a positive real parameter and $h \in K$.

As it is mentioned in the section Introduction, we are concerned by the solvability of the equation $(\mathcal{E}_{\mu,h,L})$ in $K$. For this reason, we introduce the following constants $\gamma^-_{L,K}$ and $\gamma^+_{L,K}$ that are related to the positive solvability of Equation $(\mathcal{E}_{\mu,h,L})$.

$$\gamma^-_{L,K} = \sup \left\{ \begin{array}{l} \gamma \geq 0 : \text{ for all } \mu \in (0,\gamma) \text{ and } h \succ_K 0_E, \text{ Equation} \\ (\mathcal{E}_{\mu,h,L}) \text{ has a unique solution in } K \end{array} \right\}$$

$$\gamma^+_{L,K} = \inf \left\{ \begin{array}{l} \gamma \geq 0 : \text{ for all } \mu \in (\gamma, +\infty) \text{ and } h \succ_K 0_E, \text{ Equation} \\ (\mathcal{E}_{\mu,h,L}) \text{ has no solution in } K \end{array} \right\}.$$

Notice that for all $\mu \in (0, +\infty) \setminus CV(L)$ and $h \succ_K 0_E$, $u = R(\mu, L)\, h$ is the unique solution of Equation $(\mathcal{E}_{\mu,h,L})$. This means that the constant $\gamma^+_{L,K}$ represents the nonnegative real number at which the resolvent mapping stops to be positive, i.e. $R(\mu, L)(K) \nsubseteq K$ for $\mu > \gamma^+_{L,K}$. Furthermore, since (2.1) guarantees that for all $\mu < 1/r(L)$, $R(\mu, L)$ is a positive mapping, we have $\gamma^+_{L,K} \geq 1/r(L)$. The definition of the constant $\gamma^-_{L,K}$ implies that for all $\mu > \gamma^-_{L,K}$, the set $R(\mu, L)(K) \cap (K \setminus \{0_E\})$ is nonempty, we can say that the constant $\gamma^-_{L,K}$ represents the first nonnegative real number at which the resolvent mapping starts to be positive. Clearly, it yields from their definitions that $\gamma^-_{L,K} \leq \gamma^+_{L,K}$.

Let $\lambda^-_{L,K}$ and $\lambda^+_{L,K}$ be the constants associated with the cone $K$ in $E$ and the operator $L$ in $\mathcal{L}_K(E)$, defined by

$$\lambda^+_{L,K} = \sup \Lambda^+_{L,K},$$

$$\lambda^-_{L,K} = \begin{cases} \inf \Lambda^-_{L,K} & \text{if } \Lambda^-_{L,K} \neq \emptyset, \\ +\infty & \text{if } \Lambda^-_{L,K} = \emptyset. \end{cases}$$

where

$$\begin{array}{l} \Lambda^-_{L,K} = \{\lambda > 0 : \ \exists u \succ_K 0_E \ \text{ such that } \ \lambda L u \succeq u\}, \\ \Lambda^+_{L,K} = \{\lambda \geq 0 : \ \exists u \succ_K 0_E \ \text{ such that } \ \lambda L u \preceq u\}. \end{array}$$

It is proved in [2] that if $L$ is compact then $\lambda^-_{L,K}$ and $\lambda^+_{L,K}$ have the following properties:

**G)** $\lambda^-_{L,K} \leq \lambda^+_{L,K}$,

**H)** If $\mu \in CV_K(L)$ then $\lambda^-_{L,K} \leq \mu \leq \lambda^+_{L,K}$,

**I)** if $\Lambda^-_{L,K} \neq \emptyset$ then $\lambda^-_{L,K}$ is the smallest pcv of $L$ and if $\lambda^+_{L,K} < +\infty$ then $\lambda^+_{L,K}$ is the largest pcv of $L$ (see Propositions 3.14 and 3.15 in [5]).

For $\mu \in CV_K(L)$ we denote by $P_{N_\mu}$ and $P_{R_\mu}$ the projections on the Banach subspaces of $E$, $N_\mu$ and $R_\mu$, respectively.

The following lemmas will be used in the proof of the main results in this paper.

**Lemma 1.** *If* $r(L_F) > 0$ *then* $\lambda^-_{L,K} \geq 1/r(L_F)$.

*Proof.* We distinguish two cases:

i) $\Lambda^-_{L,K} = \emptyset$, in this case we have $\lambda^-_{L,K} = +\infty \geq 1/r(L_F)$.

ii) $\Lambda^-_{L,K} \neq \emptyset$, in this case let $\lambda_0 > 0$, $u_0 \succ_K 0_E$ be such that $\lambda_0 L_F u_0 = \lambda_0 L u_0 \succeq_K u_0$ and let us show that $\lambda_0 \geq 1/r(L_F)$. By the contrary, suppose that $\lambda_0 < 1/r(L_F)$ and set $h_0 = \lambda_0 L u_0 - u_0 = \lambda_0 L_F u_0 - u_0$. We obtain then the contradiction

$$0_E \prec_K u_0 = -R(\lambda_0, L_F) h_0 = -\sum_{n \geq 0} \lambda_0^n L_F^n h_0 \preceq_K 0_E.$$

This ends the proof. □

**Lemma 2.** *Let $K$ be a cone in $E$ and let $L$ be a compact operator in $\mathcal{L}_K(E)$. If for $\mu \in CV_K(L)$, $K \cap R_\mu = \{0_E\}$, then $P_{N_\mu}(K)$ is a cone in $N_\mu$.*

*Proof.* Since Assertion **E** guarantees that the linear mapping $\Pi : E \to N_\mu \times R_\mu$ defined by $\Pi(x) = \left(P_{N_\mu}(x), P_{R_\mu}(x)\right)$ is a continuous bijection between Banach spaces, we conclude by the open mapping theorem that $\Pi$ is a linear homeomorphism between Banach spaces. Therefore, $\Pi(K)$ is a closed set in $N_L \times R_L$ and in particular $P_{N_\mu}(K)$ is a closed set in $N_\mu$. Because of the linearity of the projection $P_{N_\mu}$, we have that $P_{N_\mu}(K)$ is a convex set and $\left(tP_{N_\mu}(K)\right) \subset P_{N_\mu}(K)$ for all $t \geq 0$.

Let $x \in P_{N_\mu}(K) \cap \left(-P_{N_\mu}(K)\right)$, then there exist $y, z \in K$ such that $x = P_{N_\mu}(y)$ and $-x = P_{N_\mu}(z)$. This implies that $P_{N_\mu}(y+z) = 0_E$ and $y + z \in K \cap R_\mu = \{0_E\}$, that is $y + z = y = z = 0_E$ proving that $x = 0_E$, ending the proof of $P_{N_\mu}(K)$ is a cone. □

## 3.2 A variant of Banach contraction principle

In all this subsection, we let $K$ be a normal and minihedral cone in $E$.

**Theorem 2.** *Let $T : E \to E$ be a continuous operator such that for all $u, v \in E$*

$$|Tu - Tv| \preceq_K cL(|u - v|)$$

*where $L \in \mathcal{L}_K(E)$ and $c \geq 0$. If $cr(L_F) < 1$, then $T$ admits a unique fixed point.*

*Proof.* The case $c = 0$ is obvious, so we suppose that $c > 0$.

**Uniqueness.** If $u_1$ and $u_2$ are two fixed point of $T$ with $u_1 \neq u_2$, then $w = |u_1 - u_2| \succ_K 0_E$ and satisfies

$$w = |u_1 - u_2| = |Tu_1 - Tu_2| \preceq_K cL(|u_1 - u_2|) = cLw.$$

Hence, $c \in \Lambda^-_{L,K} \neq \emptyset$ and $r(L_F) > 0$. Indeed, we have by induction

$$L^n w \succeq_K \frac{1}{c^n} w,$$

and then the normality of the cone $K$ leads to

$$\|L_F^n\| \|w\| \geq \|L_F^n w\| = \|L_F^n w\| \geq \frac{1}{n_K c^n} \|w\|.$$

From which we see that

$$r(L_F) = \lim_{n \to +\infty} \sqrt[n]{\|L_F^n\|} \geq \frac{1}{c} > 0.$$

Therefore, this together with Lemma 1 lead to the contradiction

$$\lambda^-_{L,K} > 1/r(L_F) > c \geq \inf\{\lambda \geq 0 : \exists u \succ_K 0_E \text{ such that } \lambda Lu \succeq_K u\} = \lambda^-_{L,K}.$$

The uniqueness is proved.

**Existence.** Let $u_0 \in E$ and consider the sequence $(u_n)$ defined by $u_n = Tu_{n-1}$. We have then for all $n \geq 1$,

$$|u_{n+1} - u_n| = |T(u_n) - T(u_{n-1})| \preceq_K cL(|u_n - u_{n-1}|),$$

Since the operator $L$ is increasing, we obtain

$$|u_{n+1} - u_n| \preceq_K c^n L^n (|u_1 - u_0|)$$

Therefore, if $m, n$ are two integers with $m > n \geq 1$, then

$$\begin{aligned} |u_m - u_n| &\preceq_K |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + ... + |u_{n+1} - u_n| \\ &\preceq_K c^{m-1}L^{m-1}v + c^{m-2}L^{m-2}v + ... + c^n L^n v \end{aligned}$$

where $v = |u_1 - u_0|$.

Thus, the normality of the cone $K$ leads to

$$\begin{aligned} ||u_m - u_n|| &\leq n_K \left(c^{m-1}||L^{m-1}v|| + c^{m-2}||L^{m-2}v|| + ... + c^n||L^n v||\right) \\ &= n_K (S_{m-1} - S_{n-1}), \end{aligned}$$

where $S_n = \sum_{k=0}^{k=n} c^k||L^k v||$.

Since $v \succ_K 0_E$, we have that

$$\lim_{n\to+\infty} \sqrt[n]{||c^n L^n v||} = \lim_{n\to+\infty} \sqrt[n]{||c^n L_F^n v||} \leq c \lim_{n\to+\infty} \sqrt[n]{\|v\|} \sqrt[n]{||L_F^n||} = cr(L_F) < 1.$$

that is $(S_n)$ converges and

$$\lim_{n\to+\infty} ||u_m - u_n|| \leq \lim_{n\to+\infty} ||S_{m-1} - S_{n-1}|| = 0.$$

Therefore, the sequence $(u_n)_n$ is a Cauchy sequence and the completeness of $E$ leads to $\lim_{n\to+\infty} u_n = u \in E$. At the end, passing to the limit in $u_{n+1} = Tu_n$, we obtain $u = Tu$. Ending the proof. □

**Remark 4.** *Obseve that Theorem 2 holds true if $T : E \to E$ is replaced by $T : \Omega \to \Omega$ where $\Omega$ is a nonemplty closed convex subset of $E$.*

**Remark 5.** *Clearly if $L = I$ then $r(L_F) = 1$ and Theorem 2 coincide with the standard Banach contraction principle. Notice that if $L$ is compact then the condition $r(L_F) > 0$ in Theorem 2 is equivalent to that $L$ has a positive eigenvalue, i.e. there are $\lambda > 0$ and $u \succ_K 0_E$ such that $Lu = \lambda u$.*

**Remark 6.** *An operator satisfying the condition of Theorem 2 is not necessaraly a contraction. Indeed, let $E = C([0,1])$ equipped with its sup-norm $\|\cdot\|_\infty$, $K$ the cone of nonnegative functions lying in $E$ and $G : [0,1] \times [0,1] \to \mathbb{R}$ the Green's function associated with the differential operator $\frac{d^2}{dt^2}$ and the Dirichlet boundary conditions, given by*

$$G(t,s) = \begin{cases} s(1-t), \text{ if } 0 \leq s \leq t \leq 1, \\ t(1-s), \text{ if } 0 \leq t \leq s \leq 1. \end{cases}$$

*Let $L, T : E \to E$ be the operators defined for $u \in E$ by*

$$\begin{array}{c} Lu(t) = \int_0^1 G(t,s)u(s)ds, \\ Tu(t) = \int_0^1 G(t,s)\left(cu(s) + h(s)\right)ds, \end{array}$$

*where $c > 0$ and $h \in E$.*

*It well known that $K$ is normal genartor and minihedral with*

$$u^+(t) = \max(u(t), 0),\ \ u^-(t) = \max(-u(t), 0) \ \text{and}\ \left|u\right|(t) = u^+(t) + u^-(t)$$

*and $L \in \mathcal{L}_K(E)$ with*

$$r(L) = \frac{1}{\pi^2} \ \text{and}\ \|L\| = \sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

*After simple computations we obtain that for all $u, v \in E$*

$$\|Tu - Tv\|_\infty = c\left\|L\left(u - v\right)\right\|_\infty \le c\left\|L\right\|\left\|u - v\right\|_\infty = \frac{c}{8}\left\|u - v\right\|_\infty \tag{3.1}$$

*and*

$$\left|T\left(u - v\right)\right| = \left|cL\left(u - v\right)\right| \le cL\left(\left|u - v\right|\right) \tag{3.2}$$

*Being the best constant realizing (3.1), i.e. $\|L\| = \inf\left\{k > 0 : \|Lu\|_\infty \le k\left\|u\right\|_\infty \forall u \in E\right\}$, we conclude that the mapping $T$ is a contraction if and only if $c < 8$. Moreover, we deduce from (3.2) that the condition of Theorem 2 is satisfied if and only if $c < \pi^2$.*

*Notice that for $c \in \left(8, \pi^2\right)$ the operator $T$ satisfies the condition of Theorem 2 and it is not a contraction.*

**Corollary 1.** *Assume that the mapping $|\cdot| : E \to K$ is continuous and let $T : E \to E$ be such that for all $u, v \in E$*

$$\left|Tu - Tv\right| \preceq_K cL\left(\left|u - v\right|\right)$$

*where $L \in \mathcal{L}_K\left(E\right)$ and $c \ge 0$. If $r\left(L_F\right) > 0$ and $cr\left(L_F\right) < 1$, then $T$ admits a unique fixed point.*

*Proof.* We have to prove that the mapping $T$ is continuous. For all $u, v \in E$, we have

$$\begin{array}{rcl} \|Tu - Tv\| & \le & n_K \left\|\left|Tu - Tv\right|\right\| \\ & \le & n_K^2 c\left\|L\left(\left|u - v\right|\right)\right\| \\ & \le & n_K^2 c\left\|L\right\|\left\|\left|u - v\right|\right\| \\ & \le & n_K^2 \eta c\left\|L\right\|\left\|u - v\right\|, \end{array}$$

where $\eta = \sup_{\|u\|=1}\left\|\left|u\right|\right\| < \infty$ (see Remark 3).

Hence the mapping $T$ is continous. ∎

### 3.3 Krein-Rutman operators

In all this subsection, we let $K$ be a cone in $E$ and $L$ a compact operator in $\mathcal{L}_K(E)$. The first result of this subsection concerns constants $\lambda^-_{L,K}$, $\gamma^-_{L,K}$, $\mu^-_{L,K}$, $\mu^+_{L,K}$, $\lambda^+_{L,K}$, $\gamma^+_{L,K}$.

**Proposition 3.** *We have*

$$\lambda^-_{L,K} = \gamma^-_{L,K} = \mu^-_{L,K} \leq \mu^+_{L,K} \leq \lambda^+_{L,K} = \gamma^+_{L,K}.$$

*Moreover, if* $\lambda^+_{L,K} < +\infty$ *then* $\lambda^+_{L,K} = \gamma^+_{L,K} = \mu^+_{L,K}$.

*Proof.* We distinguish the following two cases:

**i)** $\Lambda^-_{L,K} = \emptyset$, in this case we have from Assertion **H** $CV_K(L) = \emptyset$ and $\lambda^-_{L,K} = \mu^-_{L,K} = \mu^+_{L,K} = +\infty$. Moreover, we have $R(\mu, L_F)$ is defined and positive for all $\mu > 0$ and so, for all $h \succ_K 0_E$, $u = R(\mu, L_F)\, h$ is the unique positive solution to Equation $(\mathcal{E}_{\theta,h_0,L})$, proving that $\gamma^-_{L,K} = \lambda^-_{L,K} = \mu^-_{L,K} = \mu^+_{L,K} = +\infty$. Therefore, Assertion **G** leads to $\lambda^+_{L,K} = \mu^+_{L,K} = \gamma^+_{L,K} = +\infty$.

**ii)** $\Lambda^-_{L,K} \neq \emptyset$, we prove that $\gamma^-_{L,K} \leq \mu^-_{L,K} \leq \mu^+_{L,K} \leq \gamma^+_{L,K}$. To this aim, let $\mu$ be an arbitrary pcv of $L$ having a positive eigenvector $\psi$ and let $(\epsilon_n)$ be a sequence such that $\lim \epsilon_n = 0$ and for all $n \geq 1$, $(\mu - \epsilon_n) \notin CV(L)$. Such a choice is possible because of Assertion **A**. Since for all $n \geq 1$, $\psi - (\mu - \epsilon_n) L\psi = \frac{\epsilon_n}{\mu}\psi$, we have $\mu - \epsilon_n \leq \gamma^+_{L,K}$ leading to $\mu = \lim(\mu - \epsilon_n) \leq \gamma^+_{L,K}$.

Suppose that $\mu < \gamma^-_{L,K}$ and let $\theta \in \left(\mu, \gamma^-_{L,K}\right) \smallsetminus CV_K(L)$, then the nonhomogeneous equation $(\mathcal{E}_{\theta,h_0,L})$ with $h_0 = \left(\frac{\theta-\mu}{\mu}\right)\psi$, admits a unique positive solution $u_0$. Consequently, $v_0 = u_0 + \psi \succ 0_E$ solves the equation $v_0 - \theta L v_0 = 0_E$, contradicting $\theta \notin CV_K(L)$ and proves that $\mu \geq \gamma^-_{L,K}$. Since $\mu$ is arbitrary, all the above proves that $\gamma^-_{L,K} \leq \mu^-_{L,K} \leq \mu^+_{L,K} \leq \gamma^+_{L,K}$.

At this stage, we conclude by means of Assertion **I**, that $\lambda^-_{L,K} = 1/r(L_F) = \mu^-_{L,K}$. We have to prove now that $\lambda^-_{L,K} \leq \gamma^-_{L,K}$. Let $\theta < \lambda^-_{L,K} = 1/r(L_F)$ be arbitrary and notice that $u$ is a positive solution to Equation $(\mathcal{E}_{\theta,h,L})$ if and only if $u$ is a positive solution of Equation $(\mathcal{E}_{\theta,h,L_F})$. Because $u_h = R(\theta, L_F)h$ is the unique positive solution, Equation $(\mathcal{E}_{\theta,h,L_F})$ has a unique positive solution. This shows that $\lambda^-_{L,K} = \mu^-_{L,K} = \gamma^-_{L,K}$.

Because of Assertion **H**, we have $\mu^+_{L,K} \leq \lambda^+_{L,K}$ and if $(\lambda_n)$, $(u_n)$ are two sequences such that $\lim \lambda_n = \lambda^+_{L,K}$, $u_n \succ_K 0_E$ and $u_n - \lambda_n L u_n \succ_K 0_E$, then for all $n \geq 1$ the equation $(\mathcal{E}_{\lambda_n,h_n,L})$ where $h_n = u_n - \lambda_n L u_n$ has a positive solution and $\lambda_n \leq \gamma^+_{L,K}$. Thus, passing to the limit, we obtain $\lambda^+_{L,K} = \lim \lambda_n \leq \gamma^+_{L,K}$.

On the contrary, suppose that $\lambda^+_{L,K} < \gamma^+_{L,K}$. Since for all $\theta \in \left(\lambda^+_{L,K}, \gamma^+_{L,K}\right)$ and all $u \succ_K 0_E$, $u - \theta L u \not\succeq 0_E$, Equation $(\mathcal{E}_{\theta,h,L_F})$ has no positive solution for all $h \succ_K 0_E$. This contradicts the definition of the constant $\gamma^+_{L,K}$. Thus, we have proved that $\lambda^-_{L,K} = \gamma^-_{L,K} = \mu^-_{L,K} = 1/r(L_F) \leq \mu^+_{L,K} \leq \lambda^+_{L,K} = \gamma^+_{L,K}$.

At the end, for the particular case where $\lambda^+_{L,K} < +\infty$, we have from Assertion **I** that $\lambda^+_{L,K} = \mu^+_{L,K} = \gamma^+_{L,K}$. □

**Definition 8.** *The operator $L$ is said to be a KRO if it has a pcv $\lambda_L$ such that Equation $(\mathcal{E}_{\mu,h,L})$ has a unique positive solution if $\mu < \lambda_L$ and Equation $(\mathcal{E}_{\mu,h,L})$ has no positive solution if $\mu \geq \lambda_L$.*

We obtain from Proposition 3 and the above definition the following proposition.

**Proposition 4.** *If $L$ is a KRO, then*

$$\lambda_L = \lambda^-_{L,K} = \gamma^-_{L,K} = \mu^-_{L,K} = \lambda^+_{L,K} = \gamma^+_{L,K} = \mu^+_{L,K}$$

*is the unique pcv of $L$.*

*Reciprocally, if*

$$\lambda_L = \lambda^-_{L,K} = \gamma^-_{L,K} = \mu^-_{L,K} = \lambda^+_{L,K} = \gamma^+_{L,K} = \mu^+_{L,K} < \infty$$

*and for all $h \succ_K 0_E$ Equation $(\mathcal{E}_{\lambda_L,h,L})$ has no positive solution then $L$ is a KRO having a unique pcv $\lambda_L$.*

**Definition 9.** *The operator $L$ is said to have the property $(\mathcal{H})$ if*

$$\mu \in CV_K(L) \text{ implies } \mu \text{ has a strictly positive eigenvector.}$$

The first main result of this subsection is:

**Theorem 3.** *Assume that $\lambda^-_{L,K} < \infty$ and $L^*_F$ has the propety $(\mathcal{H})$, then $L$ is a KRO.*

*Proof.* Since $K$ is total in $F$ and $L^*_F$ has the property $(\mathcal{H})$, we have from Theorem 1 that $\lambda^-_{L,K} = 1/r(L_F)$ is a pcv of $L_F$ and $L^*_F$ and $\lambda^-_{L,K}$ has a strictly positive eigenvector $\varphi \in K^*$.

Let for $i = 1, 2$, $\lambda_i > 0$, $u_i \succ 0_E$ be such $\lambda_1 L u_1 \preceq_K u_1$ and $\lambda_2 L u_2 \succeq_K u_2$, we have then

$$\begin{aligned} 0 &< (\varphi, u_1) \leq (\varphi, \lambda_1 L u_1) = (\varphi, \lambda_1 L_F u_1) \\ &= \lambda_1 (L^*_F \varphi, u_1) = \frac{\lambda_1}{\lambda^-_{L,K}} (\varphi, u_1) \end{aligned}$$

and

$$\begin{aligned} 0 &< (\varphi, u_2) \geq (\varphi, \lambda_2 L u_2) = (\varphi, \lambda_2 L_F u_2) \\ &= \lambda_2 (L^*_F \varphi, u_2) = \frac{\lambda_1}{\lambda^-_{L,K}} (\varphi, u_2), \end{aligned}$$

leading to $\lambda_2 \leq \lambda^-_{L,K} \leq \lambda_1$.

Because that $\lambda_2, \lambda_1$ are arbitrary, we obtain from Assertion **G** and Proposition 3 that

$$\lambda_L = \lambda^-_{L,K} = \gamma^-_{L,K} = \mu^-_{L,K} = \lambda^+_{L,K} = \gamma^+_{L,K} = \mu^+_{L,K}$$

is the unique pcv of $L$.

It remains to prove that for all $h \succ_K 0_E$, equation $u - \lambda_L L u = h$ has no positive solution. On the contrary, if this fails and for some $h \succ_K 0_E$ there is $u \succ_K 0_E$ such that $u - \lambda_L L u = h$, we obtain the contradiction

$$\begin{aligned} 0 &< (\varphi, h) = (\varphi, u) - (\varphi, \lambda_L L u) \\ &= (\varphi, u) - (\varphi, \lambda_L L_F u) \\ &= (\varphi, u) - (\lambda_L L^*_F \varphi, u) \\ &= (\varphi, u) - (\varphi, v) = 0. \end{aligned}$$

Thus, Proposition 4 leads to $L$ is a KRO, and this ends the proof. □

**Theorem 4.** *Assume that* $\lambda^{-}_{L,K} < \infty$ *and the operator* $L$ *satisfies the condition* $(\mathcal{H})$*, then* $\lambda^{-}_{L,K}$ *is the unique pcv of* $L$ *and* $L^*_F$ *is a KRO.*

*Proof.* Notice that $\mu$ is a pcv of $L$ if and only if $\mu$ is a pcv of $L_F$. Since $K$ is total in $F$ and $\Lambda^{-}_{L,K} \neq \emptyset$, by Theorem 1 we conclude that $\lambda^{-}_{L,K} = 1/r(L_F)$ is a pcv of $L$ and of $L^*_F$. Let $u$ be the strictly positive eigenvector associated with $\lambda^{-}_{L,K}$ as a pcv of $L$ and let $\varphi \in K^*$ be the eigenvector associated with $\lambda^{-}_{L,K}$ as a pcv of $L^*_F$. Thus, if $\mu \in CV_K(L)$ has a strictly positive eigenvector $v$, we obtain

$$0 < (\varphi, v) = (\varphi, \mu L v) = (\mu L^* \varphi, v) = \frac{\mu}{\lambda^{-}_{L,K}} (\varphi, v),$$

leading to $\mu = \lambda^{-}_{L,K}$.

Now, let $h \succ_{K^*} 0_{F^*}$ and suppose that Equation $(\mathcal{E}_{\lambda^{-}_{L,K},h,L^*_F})$ has a positive solution $\psi$. We obtain then the contradiction

$$\begin{aligned} 0 < (h, u) &= (\psi, u) - \left(\lambda^{-}_{L,K} L^*_F \psi, u\right) \\ &= (\psi, u) - \left(\psi, \lambda^{-}_{L,K} L_F u\right) \\ &= (\psi, u) - (\psi, u) = 0. \end{aligned}$$

This proves that for all $h \succ_{K^*} 0_{F^*}$, Equation $(\mathcal{E}_{\lambda^{-}_{L,K},h,L^*_F})$ has no positive solution. Therefore, we conclude from Proposition 4 the operator $L^*_F$ is a KRO, and the proof is finished. □

We obtain from Theorem 1 and Theorem 4 the following corollary.

**Corollary 2.** *Assume that the cone* $K$ *is total and* $r(L) > 0$*. If the operators* $L$ *and* $L^*$ *have the property* $(\mathcal{H})$*, then* $L$ *and* $L^*$ *are KRO.*

The following theorem consider the case where $E$ is a Hilbert space.

**Theorem 5.** *Assume that* $E$ *is a Hilbert space and the operator* $L$ *has the property* $(\mathcal{H})$*. If* $\lambda^{-}_{L,K} < \infty$ *then* $\lambda^{-}_{L,K}$ *is a pcv of* $L^*$ *and* $L^*$ *is a KRO.*

*Proof.* Denote by $< \cdot, \cdot >$ the inner product in $E$ and let $W$ be the orthogonal subspace of $F$. Because that $K$ is a total cone in $F$ and $\lambda^{-}_{L,K} = 1/r(L_F) = 1/r(L^*_F) > 0$, we have from Theorem 1, that $\lambda^{-}_{L,K}$ is a pcv of $L^*_F$ having an eigenvector $u \succ_{K^*} 0_{F^*}$, namely, for all $v \in F$, we have

$$< u, v > = < \lambda^{-}_{L,K} L^*_F u, v > = < u, \lambda^{-}_{L,K} L_F v > .$$

Since $L^*(K^*) \subset K^*$ and $u \in K^*$, we have $L^*(u) \in F^* = F$ and so, $L^*u - L_F^*u \in F$. Moreover, we have for all $v \in F$

$$\begin{aligned}\lambda_{L,K}^- < L^*u - L_F^*u, v >&= < \lambda_{L,K}^- L^*u, v > - < \lambda_{L,K}^- L_F^*u, v >\\ &= < \lambda_{L,K}^- u, Lv > - < \lambda_{L,K}^- u, L_F v >\\ &= < \lambda_{L,K}^- u, L_F v > - < \lambda_{L,K}^- u, L_F v >= 0,\end{aligned}$$

proving that $L^*u - L_F^*u \in W$. Therefore, $L^*u - L_F^*u \in W \cap F = \{0_E\}$, leading to $L^*u = L_F^*u = \lambda_{L,K}^- u$. This ends the proof. □

We deduce from Theorem 5 the following corollary.

**Corollary 3.** *Assume that $E$ is a Hilbert space and the operator $L$ is selfadjoint having the property $(\mathcal{H})$. If $\lambda_{L,K}^- < \infty$ then $L$ is a KRO.*

**Theorem 6.** *Assume that $\lambda_{L,K}^- < \infty$, $m(\lambda_{L,K}^-) = \varkappa(\lambda_{L,K}^-)$ and for all $h \succ_K 0_E$ Equation $(\mathcal{E}_{\lambda_{L,K}^-,h,L})$ has no solution in $E$. Then $L$ is a KRO.*

*Proof.* We have from Assertion **E** that $E = N_{\lambda_{L,K}^-} \oplus R_{\lambda_{L,K}^-}$ and notice that $K \cap R_{\lambda_{L,K}^-} = \{0_E\}$. Indeed, if for $h \succ_K 0_E$ there is $v \in E$ and $k \geq 1$ such that $h = \left(I - \lambda_{L,K}^- L\right)^k (v)$, then $h = w - \lambda_{L,K}^- Lw$ where $w = \left(\lambda_{L,K}^- I - L\right)^{k-1}(v)$, contradicting Hypothesis that Equation $(\mathcal{E}_{\lambda_{L,K}^-,h,L})$ has no solution in $E$. Therefore, we conclude by Lemma 2 that $C = P_{N_{\lambda_{L,K}^-}}(K)$ is a cone in $N_{\lambda_{L,K}^-}$.

Since $\dim(N_{\lambda_{L,K}^-}) < \infty$, namely $N_{\lambda_{L,K}^-}$ is separable, we conclude from Assertion c) of Proposition 19.3 in [7], that there is $\varphi \in N_{\lambda_{L,K}^-}^*$ such that $(\varphi, u) > 0$ for all $u \succ_C 0_E$.

Consider $\widetilde{\varphi} = \varphi \circ P_{N_{\lambda_{L,K}^-}}$, clearly $\widetilde{\varphi} \in E^*$ and observe that for all $u \succ_K 0_E$, $P_{N_{\lambda_{L,K}^-}} u \succ_C 0_E$ and $(\widetilde{\varphi}, u) = (\varphi, P_{N_{\lambda_{L,K}^-}} u) > 0$. Indeed, $P_{N_{\lambda_{L,K}^-}} u = 0_E$ for some $u \succ_K 0_E$ leads to the contradiction $u = P_{R_{\lambda_{L,K}^-}} u \in K \cap R_{\lambda_{L,K}^-} = \{0_E\}$. Thus, taking in consideration $m(\lambda_{L,K}^-) = \varkappa(\lambda_{L,K}^-)$ and Assertion **F**, we obtain for all $u \in E$

$$\begin{aligned}(\lambda_{L,K}^- L^*\widetilde{\varphi}, u) &= (\widetilde{\varphi}, \lambda_{L,K}^- Lu) = (\varphi, \lambda_{L,K}^- P_{N_{\lambda_{L,K}^-}} Lu) = (\varphi, \lambda_{L,K}^- L P_{N_{\lambda_{L,K}^-}} u)\\ &= (\varphi, \lambda_{L,K}^- L_{N_{\lambda_{L,K}^-}} P_{N_{\lambda_{L,K}^-}} u) = (\varphi, P_{N_{\lambda_{L,K}^-}} u) = (\varphi, P_{N_{\lambda_{L,K}^-}} u)\\ &= (\widetilde{\varphi}, u),\end{aligned}$$

proving that $\widetilde{\varphi}$ is a strictly positive eigenvector associated with the pcv $\lambda_{L,K}^-$ of $L^*$.

Thus, we conclude by Proposition 4 that $L$ is a KRO. This ends the proof. □

We deduce from Theorem 6 the following corollary.

**Corollary 4.** *Assume that $\lambda_{L,K}^- < \infty$, $m(\lambda_{L,K}^-) = 1$ and for all $h \succ_K 0_E$ Equation $(\mathcal{E}_{\lambda_{L,K}^-,h,L})$ has no solution in $E$. Then $L$ is a KRO.*

# 4 On the generalized Fisher equation

## 4.1 Notations and main results

Let $c$ and $\lambda$ be two positive constants and consider the linear bvp

$$\begin{cases} -u'' + cu' + \lambda u = \mu q u + h \text{ in } \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0 \end{cases} \tag{4.1}$$

where $\mu$ is a real parameter, and $q, h \in \mathcal{E}^+$ with

$$\begin{array}{l} \mathcal{E} = \left\{ u \in C(\mathbb{R}, \mathbb{R}) : \lim_{|t|\to\infty} u(t) = 0 \right\}, \\ \mathcal{E}^+ = \{ u \in \mathcal{E} : u \geq 0 \text{ in } \mathbb{R} \}. \end{array}$$

For the physical interest, we refer the reader to [8], where authors were interested by existence of positive eigenvalues to the bvp (4.1).

The statement of main results of this section and their proofs need to introduce some notations. In what follows, we let $G$ and $G^*$ be the functions defined by

$$\begin{aligned} G(t,s) &= \frac{1}{r_2 - r_1} \begin{cases} \exp(r_1(t-s)) \text{ if } s \leq t \\ \exp(r_2(t-s)) \text{ if } t \leq s \end{cases} \\ G^*(t,s) &= \frac{1}{\rho_2 - \rho_1} \begin{cases} \exp(\rho_1(t-s)) \text{ if } s \leq t \\ \exp(\rho_2(t-s)) \text{ if } t \leq s \end{cases} \end{aligned}$$

where $r_1 < 0 < r_2$ are the solutions of the characteristic equation $-X^2 + cX + \lambda = 0$ and $\rho_1 < 0 < \rho_2$ are the solutions of the characteristic equation $-X^2 - cX + \lambda = 0$.

We let $E_p$ be the linear space defined by

$$E_p = \left\{ u \in C(\mathbb{R}, \mathbb{R}) : \lim_{|t|\to\infty} p(t) u(t) = 0 \right\}$$

where $p(t) = e^{-r_2|t|}$. Equipped with the norm $\|\cdot\|_p$, where for $u \in E_p$ $\|u\|_p = \sup_{t\in\mathbb{R}} p(t) |u(t)|$, $E_p$ becomes a Banach space.

The subsets $E_p^+$ and $K_p$ of $E_p$ given by

$$\begin{array}{l} E_p^+ = \{ u \in E : u \geq 0 \text{ in } \mathbb{R} \}, \\ K_p = \{ u \in E : u(t) \geq \gamma(t) \|u\| \text{ for all } t \in \mathbb{R} \} \end{array}$$

where

$$\gamma(t) = p(t) \inf(1, e^{2r_2 t}, e^{(r_1 - r_2)t}, e^{(r_1 + r_2)t}) = \begin{cases} e^{r_1 t} \text{ if } t \geq 0, \\ e^{r_2 t} \text{ if } t \leq 0. \end{cases}$$

are obviously total cones of $E$.

In the case where $q$ does not vanish identically on any interval, we let

$$L_q^1 = \left\{ u : \mathbb{R} \to \mathbb{R} \text{ measurable such that } \int_{\mathbb{R}} q |u| < \infty \right\}$$

equipped with natural norm $|\cdot|_{1,q}$ where for $u \in L_q^1$, $|u|_{1,q} = \int_{\mathbb{R}} q |u|$.

The subset

$$K = \left\{u \in L^1_q : u(t) \geq 0 \text{ a.e } t \in \mathbb{R}\right\}$$

is a total cone in $L^1_q$.

The dual space and the dual cone associated respectively to respectively of $L^1_q$ and $K$ are then

$$L^\infty_q = \left\{u : \mathbb{R} \to \mathbb{R} \text{ measurable such that } \sup_{t\in\mathbb{R}} q\,|u| < \infty\right\}$$

and

$$K^* = \left\{u \in L^\infty_q : u(t) \geq 0 \text{ a.e } t \in \mathbb{R}\right\}.$$

**Theorem 7.** *Assume that $q$ does not vanish identically on any subinterval, then there exists a unique real number $\mu_1 = \mu_1(q, \lambda, c) > 0$ such that:*

**i)** *If $h = 0$, then the bvp (4.1) admits a solution in $K \smallsetminus \{0\}$ if and only if $\mu = \mu_1$.*

**ii)** *For all $h \in L^1 \smallsetminus \{0\}$ the bvp (4.1) admits a unique solution in $K$ if $\mu < \mu_1$ and no solution in $K$ if $\mu \geq \mu_1$.*

**Theorem 8.** *Assume that $q$ does not vanish identically and the ratio $p/q$ belongs to $\mathcal{E}^+$, then there exists a unique real number $\mu_1 = \mu_1(q, \lambda, c) > 0$ such that:*

**i)** *If $h = 0$, then the bvp (4.1) admits a solution in $K_p \smallsetminus \{0\}$ if and only if $\mu = \mu_1$.*

**ii)** *For all $h \neq 0$ the bvp (4.1) admits a unique solution in $K_p$ if $\mu < \mu_1$ and no solution in $K_p$ if $\mu \geq \mu_1$.*

## 4.2 Preliminary Lemmas

We begin this section, by two results of compactness that are respectively versions of Frechet-Kolmogrov theorem ([11] p. 275) and Corduneanu theorem ([6], p. 62).

**Lemma 3.** *Let $q \in C(\mathbb{R}, \mathbb{R})$ with $q > 0$ a.e. in $\mathbb{R}$. A nonempty set $S$ in $L^1_q$, is relatively compact if and only if the following conditions hold:*

*(a) $S$ is bounded in $L^1_q$,*

*(b) for all $\epsilon > 0$ there is $\delta > 0$ such that for all $u \in S$ and all $\eta \in (0, \delta)$, $\int_{\mathbb{R}} |q(t+h)u(t+h) - q(t)u(t)|\, dt < \epsilon$, and*

*(c) for all $\epsilon > 0$ there is $\xi > 0$ such that for all $u \in S$ $\int_{\mathbb{R}\smallsetminus[-\xi,\xi]} q(t)\,|u(t)|\, dt < \epsilon$.*

**Lemma 4.** *A nonempty set $S$ in $E_p$ is relatively compact if and only if the following conditions hold:*

*(a) $S$ is bounded in $E_p$,*

*(b) functions in $\widetilde{S} = \{pu : u \in S\}$ are equicontinuous in compact intervals, and*

*(c) for all $\epsilon > 0$ there is there is $T > 0$ such that for all $u \in S$ $e^{-\theta t}\,|u(t)| < \epsilon$ for all $t$, $|t| > T$.*

**Lemma 5.** *Assume that $q \in E$, then for all $\epsilon > 0$ there is $\delta_{q,\epsilon} > 0$ such that for all $t \in \mathbb{R}$ and all $\eta \in (0, \delta_{q,\epsilon})$ $|q(t+\eta) - q(t)| < \epsilon$.*

*Proof.* Let $\epsilon > 0$, since $\lim_{|t|\to\infty} q(t) = 0$ there is $T > 0$ such that $q(t) < \epsilon/2$ for all $t$, $|t| > T$. Let $t \in \mathbb{R}$, if $t > T$ then $|q(t+\eta) - q(t)| \leq |q(t+\eta)| + |q(t)| < \epsilon$ and if $t < -2T$ and $h \in (0, T)$ then $|q(t+\eta) - q(t)| \leq |q(t+\eta)| + |q(t)| < \epsilon$. Now, because the function $q$ is uniformly continuous on the interval $[-2T, T]$, there is $\delta_{T,\epsilon} > 0$ such that for all $t \in [-2T, T]$ and all $\eta \in (0, \delta_{T,\epsilon})$ $|q(t+\eta) - q(t)| < \epsilon$.

We conclude from the above discussion that for all $t \in \mathbb{R}$ and all $\eta \in (0, \delta_{q,\epsilon})$, $|q(t+\eta) - q(t)| < \epsilon$ where $\delta_{q,\epsilon} = \inf(\delta_{T,\epsilon}, T)$. □

**Lemma 6.** *The function $G$ has the following properties:*

***i)*** *$0 < G(t,s) \leq \frac{1}{r_2 - r_1}$ for all $t, s \in \mathbb{R}$,*

***ii)*** *For all $t, \tau, s \in \mathbb{R}$*

$$p(t)G(t,s) \geq \widetilde{\gamma}(t)\, p(\tau)G(\tau, s)$$

*where $\widetilde{\gamma}(t) = \inf(1, e^{2r_2 t}, e^{(r_1 - r_2)t}, e^{(r_1 + r_2)t})$.*

***iii)*** *For all $u \in E$, $v(t) = \int_{\mathbb{R}} G(t,s)u(s)ds$ belongs to $E \cap C^2(\mathbb{R}) \cap E_p$ and satisfies $-v'' + cv' + \lambda v = u$. Moreover, if $u \in L^1$ then $v \in L^1$.*

*Proof.*

**i)** is obvious.

**ii)** Set $Q(t, \tau, s) = \dfrac{p(t)G(t,s)}{p(\tau)G(\tau,s)}$. We distinguish then four cases.

**a)** $\tau, t \geq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(-\left(r_2 - r_1\right)t + \left(r_2 - r_1\right)\tau\right) \geq e^{-(r_2 - r_1)t} \text{ if } s \leq \tau \leq t, \\ \exp\left(-\left(r_2 - r_1\right)t + \left(r_2 - r_1\right)s\right) \geq e^{-(r_2 - r_1)t} \text{ if } \tau \leq s \leq t, \\ 1 \text{ if } \tau \leq t \leq s, \\ \exp\left(-\left(r_2 - r_1\right)t + \left(r_2 - r_1\right)\tau\right) \geq e^{-(r_2 - r_1)t} \text{ if } s \leq t \leq \tau, \\ \exp\left(\left(r_2 - r_1\right)\tau - \left(r_2 - r_1\right)s\right) \geq 1 \text{ if } t \leq s \leq \tau, \\ 1 \text{ if } t \leq \tau \leq s \end{cases} \geq \widetilde{\gamma}(t).$$

**b)** $\tau, t \leq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(\left(r_2 + r_1\right)t - \left(r_2 + r_1\right)\tau\right) \geq e^{(r_2 + r_1)t} \text{ if } s \leq \tau \leq t, \\ \exp\left(-\left(r_2 - r_1\right)t - 2r_2\tau + \left(r_2 - r_1\right)s\right) \geq e^{-(r_2 - r_1)t} \text{ if } \tau \leq s \leq t, \\ \exp\left(2r_2 t - 2r_2\tau\right) \geq e^{2r_2 t} \text{ if } \tau \leq t \leq s, \\ \exp\left(\left(r_2 + r_1\right)t - \left(r_2 + r_1\right)\tau\right) \geq e^{(r_2 + r_1)t} \text{ if } s \leq t \leq \tau, \\ \exp\left(2r_2 t - \left(r_2 + r_1\right)\tau - \left(r_2 - r_1\right)s\right) \geq e^{2r_2 t} \text{ if } t \leq s \leq \tau, \\ \exp\left(2r_2 t - 2r_2\tau\right) \geq e^{2r_2 t} \text{ if } t \leq \tau \leq s \end{cases} \geq \widetilde{\gamma}(t).$$

**c)** $\tau \leq 0, t \geq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(-\left(r_2 - r_1\right)t - \left(r_2 + r_1\right)\tau\right) \geq e^{-(r_2 - r_1)t} \text{ if } s \leq \tau \leq t, \\ \exp\left(-\left(r_2 - r_1\right)t - 2r_2\tau + \left(r_2 - r_1\right)s\right) \geq e^{-(r_2 - r_1)t} \text{ if } \tau \leq s \leq t, \\ \exp\left(-2r_2\tau\right) \geq 1 \text{ if } \tau \leq t \leq s, \end{cases} \geq \widetilde{\gamma}(t).$$

**d)** $\tau \geq 0,\ t \leq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(\left(r_2+r_1\right)t + \left(r_2-r_1\right)\tau\right) \geq e^{(r_2+r_1)t} \text{ if } s \leq t \leq \tau, \\ \exp\left(2r_2 t + \left(r_2-r_1\right)\tau - \left(r_2-r_1\right)s\right) \geq e^{2r_2 t} \text{ if } t \leq s \leq \tau, \\ \exp\left(2r_2 t\right) \text{ if } t \leq \tau \leq s \end{cases} \geq \widetilde{\gamma}(t).$$

**iii)** We have

$$\begin{aligned} v(t) &= \frac{1}{r_2-r_1}\left(e^{r_1 t}\int_{-\infty}^{t} e^{-r_1 s}u(s)ds + e^{r_2 t}\int_{t}^{+\infty} e^{-r_2 s}u(s)ds\right) \\ &= \frac{1}{r_2-r_1}\left(\frac{\int_{-\infty}^{t} e^{-r_1 s}u(s)ds}{e^{-r_1 t}} + \frac{\int_{t}^{+\infty} e^{-r_2 s}u(s)ds}{e^{-r_2 t}}\right). \end{aligned}$$

This shows that $v \in C^2(\mathbb{R},\mathbb{R})$ and by means of L'Hopital's rule we see that $\lim_{|t|\to\infty} v(t) = 0$. Thus, we have proved that $v \in \left(E \cap C^2(\mathbb{R},\mathbb{R})\right) \subset E_p$.

Moreover if $u \in L^1$, we obtain by means of Fubbini's rule

$$\begin{aligned} \int_{\mathbb{R}} |v(t)|\,dt &\leq \frac{1}{r_2-r_1}\int_{\mathbb{R}}\left(e^{r_1 t}\int_{-\infty}^{t} e^{-r_1 s}|u(s)|\,ds + e^{r_2 t}\int_{t}^{+\infty} e^{-r_2 s}|u(s)|\,ds\right)dt \\ &= \frac{1}{r_2-r_1}\int_{\mathbb{R}} |u(s)|\left(e^{-r_1 s}\int_{s}^{+\infty} e^{r_1 t}dt + e^{-r_2 s}\int_{-\infty}^{s} e^{r_2 t}dt\right)ds \\ &= \frac{1}{-r_1 r_2}\int_{\mathbb{R}} |u(s)|\,ds. \end{aligned}$$

The proof is complete. □

Set for $u \in L_q^1$, $Lu(t) = \int_{\mathbb{R}} G(t,s)q(s)u(s)ds$

**Lemma 7.** *Assume that $q \in E$ and does not vanish identically on any inerval, then $L$ define a compact operator in $\mathcal{L}_K\left(L_q^1\right)$.*

*Proof.* Since Assertion **iii** in Lemma 6 states that for all $u \in L_q^1$, $Lu \in L^1$ and $q$ is a continuous bounded function, we have that $qLu \in L^1$, that is $Lu \in L_q^1$. The linearity and the positivity of $L$ are obvious, so let us prove the compactness of $L$. Let $\Omega \subset B(0,R)$, as in proof of Assertion **iii** in Lemma 6, we have for all $u \in \Omega$

$$\int_{\mathbb{R}} q(t)\,|Lu(t)|\,dt \leq q_\infty \int_{\mathbb{R}} |Lu(t)|\,dt = \frac{q_\infty}{-r_1 r_2}\int_{\mathbb{R}} q(s)\,|u(s)|\,ds \leq \frac{q_\infty R}{-r_1 r_2}.$$

We have for all $u \in \Omega$ and all $\eta > 0$

$$\int_{\mathbb{R}} |q(t+\eta)Lu(t+\eta) - q(t)Lu(t)|\,dt \leq I_1(\eta) + I_2(\eta) + J_1(\eta) + J_2(\eta)$$

where

$$\begin{aligned}
I_1(\eta) &= \int_{\mathbb{R}}\int_{-\infty}^{t} \left|q(t+\eta)e^{r_1\eta} - q(t)\right| e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt, \\
I_2(\eta) &= \int_{\mathbb{R}}\int_{t}^{+\infty} \left|q(t+\eta)e^{r_2\eta} - q(t)\right| e^{r_2(t-s)}q(s)\left|u(s)\right| dsdt, \\
J_1(\eta) &= \int_{\mathbb{R}}\int_{t}^{t+\eta} q(t+\eta)e^{r_1\eta}e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt, \\
J_2(\eta) &= \int_{\mathbb{R}}\int_{t}^{t+\eta} q(t+\eta)e^{r_2\eta}e^{r_2(t-s)}q(s)\left|u(s)\right| dsdt.
\end{aligned}$$

Straightforward computations lead to

$$\begin{aligned}
I_1(\eta) \leq &\int_{\mathbb{R}} \left|q(t+\eta) - q(t)\right| e^{r_1\eta}\int_{-\infty}^{t} e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt \\
&+ \int_{\mathbb{R}} q(t)\left|e^{r_1\eta} - 1\right| \int_{-\infty}^{t} e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt,
\end{aligned}$$

$$\begin{aligned}
I_2(\eta) \leq &\int_{\mathbb{R}} \left|q(t+\eta) - q(t)\right| e^{r_2\eta}\int_{t}^{+\infty} e^{r_2(t-s)}q(s)\left|u(s)\right| dsdt, \\
&+ \int_{\mathbb{R}} q(t)\left(e^{r_2\eta} - 1\right) \int_{t}^{+\infty} e^{r_2(t-s)}q(s)\left|u(s)\right| dsdt,
\end{aligned}$$

$$\begin{aligned}
J_1(\eta) &\leq q_\infty e^{r_1\eta}\int_{\mathbb{R}}\int_{t}^{t+\eta} e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt \\
&= q_\infty e^{r_1\eta}\int_{\mathbb{R}} e^{-r_1 s}q(s)\left|u(s)\right| \int_{s-\eta}^{s} e^{r_1(t-s)}dtds \\
&= q_\infty \frac{1-e^{r_1\eta}}{r_1}\int_{\mathbb{R}} q(s)\left|u(s)\right| ds \\
&\leq q_\infty \frac{e^{r_1\eta}-1}{r_1}R
\end{aligned}$$

and

$$J_2(\eta) \leq q_\infty \frac{e^{r_2\eta}-1}{r_2}R$$

where $q_\infty = \sup_{t\in\mathbb{R}} q(t)$.

Let $\epsilon > 0$, we obtain from Lemma 5 there is $\delta_{q,\epsilon} > 0$ such that for all $t \in \mathbb{R}$ and all $\eta \in (0, \delta_{q,\epsilon})$, $\left|q(t+\eta) - q(t)\right| < \epsilon$ and there is $\delta_\epsilon > 0$ such that for all $\eta \in (0, \delta_\epsilon)$, $\sup\left(e^{r_2\eta} - 1, 1 - e^{r_1\eta}\right) < \epsilon$. Thus, for all $\eta \in \left(0, \widetilde{\delta}_\epsilon\right)$ where $\widetilde{\delta}_\epsilon = \inf\left(\delta_{q,\epsilon}, \delta_\epsilon, 1\right)$ we have

$$\begin{aligned}
I_1(\eta) &\leq (1+q_\infty)\,\epsilon\int_{\mathbb{R}}\int_{-\infty}^{t} e^{r_1(t-s)}q(s)\left|u(s)\right| dsdt \\
&= \frac{(1+q_\infty)}{-r_1}\epsilon\int_{\mathbb{R}} q(s)\left|u(s)\right| ds \leq \frac{(1+q_\infty)\,R}{-r_1}\epsilon,
\end{aligned}$$

$$I_2(\eta) \leq \frac{(e^{r_2} + q_\infty) R}{r_2} \epsilon$$

$$J_1(\eta) \leq q_\infty \frac{R}{-r_1} \epsilon$$

and

$$J_2(\eta) \leq q_\infty \frac{R}{r_2} \epsilon.$$

All the above estimates show that Condition b in Corollary 3 is satisfied.

Now, let $T > 0$ such that $q(t) < \epsilon$ for all $t,\ |t| > T$. We have

$$\begin{array}{l} \int_{\mathbb{R}\setminus[-T,T]} q(t) |Lu(t)| \, dt \leq \epsilon \int_{\mathbb{R}\setminus[-T,T]} |Lu(t)| \, dt \leq \epsilon \int_{\mathbb{R}} |Lu(t)| \, dt \\ \leq \epsilon \left( \int_{\mathbb{R}} \int_{-\infty}^{t} e^{r_1(t-s)} q(s) |u(s)| \, ds dt + \int_{\mathbb{R}} \int_{t}^{+\infty} e^{r_2(t-s)} q(s) |u(s)| \, ds dt \right) \\ = \epsilon \left( \int_{\mathbb{R}} \int_{s}^{+\infty} e^{r_1(t-s)} q(s) |u(s)| \, dt ds + \int_{\mathbb{R}} \int_{-\infty}^{s} e^{r_2(t-s)} q(s) |u(s)| \, dt ds \right) \\ \leq \left( \frac{r_1 - r_2}{r_1 r_2} \right) R\epsilon. \end{array}$$

This show that Condition c in Lemma 3 is satisfied and achieve the proof of compactness of the operator $L$. □

**Lemma 8.** *Assume that $q \in E$ and does not vanish identically on $\mathbb{R}$ (it may vanish on some intervals), then $L$ define a compact operator in $\mathcal{L}_{K_p}(E_p)$.*

*Proof.* Since for all $u \in E_p$ $qu = \frac{q}{p}(pu) \in \mathcal{E}$, Assertion **iii** in Lemma 6 guarantees that $Lu \in E_p$. The linearity of $L$ is obvious. Now, we prove that $L(K_p) \subset K_p$, that is $L \in \mathcal{L}_{K_p}(E_p)$. Let $u \in E_p^+$, for all $t, \tau \in \mathbb{R}$ Assertion ii in Lemma 6 gives

$$p(t)Lu(t) = \int_{\mathbb{R}} p(t)G(t,s)q(s)u(s)ds \geq \widetilde{\gamma}(t) \int_{\mathbb{R}} p(\tau)G(\tau,s)q(s)u(s)ds,$$

leading to

$$Lu(t) \geq \frac{\widetilde{\gamma}(t)}{p(t)} \int_{\mathbb{R}} p(\tau)G(\tau,s)q(s)u(s)ds = \gamma(t)\, p(\tau)Lu(\tau).$$

Since $\tau$ is arbitrary in $\mathbb{R}$, we obtain that for all $t \in \mathbb{R}$, $Lu(t) \geq \|Lu\|_p$. Proving that $L(E_p^+) \subset K_p$.

Now, let $\Omega$ be a subset in $E_p$ bounded by $R > 0$ and set $\widetilde{\Omega} = \{pLu : u \in \Omega\}$. We have for any $u \in \Omega$ and $t \geq 0$

$$\begin{array}{c} |p(t)Lu(t)| \leq \int_{\mathbb{R}} \frac{p(t)}{p(s)} G(t,s)q(s) |p(s)u(s))| \, ds \\ \leq \|u\|_p \, p(t) \int_{\mathbb{R}} \frac{q(s)}{p(s)} G(t,s)ds \\ \leq R \sup_{t\in\mathbb{R}} \left( p(t) \int_{\mathbb{R}} \frac{q(s)}{p(s)} G(t,s)ds \right). \end{array}$$

This proves that $L(\Omega)$ is bounded in $E_P$.

Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}$, for all $u \in \Omega$ we have

$$\begin{aligned}|p(t_2)Lu\,(t_2) - p(t_1)Lu\,(t_1)| \le{}& |p_1(t_2) - p_1(t_1)| \textstyle\int_{-\infty}^{\zeta} e^{-r_1 s} \frac{q(s)}{p(s)} ds \\ &+ |p_2(t_2) - p_2(t_1)| \textstyle\int_{\eta}^{+\infty} e^{-r_2 s} \frac{q(s)}{p(s)} ds \\ &+ \left(\sup_{t\in[\eta,\zeta]} \Gamma(t)\right) |t_2 - t_1|\end{aligned}$$

where for $i = 1, 2,\ p_i(t) = e^{-r_2|t| + r_i t}$ and $\Gamma(t) = \left(e^{2(r_2 - r_1)|t|} + e^{4r_2|t|}\right) q(t)$.

Because $p_1, p_2$ are uniformly continuous on compact intervals, the above estimates prove that $\widetilde{\Omega}$ is equicontinuous on compact intervals.

We have for all $u \in \Omega$ and $t \ge 0$ :

$$|p(t)Lu(t)| \le Rp(t) \int_{\mathbb{R}} \frac{q(t)}{p(s)} G(t, s) ds := R\widehat{H}(t).$$

Since $\lim_{|t|\to+\infty} \widehat{H}(t) = 0$, we have that $\widetilde{\Omega}$ is equiconvergent at $\pm\infty$. This ends the proof. □

## 4.3 Proof of Theorem 7

Notice that $u$ is a positive solution to the bvp (4.1) if and only if $u - \mu Lu = \widetilde{h}$ where $\widetilde{h}(t) = \int_{-\infty}^{+\infty} G(t, s)h(s)ds$. Since Lemma 6 guarantees that $\widetilde{h}$ belongs to $K$ and Lemma 7 states that $L$ is a compact operator in $\mathcal{L}_K(L_q^1)$, we have to prove that $L$ is a KRO.

Let $[a, b]$ be any interval in $\mathbb{R}$ and consider the function

$$u_0(s) = \begin{cases} (s-a)(b-s) & \text{if } s \in [a, b] \\ 0 & \text{if } s \notin [a, b]. \end{cases}$$

We have then, $Lu_0 \ge \lambda_0 u_0$ where $\lambda_0 = \min_{t\in[a,b]} (Lu_0(t)/u_0(t)) > 0$, proving that $\Lambda_{L,K}^{-} \ne \emptyset$ and $\mu_1 = \mu_1(q, \lambda, c) = \lambda_{L,K_\theta}^{-} = 1/r(L)$ is a pcv of $L$. It is easy to see that that adjoint operator $L^*$ of $L$ where $L^* \in \mathcal{L}\left(L_q^\infty\right)$ is defined for $u \in L_q^\infty$ by $L^*u(s) = \int_{\mathbb{R}} G(t, s)q(t)u(t)dt$. Moreover, Theorem 1 states that $\mu_1$ is a pcv of $L^*$ having a positive eigenfunction $\phi$.

Now, we have just to prove that $L^*$ has the property $(\mathcal{H})$. First, notice that a function $u$ in $K$ is strictly positivity if and only if $u > 0$ a.e. in $\mathbb{R}$. Consequently, we have to prove that if $\mu$ is a pcv of $L^*$,having a positive eigenfunction $\phi$, then $\phi > 0$ in $\mathbb{R}$. Since $\mu > 0$ and $q > 0$ a.e. in $\mathbb{R}$, we have

$$\phi(t) = \mu \int_{\mathbb{R}} G^*(t, s)q(s)\phi(s)ds > 0 \text{ for all } t \in \mathbb{R}.$$

Thus, Theorem 7 follows from a direct application of Theorem 3.

## 4.4 Proof of Theorem 8

Let $[a, b]$ be an interval such that $q(s) > 0$ for all $s \in [a, b]$ and consider the function

$$u_0(s) = \begin{cases} (s-a)(b-s) & \text{if } s \in [a, b] \\ 0 & \text{if } s \notin [a, b]. \end{cases}$$

We have then, $Lu_0 \geq \lambda_0 u_0$ where $\lambda_0 = \min_{t\in[a,b]} (Lu_0(t)/u_0(t)) > 0$. This proves that $\Lambda^-_{L,K_\theta} \neq \emptyset$ and $\mu_1 = \mu_1(q,\lambda,c) = \lambda^-_{L,K_\theta} = 1/r(L)$ is a pcv of $L$ having a positive eigenfunction $\psi$.

At this stage let us prove that $\mu_1$ is geometrically simple. Suppose that $\psi_1$ is an eigenfunction associated with $\mu_1$ and $W$ be the Wronksian of $\psi$ and $\psi_1$. We have then $W' - cW = 0$, leading to $W = \alpha e^{ct}$. Since

$$\begin{aligned} \psi'(t) &= \frac{\mu_1 r_1 \int_{-\infty}^{t} e^{-r_1 s} q(s)\psi(s)ds}{(r_2-r_1)\, e^{-r_1 t}} + \frac{\mu_1 r_2 \int_{t}^{+\infty} e^{-r_2 s} q(s)\psi(s)ds}{(r_2-r_1)\, e^{-r_2 t}}, \\ \psi_1'(t) &= \frac{\mu_1 r_1 \int_{-\infty}^{t} e^{-r_1 s} q(s)\psi_1(s)ds}{(r_2-r_1)\, e^{-r_1 t}} + \frac{\mu_1 r_2 \int_{t}^{+\infty} e^{-r_2 s} q(s)\psi_1(s)ds}{(r_2-r_1)\, e^{-r_2 t}}, \end{aligned}$$

we have $\lim_{t\to+\infty} \psi'(t) = \lim_{t\to+\infty} \psi_1'(t) = 0$ and

$$\lim_{t\to+\infty} \alpha e^{ct} = \lim_{t\to+\infty} W(t) = \lim_{t\to+\infty} \psi(t)\psi_1'(t) - \psi'(t)\psi_1(t) = 0.$$

Leading to $\alpha = 0$ and so, to $W = 0$. Thus, we have proved that $\mu_1$ is geometrically simple.

Consider now the eigenvalue problem

$$\begin{cases} -u'' - cu' + \lambda u = \mu q u \text{ in } \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0. \end{cases} \tag{4.2a}$$

We claim that taking $\mu_1 = \mu_1(q,\lambda,c) = \lambda^-_{L,K_\theta}$ is a positive of the bvp (4.2a). Indeed, we have that $\mu$ is a positive eigenvalue of the bvp (4.2a) if and only if $\mu$ is a pcv of the linear compact operator $L^* \in \mathcal{L}(E)$ where

$$L^* u(t) = \int_{\mathbb{R}} G^*(t,s) q(s) u(s) ds,$$

As for $L$, $r(L^*) > 0$ and $\lambda^-_{L^*,K_\theta}$ is a pcv of $L^*$ having a positive eigenvector $\psi$. Moreover, we have

$$\int_{\mathbb{R}} (-\phi'' + c\phi' + \lambda\phi)\, \psi = \mu_1 \int_{\mathbb{R}} q\phi\psi.$$

Integrating by parts, we obtain

$$\begin{aligned} \mu_1 \int_{\mathbb{R}} q\phi\psi &= \int_{\mathbb{R}} (-\phi'' + c\phi' + \lambda\phi)\, \psi \\ &= \int_{\mathbb{R}} (-\psi'' - c\psi' + \lambda\psi)\, \phi \\ &= \lambda^-_{L^*,K_\theta} \int_{\mathbb{R}} q\phi\psi \end{aligned}$$

leading to $\lambda^-_{L^*,K} = \mu_1$.

Let $h \in K_p \setminus \{0\}$ and suppose that the equation $u - \mu_1 L u = h$ has a solution $v$ and let $\omega = v - h$. Thus $\omega$ satisfies

$$\begin{cases} -\omega'' + c\omega' + \lambda\omega = \mu_1 q\omega + \mu_1 q h \text{ in } \mathbb{R}, \\ \omega(-\infty) = \omega(+\infty) = 0 \end{cases}$$

Therefore, we have

$$\int_{\mathbb{R}} (-\omega'' + c\omega' + \lambda\omega)\,\phi = \mu_1 \int_{\mathbb{R}} q\omega\phi + \mu_1 \int_{\mathbb{R}} qh\phi$$

and two integrations by parts lead to

$$\mu_1 \int_{\mathbb{R}} qh\phi \ = 0.$$

But this is impossible since $\mu_1 > 0$, $h \in K_p \setminus \{0\}$ and

$$\phi\,(t) = \mu_1 \int_{\mathbb{R}} G(t,s)q(s)\phi(s)ds > 0 \text{ for all } t \in \mathbb{R}.$$

Indeed, if $\phi\,(t_0) = 0$ for some $t_0 \in \mathbb{R}$ then $q(s)\phi(s) = 0$ for all $s \in \mathbb{R}$, and there is an interval $[\alpha, \beta]$ such that $\phi(s) = 0$ for all $s \in [\alpha, \beta]$. In particular there is $t_*$ such that $\phi\,(t_*) = \phi'\,(t_*) = 0$ and since $\phi$ satisfies a linear second order ordinary differential equation, $\phi$ vanishes identically. This contradicts the fact that $\phi$ is an eigenfunction associated with $\mu_1$. Thus, Theorem 7 follows from a direct application of Corollary 4.

## 5 Existence and uniqueness for a third order bvp

We are concerned in this section by existence and uniqueness of solution to the third-order bvp

$$\begin{cases} -u''' + au'' + bu' = f(t,u) \text{ in } (0,+\infty), \\ u(0) = \alpha, \ u'(0) = \beta, \ u'(+\infty) = 0 \end{cases} \tag{5.1}$$

where $a, b, \alpha, \beta$ are real numbers with $a, b > 0$ and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In what follows, we let

$$\begin{aligned} \mathcal{P} &= \left\{ \begin{array}{l} m \in C(\mathbb{R}^+, \mathbb{R}^+) : m(t_0) > 0 \text{ for some } t_0 \geq 0 \\ \text{and } \lim_{t \to +\infty}(1+t)m(t) = 0 \end{array} \right\}, \\ E &= \left\{ u \in C(\mathbb{R}^+, \mathbb{R}) : \sup_{t \geq 0} \frac{|u(t)|}{1+t} < \infty \right\}. \end{aligned}$$

Equipped with the norm $\|u\| = \sup_{t\geq 0} \frac{|u(t)|}{1+t}$, $E$ becomes a Banach space. We denote then by $E^+$ the cone of nonnegative function in $E$.

The Green's function $G$ associated with the bvp (5.1) is given by

$$G(t,s) = \frac{1}{W(s)} \begin{cases} -\phi_1'(s)\phi_2(t) + \phi_1'(s)\phi_2(s) - \phi_1(s)\phi_2'(s) & \text{if } s \leq t \\ -\phi_2'(s)\phi_1(t) & \text{if } t \leq s \end{cases}$$

where

$$\begin{aligned} &W(s) = r_1^2 r_2\,(r_2 - r_1)\exp((r_2 + r_1)\,s), \\ &\phi_1(s) = (r_2 - r_1) - r_2 \exp(r_1 s) + r_1 \exp(r_2 s), \\ &\phi_2(s) = 1 - \exp(r_1 s), \end{aligned}$$

and $r_1,\ r_2$ are the solutions of the equation $-X^2+aX+b=0$ and are such that $r_1<0<r_2$. We have

$$\begin{aligned}\frac{\partial G}{\partial t}(t,s) &= \frac{1}{W(s)}\left\{\begin{array}{l} -\phi_1'(s)\phi_2'(t) \text{ if } s\le t\\ -\phi_2'(s)\phi_1'(t) \text{ if } t\le s\end{array}\right.\\ &= \frac{1}{r_2-r_1}\left\{\begin{array}{l} (e^{-r_1 s}-e^{-r_2 s})e^{-r_1 t} \text{ if } s\le t\\ (e^{r_2 t}-e^{r_1 t})e^{-r_2 s} \text{ if } t\le s.\end{array}\right.\end{aligned}$$

Arguing as in the proof of Assertion **iii** in Lemma 6, we see that for $h\in C(\mathbb{R}^+,\mathbb{R})$ with $\lim_{t\to+\infty} h(t)=0$, the function $v(t)=\int_0^{+\infty}G(t,s)u(s)ds$ belongs to $E$ and satisfies

$$\left\{\begin{array}{l} -v'''+av''+bv'=h \text{ in } (0,+\infty),\\ v(0)=v'(0)=v'(+\infty)=0.\end{array}\right.$$

Consequently, for $q\in\mathcal{P}$ the operator $L_q: E\to E$ where for $u\in E$, $L_q u(t)=\int_0^{+\infty}G(t,s)q(s)u(s)ds$, is well defined and is linear bounded and positive, that is $L_q(E^+)\subset E^+$. Moreover, arguing as in the proof of Theorem 8 we see that $L_q u_0\ge \mu_0 u_0$ for some $\mu_0>0$ and $u_0\in E^+\setminus\{0\}$ and so, $r(L_q)>0$.

Let $\phi(t)=\alpha-\dfrac{\beta}{r_1}(1-\exp(r_1 t))$. It is easy to see that $\phi$ is the unique solution to

$$\left\{\begin{array}{l} -u'''+au''+bu'=0, \ \text{ in } (0,+\infty)\\ u(0)=\alpha,\ u'(0)=\beta,\ u'(+\infty)=0.\end{array}\right.$$

The main result in this section is:

**Theorem 9.** *Assume that* $\lim_{t\to+\infty} f(t,\phi(t))=0$*, there exists* $q\in\mathcal{P}$ *and* $c>0$ *such that for all* $u,v\in\mathbb{R}$ *and all* $t\in\mathbb{R}^+$,

$$|f(t,u)-f(t,v)|\le cq(t)\,|u-v|\,.$$

*If* $cr(L_q)<1$ *then the bvp (5.1) admits a unique solution.*

*Proof.* Notice that $u\in E$ is a solution to the bvp (5.1) if and only if $v=u-\phi$ satisfies

$$\left\{\begin{array}{l} -v'''+av''+bv'=f(t,v+\phi) \text{ in } (0,+\infty)\\ v(0)=v'(0)=v'(+\infty)=0,\end{array}\right. \tag{5.2a}$$

Set for $v\in E$, $Tv(t)=\int_0^{+\infty}G(t,s)f(s,v(s)+\phi(s))ds$. We have from the hypotheses in Theorem 9,

$$|Tv(t)|\le c\int_0^{+\infty}G(t,s)q(s)\,|v(s)|\,ds+\int_0^{+\infty}G(t,s)f(s,\phi(s))ds$$

and

$$\left|(Tv)'(t)\right|\le c\int_0^{+\infty}\frac{\partial G}{\partial t}(t,s)q(s)\,|v(s)|\,ds+\int_0^{+\infty}\frac{\partial G}{\partial t}(t,s)f(s,\phi(s))ds.$$

The above two estimates prove that $Tv\in E$ and $T$ define a self-mapping on $E$. Moreover, $v\in E$ is a solution to the bvp (5.2a) if and only if $v$ is a fixed point of $T$.

At the end, since $E^+$ is a normal and minihedral cone in $E$ and for all $v_1, v_2 \in E$

$$|Tv_1 - Tv_2| \leq cL_q |u - v| \text{ with } c < 1/r(L_q),$$

we conclude from Theorem 2 that the mapping admits a unique fixed point $v$, then $u = v + \phi$ is the unique solution to the bvp (5.1). □

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