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A note on roots of Möbius transformations in discrete groups

by
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Abstract

This short article treats a very naive question concerning roots of Möbius transformations in discrete groups. We exhibit, according to the standard classification of Möbius transformations, formulations depending on the types of given transformations. Along the way, algebraic natures of relevant subgroups of a discrete group under consideration are discussed, which allows us to provide intuitive geometrical interpretations. We indicate how the discreteness plays a substantial role during the study while exploring folklore or classical results for these subgroups. We conclude by remarking on the original motivation for the study and point out possible directions for future research.

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1 Introduction

In order to set up terminology to be adhered to throughout the article, we briefly recall standard notation and establish our definitions.

First $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. For instance, $\mathbb{C}$ is understood to be a topological field which is equipped with the topology induced by the absolute-value norm $|\cdot|$. We shall, without further comment, assume such familiar underlying topological structures. The complex numbers $\mathbb{C}$ is from time to time thought of as the complex plane. In this context, we use $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the extended plane (i.e., the Riemann sphere), and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for the punctured plane. $\mathbb{Z}$ denotes the integers (positive, zero, and negative), which is to be in this paper viewed as the infinite cyclic group or the free abelian group of rank 1 through the ordinary addition operation; we denote by $\mathbb{Z}_q$ the cyclic group $\mathbb{Z}/q\mathbb{Z}$ for a (positive) integer q.

We write the group of all Möbius (or linear-fractional) transformations with complex coefficients

$$z \mapsto \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \tag{1.1}$$

as $\text{Möb}(\mathbb{C}) \cong \mathrm{Aut}(\hat{\mathbb{C}})$, which is identified with the three-dimensional complex Lie group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm I$. There is no opportunity to make use of the manifold structure of $\mathrm{PSL}_2(\mathbb{C})$ in what follows, so it would simply be regarded as a topological group with the quotient topology from $\mathrm{SL}_2(\mathbb{C})$, which is endowed with a subspace topology sitting in a Euclidean space of 2×2 matrices over $\mathbb{C}$.

Let n be a positive integer. For $g \in \text{Möb}(\mathbb{C})$, consider a simple algebraic equation

$$\gamma^n = g. \tag{1.2}$$

A solution of the equation (1.2) always exists in $\text{Möb}(\mathbb{C})$. Such a solution is called an nth root (or, more shortly but less informatively, a root) of g; see (for example) Beardon's text [1, p. 74]. The situation is different, however, when we seek a solution in a discrete group $\Gamma \subset \text{Möb}(\mathbb{C})$. This is partly because topology intervenes.

A series of observations indicate how the topology or discreteness plays a role in the study. Along the way, we occasionally discuss somewhat folklore results for relevant subgroups of a discrete group Γ, which have, to the best of the author's knowledge, not been formulated in quite the same manner. Such subgroups are naturally connected with centralizers of $g \in \Gamma$ that are typically abelian and easy to compute, as γ is in the centralizer of g if g is a power of γ. In the last section we point out the origin of the study and possible directions for further research in the future.

Remark. A group $\Gamma \subset \text{Möb}(\mathbb{C})$ is classically called, by definition, a *Kleinian group* if the *region of discontinuity of* Γ

$$\Omega(\Gamma) = \{ z \in \hat{\mathbb{C}} \mid \Gamma \text{ acts discontinuously at } z \} \tag{1.3}$$

is nonempty. Some of the works refer to a discrete group as a Kleinian group, which is necessarily discrete by the very definition above. Nevertheless, it is possible for $\Omega(\Gamma)$ to be empty even though Γ is discrete. We avoid (except in the last section) the use of the term "Kleinian," because for our assertions Γ need only be discrete but is not at all required to be Kleinian in the sense that its region of discontinuity is nonempty.

2 Fixed points of Möbius transformations

We further introduce some notation that we follow for the remainder of the present paper. F_g is the set of the fixed point(s) of a transformation g. As before Γ is a subgroup of Möb($\mathbb{C}$), which is usually (but not always) discrete, where being discrete means discrete as a subset with respect to the Lie-group topology of Möb($\mathbb{C}$). Furthermore, $Z_\Gamma(g)$ denotes the centralizer of g of Γ, for which the group Γ is dropped from the symbols when $\Gamma = $ Möb($\mathbb{C}$).

Let $1 \neq g \in$ Möb($\mathbb{C}$). (1 denotes for us appropriate identity elements.) With the aid of the quantity $\mathrm{tr}^2\, g = (a+d)^2$, which is well defined, we classify standardly:

(i) g is *parabolic* if $\mathrm{tr}^2\, g = 4$. This happens precisely when g is conjugate to a translation $z \mapsto z+1$ in Möb($\mathbb{C}$).

(ii) g is *elliptic* if $0 \leq \mathrm{tr}^2\, g < 4$, which happens if and only if g is conjugate to a Euclidean rotation $z \mapsto \alpha z$ with $|\alpha| = 1$ in Möb($\mathbb{C}$).

(iii) g is *loxodromic* if it is neither parabolic nor elliptic.

It is easy to check that $\#F_g \geq 1$ and the equality holds if and only if g is parabolic. (Here, and in the sequel, # denotes the cardinality of a set.) Also $\#F_g \leq 2$ for every non-identity transformation, because a Möbius transformation is specified by its action at just three points. Using chiefly those well-known facts, we proceed to recall some (algebraic) information for convenience.

As we have already mentioned, we wish to solve the equation (1.2) in a discrete group Γ. Clearly, the type of a Möbius transformation that has just been introduced, and the fixed points as well, are preserved by raising to a power, although iterations of an elliptic transformation can possibly be the identity transformation. This too is useful in the (initial) analysis below for our discussions of the determination of "appropriate" subgroups of Γ to work on.

We are now going to examine in detail the fixed-point sets of two commuting transformations.

One will be able to generally show that, if two transformations commute with each other, every one of the transformations, when they are bijections, leaves invariant the fixed points of the other of the transformations; in particular, for two Möbius transformations g and h, the commutative property $gh = hg$ implies

$$g(F_h) = F_h, \quad h(F_g) = F_g. \tag{2.1}$$

Thus, g (resp., h) fixes $F_g \cup F_h$ setwise while fixing F_g (resp., F_h) pointwise.

Now suppose that $1 \neq h \in Z(g)$. We claim that, when g is parabolic, h is also parabolic such that g and h have the same fixed point. Indeed, if not, at least g^2 has to fix a point of F_h which is not in F_g. This contradicts the assumption that g is parabolic.

We record for frequent use later the next

Assertion. *Let g and h be nontrivial Möbius transformations. Assume that $gh = hg$. Then*

$$\#F_g = \#F_h. \tag{2.2}$$

Moreover, we have:

(i) *when g and h are parabolic, $F_g = F_h$;*

(ii) *when g and h are non-parabolic, either $F_g = F_h$ or $F_g \cap F_h = \emptyset$. In the latter case, g and h are both elliptic of order two and g (resp., h) interchanges the two points of F_h (resp., F_g).*

In particular, when g is not elliptic of order two, the centralizer $Z(g)$ coincides with the abelian group comprising the transformations each of which has exactly the same fixed-point set as g.

In the sequel, the abelian group found in the statement of Assertion will be written as $S(g)$. For the remainder of the current note it frequently shows up due to the nature in its own right.

Proof of Assertion. Let us make a comment on the second part of the assertion. When g and h are non-parabolic, we have

$$\#F_g = \#F_h = 2. \tag{2.3}$$

Assume for contradiction $\#(F_g \cap F_h) = 1$ (by (2.3) no other possibilities exist). Then,

$$\#(F_g \cup F_h) = \#F_g + \#F_h - \#(F_g \cap F_h) = 3 \tag{2.4}$$

and one can verify that g (and h) have to fix $F_g \cup F_h$ pointwise. This contradiction shows that either $F_g = F_h$ or $F_g \cap F_h = \emptyset$.

The last statement is now practically obvious. See also the remarks below. □

Remarks. (1) If $F_g = F_h$, direct calculations yield the commutativity of g and h (see the subsequent sections).

(2) When g is elliptic of order two, it involves no loss of generality to assume after a routine conjugation that $g = -z$. If h interchanges the two points of $F_g = \{\, 0, \infty \,\}$, then it has to be of the form $z \mapsto b/z$, with $b \neq 0$. Notice that from this setup h is also elliptic of order two, and g automatically interchanges the two points of $F_h = \{\, \pm\sqrt{b} \,\}$.

(3) One is thus able to convince himself or herself that the condition (2.1) actually implies that $gh = hg$ for two Möbius transformations g and h.

In the following section, we will begin to study discreteness (of discrete subgroups Γ of Möb($\mathbb{C}$)), which has not played a vital part in the discussion so far.

3 The parabolic cases

Suppose in this section that g is of parabolic type. Thus, for the moment, we assume (by conjugation for sending the sole fixed point to ∞) that $F_g = \{\, \infty \,\}$; that is, g is of the form

$$z \mapsto z + \beta_g \tag{3.1}$$

with a nonzero complex number β_g. In the light of part (i) of Assertion in Section 2, the centralizer $Z_\Gamma(g)$ comprises the transformations with the same fixed point as g, which are inevitably of parabolic type (except for identity transformation, of course). Hence, $Z_\Gamma(g)$ and the abelian subgroup* of Γ

$$S_\Gamma(g) = \{\, \gamma \in \Gamma \mid F_\gamma = F_g \,\} \tag{3.2}$$

coincide for a parabolic transformation g—and $S_\Gamma(g)$ is the desired subgroup that we are seeking.

With the foregoing notation and conventions, we define an injective homomorphism

$$\tau \colon S_\Gamma(g) \to \mathbb{C} \tag{3.3}$$

given by $\gamma \mapsto \beta_\gamma$, where $\mathbb{C}$ is regarded as an additive group topologized by the absolute-value norm. Since $S_\Gamma(g)$ is discrete, so is the image of $S_\Gamma(g)$ in $\mathbb{C}$ under the homomorphism τ. Therefore, $S_\Gamma(g)$ being nontrivial is a free abelian group of rank 1 or 2.

We have shown:

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*As (briefly) mentioned, in order not to make exceptions of the identity transformation we will and hereafter do adopt the convention that 1(= identity element) belongs to any of these kinds of sets in the present article.

Proposition 1. *Let g be a parabolic transformation of a discrete group $\Gamma \subset$ Möb$(\mathbb{C})$. Then the centralizer $Z_\Gamma(g)$ coincides with $S_\Gamma(g)$ and either $S_\Gamma(g) \cong \mathbb{Z}$ or $S_\Gamma(g) \cong \mathbb{Z} \oplus \mathbb{Z}$. The equation* (1.2) *admits a solution in Γ if and only if g belongs to the subgroup of $S_\Gamma(g)$*

$$S^n_\Gamma(g) = \{\, \gamma^n \mid \gamma \in S_\Gamma(g) \,\}\,. \tag{3.4}$$

The equation admits, if any, a unique solution in Γ.

We shall in the rest of the paper assume that $g \neq 1$ is non-parabolic, i.e., $\#F_g = 2$, unless the contrary is stated explicitly.

4 The elliptic cases

The elliptic case could be treated as a "specialized" version of the loxodromic case to be studied in the next section. However, in order to get a transparent picture of what is going on, we separately study the elliptic case.

We assume after conjugation that $F_g = \{\, 0, \infty \,\}$; that is, g is of the form

$$z \mapsto \alpha_g z. \tag{4.1}$$

The coefficient α_g is known as the multiplier of g.[†] Because g is elliptic, α_g is a nonzero complex number such that $|\alpha_g| = 1$ (also see the remark at the end of this section).

In view of part (ii) of Assertion in Section 2, the centralizer $Z_\Gamma(g)$ seems to be a larger subgroup than is required. Even if no elliptic transformations of order two are involved, the subgroup $S_\Gamma(g)$, which has been introduced by (3.2) for parabolic transformations but can also be defined for non-parabolic transformations, would be still too large. In this situation, we shall use a smaller subgroup

$$\Sigma_\Gamma(g) = \{\, \gamma \in \Gamma \mid F_\gamma = F_g \text{ and } \gamma \text{ is of elliptic type} \,\}\,, \tag{4.2}$$

as the set appearing on the right-hand side is indeed a group.

We immediately have $\Sigma(g)$ is a normal subgroup of $S(g)$ (dropping the subscript Möb$(\mathbb{C})$ for brevity).

To digress slightly, we define, again in the light of Assertion, a natural homomorphism

$$j \colon Z(g) \to \mathfrak{S}(F_g) \tag{4.3}$$

[†]Associated uniquely to a (non-parabolic) transformation g is a quantity $\alpha_g + 1/\alpha_g$. It is not hard to verify that $\alpha_g + 1/\alpha_g = \operatorname{tr}^2 g - 2$.

by viewing $\gamma \in Z(g)$ as an element of the symmetric group $\mathfrak{S}(F_g)$. We also see that γ is stationary on F_g if and only if it belongs to the kernel of j, or in other words

$$\ker j = S(g). \tag{4.4}$$

For an elliptic transformation g of order two, the relevant information is thus summarized in the following short exact sequence:

$$1 \to S(g) \hookrightarrow Z(g) \xrightarrow{j} \mathfrak{S}(F_g) \to 1. \tag{4.5}$$

Note that we are to identify the image of $Z(g)$ under the homomorphism j with the cyclic group $\mathbb{Z}_2 \cong \mathfrak{S}(F_g)$, where the nontrivial element of $\mathbb{Z}_2$ corresponds to the permutation of $\mathfrak{S}(F_g)$ that interchanges the two points of F_g. There exists obviously a cross section (i.e., right inverse)

$$s\colon \mathfrak{S}(F_g) \to Z(g) \tag{4.6}$$

of the homomorphism j. Thus, for the sequence splits, we have that $Z(g)$ is a semidirect product of $S(g)$ and $s(\mathfrak{S}(F_g))$, in symbols

$$Z(g) = S(g) \rtimes s(\mathfrak{S}(F_g)), \tag{4.7}$$

or that $Z(g)$ is a splitting $\mathbb{Z}_2$-extension of $S(g)$.

In passing, we mention that for the purpose of quickly deriving some properties of $Z(g)$, a straightforward computation could serve as an elementary but helpful tool. For example, we thereby deduce that if a non-identity transformation of $S(g) \subseteq Z(g)$ commutes with the transformation of $Z(g)$ which corresponds to the permutation of $\mathfrak{S}(F_g)$ via the cross section s as above, then it is in fact g, because its multiplier, say α, must satisfy $\alpha - 1/\alpha = 0$. See Remark 2 of Section 2. In addition, the algebraic structure of $S(g)$ and the relation between $S(g)$ and $\Sigma(g)$ are explained later (see the argument after the corollary following Theorem in Section 5).

After this short detour we now return to (4.2) and define a further injective homomorphism

$$\theta\colon \Sigma_\Gamma(g) \to \mathbb{T} \tag{4.8}$$

given by $\gamma \mapsto \alpha_\gamma$, where $\mathbb{T} \subset \mathbb{C}^*$ is the compact multiplicative group (a 1-torus) of all complex numbers with absolute value 1. The image of $\Sigma_\Gamma(g)$ in $\mathbb{T}$ under the homomorphism θ is discrete. Hence,

$$\Sigma_\Gamma(g) \cong \mathbb{Z}_q \tag{4.9}$$

for some integer $q \geq 2$ as $\Sigma_\Gamma(g)$ is a nontrivial finite cyclic group.

We have almost obtained

Proposition 2. *Let g be an elliptic transformation of a discrete group $\Gamma \subset \text{Möb}(\mathbb{C})$. Then there exists an integer $q \geq 2$ such that $\Sigma_\Gamma(g) \cong \mathbb{Z}_q$. The equation* (1.2) *admits a solution in Γ if and only if g belongs to the subgroup of $\Sigma_\Gamma(g)$*

$$\Sigma^n_\Gamma(g) = \{\, \gamma^n \mid \gamma \in \Sigma_\Gamma(g) \,\}\,. \tag{4.10}$$

The number of solutions, if any, is $\gcd(n, q)$.

Proof. The last part is a direct consequence of an exercise in basic group theory. Indeed, by raising to the nth power, we define a homomorphism

$$\Sigma_\Gamma(g) \to \Sigma^n_\Gamma(g) \tag{4.11}$$

between these two cyclic groups. The order of the kernel of the above homomorphism is $\gcd(n, q)$. □

Remark. Sometimes the identity transformation is considered to be of elliptic type. When $g = 1$, an element $\gamma \in \Gamma$ is a solution to the equation (1.2) (if any) or a root of unity if and only if it is an elliptic transformation of order dividing n, which belong to Γ.

5 The loxodromic cases

We turn now to the remaining and hopefully less obvious case: the loxodromic transformation. We assume that a loxodromic transformation g is normalized, i.e., g is of the form

$$z \mapsto \alpha_g z. \tag{5.1}$$

Here, since g is loxodromic, α_g is a nonzero complex number such that $|\alpha_g| \neq 1$.

The statement of Theorem below might appear to be a mixture of those of the preceding two propositions.

Theorem. *Let g be a loxodromic transformation of a discrete group $\Gamma \subset \text{Möb}(\mathbb{C})$. Then the centralizer $Z_\Gamma(g)$ coincides with $S_\Gamma(g)$. The equation* (1.2) *admits a solution in Γ if and only if g belongs to the subgroup of $S_\Gamma(g)$*

$$S^n_\Gamma(g) = \{\, \gamma^n \mid \gamma \in S_\Gamma(g) \,\}\,. \tag{5.2}$$

Moreover, we have:

(i) *If $S_\Gamma(g)$ does not contain an elliptic transformation, then $S_\Gamma(g) \cong \mathbb{Z}$ and the equation admits, if any, a unique solution in Γ.*

(ii) *If $S_\Gamma(g)$ contains an elliptic transformation, then there exists an integer $q \geq 2$ such that $S_\Gamma(g) \cong \mathbb{Z}_q \oplus \mathbb{Z}$ and the number of solutions, if any, is* $\gcd(n, q)$.

Proof. In view of part (ii) of Assertion in Section 2, we see that $Z_\Gamma(g)$ comprises the transformations with the same fixed points as g, which are of elliptic or loxodromic type; this time we are to work with the abelian subgroup $S_\Gamma(g)$, with which the centralizer $Z_\Gamma(g)$ coincides. So we define, by considering the "length spectrum," a homomorphism

$$\Theta\colon S_\Gamma(g) \to \mathbb{R} \tag{5.3}$$

given by $\gamma \mapsto \log|\alpha_\gamma|$, where $\mathbb{R}$ is regarded as an additive group topologized by the absolute-value norm.

The image of $S_\Gamma(g)$ in $\mathbb{R}$ under the homomorphism Θ is discrete, so this implies that

$$\operatorname{im}\Theta \cong \mathbb{Z} \tag{5.4}$$

because $\operatorname{im}\Theta \subset \mathbb{R}$, which cannot be trivial under the assumption that g is a loxodromic transformation, has to be a free abelian group of rank 1.

If $S_\Gamma(g)$ contains no elliptic transformations,

$$S_\Gamma(g) \cong \mathbb{Z} \tag{5.5}$$

from (5.4), for $\ker\Theta$ must be trivial. Otherwise, we obtain as before (Proposition 2) that there exists an integer $q \geq 2$ such that

$$\ker\Theta = \Sigma_\Gamma(g) \cong \mathbb{Z}_q. \tag{5.6}$$

(A similar construction of the subgroup $\Sigma_\Gamma(g)$ is also possible for a loxodromic transformation g and the equality on the last line clearly holds.) The following short exact sequence encodes the related data:

$$1 \to \mathbb{Z}_q \hookrightarrow S_\Gamma(g) \overset{\Theta}{\to} \mathbb{Z} \to 1; \tag{5.7}$$

in a slight abuse of language, we identify the image $\operatorname{im}\Theta$ with $\mathbb{Z}$, for example. The sequence of course splits, and hence $S_\Gamma(g)$ is isomorphic to a direct product of $\mathbb{Z}_q$ and $\mathbb{Z}$ as we are working in an abelian category. Therefore,

$$S_\Gamma(g) \cong \mathbb{Z}_q \oplus \mathbb{Z}, \tag{5.8}$$

as is to be expected.

The rest of the proof is straightforward. ∎

Most of the arguments in the proof of Theorem above is true for non-parabolic transformations, whether g be loxodromic or not. We now state the following

Corollary (of the proof)**.** *Under the hypothesis of Proposition* 2*, the centralizer* $Z_\Gamma(g)$ *is a splitting* $\mathbb{Z}_2$*-extension of* $S_\Gamma(g)$*, when* $Z_\Gamma(g)$ *does not coincide with* $S_\Gamma(g)$*. Moreover, we have*:

(i) $S_\Gamma(g) \cong \mathbb{Z}_q$ *if* $S_\Gamma(g)$ *does not contain a loxodromic transformation.*

(ii) $S_\Gamma(g) \cong \mathbb{Z}_q \oplus \mathbb{Z}$ *if* $S_\Gamma(g)$ *contains a loxodromic transformation.*

The finite cyclic group $\mathbb{Z}_q$ *as above is identified with* $\Sigma_\Gamma(g)$*.*

Proof. The first statement follows readily from the observation on the homomorphism j as introduced in (4.3), which relates the symmetric group $\mathfrak{S}(F_g)$ to $Z_\Gamma(g)$, and so we include some comments on the rest of the corollary. We for the sake of clarity demonstrate that this time (5.5) apparently never holds. (Compare with (i) of Theorem.)

The homomorphism Θ is handy hereinafter. There always exists an integer $q \geq 2$ such that (5.6) holds whereas it is not possible for $\ker \Theta$ to be trivial under the assumption that g is an elliptic transformation. On the other hand, (5.4) may or may not be true, as follows.

If $S_\Gamma(g)$ does not contain a loxodromic transformation (and hence is a purely elliptic subgroup), we have, instead of (5.5),

$$S_\Gamma(g) \cong \mathbb{Z}_q \tag{5.9}$$

from (5.6), because $\ker \Theta$ coincides with $S_\Gamma(g)$. As such $\operatorname{im} \Theta$ must be trivial and thus (5.4) is not true.

If $S_\Gamma(g)$ contains a loxodromic transformation, we see that (5.4) is true, for $\operatorname{im} \Theta$ has to be a free abelian group of rank 1. Our situation is then that it is possible to exploit the splitting exact sequence (5.7) we encountered in the course of the proof of Theorem, so (5.8) again holds. ∎

Now we mention that the short exact sequence (5.7) turns out to be

$$1 \to \mathbb{T} \hookrightarrow S(g) \stackrel{\Theta}{\to} \mathbb{R} \to 1 \tag{5.10}$$

when Γ equals Möb($\mathbb{C}$), which is not discrete, and that the 1-torus $\mathbb{T}$ is identified with $\Sigma(g)$. The set of the positive multipliers of transformations of $S(g)$ is (via the exponential function) in a natural one-to-one correspondence with $\mathbb{R}$, and this yields (by abuse of notation) a canonical cross section

$$s\colon \mathbb{R} \to S(g). \tag{5.11}$$

An easy consequence is that

$$S(g) = \Sigma(g) \oplus s(\mathbb{R}) \cong \mathbb{T} \oplus \mathbb{R} \cong \mathbb{C}^*. \tag{5.12}$$

The direct summand $\mathbb{R}$ corresponds to the subgroup $s(\mathbb{R})$ of $S(g)$ comprising the hyperbolic transformations. (A loxodromic transformation g is called hyperbolic if $\mathrm{tr}^2\, g > 4$.)

6 Conclusion

The results we derived so far show when and how a solution of the equation (1.2) exists in a discrete group $\Gamma \subset$ Möb($\mathbb{C}$). In particular, Theorem in Section 5 says that there is a maximum number $n \geq 1$, depending on g and Γ, such that an nth root of g or a solution to (1.2) exists in Γ. We shall denote the maximum number above by the symbol $n_\Gamma(g)$.

Since in view of the Poincaré–Klein–Koebe uniformization theorem any nonexceptional Riemann surface is uniformized by a torsion-free Fuchsian group, important discrete groups are the Fuchsian groups. (By definition, a Kleinian group Γ, which is a discrete group acting properly discontinuously on a nonempty open subset of $\hat{\mathbb{C}}$, is called Fuchsian if, after a global conjugation, its *limit set* lies on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and Γ preserves each of the two disks, i.e., the upper half plane U and the lower half plane L in $\mathbb{C}$.)

In fact, the interest in the topic originated with an attempt to count the number of holomorphic mappings between Riemann surfaces. Let f be a nonconstant holomorphic mapping between two compact Riemann surfaces M and N of genus greater than one, which are hyperbolic Riemann surfaces. Since such an f of M onto N is contracting in their respective hyperbolic metrics, for a closed curve C on M, the hyperbolic length of

$f(C)$ on N does not exceed that of C on M.

A holomorphic mapping between compact Riemann surfaces is determined by the way it operates on the homology of the surface, and this brings the subject down to counting homology actions. By virtue of such rigidity of holomorphic mappings, it would be natural to assign an algebraic invariant, like $n_\Gamma(g)$, to an algebraic object, for example, a homology or homotopy class. This point of view enables us to interpret $n_\Gamma(g)$ as a more geometric concept that is easier to visualize and sheds light on deformations of a Fuchsian group (or any other discrete group). However, using the same notations as before, $f(C)$ may wrap a *primitive* curve on N around itself "many times" even if C is primitive. This is where we were forced to rely on algebra and an algebraic attack is indeed effective, cf. Ito-Yamamoto [4]; see also Imayoshi-Ito-Yamamoto [3].

To be more specific, we assume that Γ is a Fuchsian group uniformizing smoothly a compact Riemann surface M, which is to be equipped with a canonical hyperbolic metric. Each element $g \in \Gamma$ is then of hyperbolic type and associated with g is an *axis* that projects to a closed geodesic on $M = U/\Gamma$. Under the assumptions, Theorem (i) implies that $S_\Gamma(g) \cong \mathbb{Z}$ and hence that $n_\Gamma(g)$ is equal to the index of a cyclic group $\langle g \rangle$ in $S_\Gamma(g)$. Thus, the index measures how many times g wraps a corresponding primitive element of Γ around itself and g is an $n_\Gamma(g)$-iterate of the primitive element (e.g., Lemma 9.2.6 of Buser [2]). Unfortunately no definitive statements can be made at present, but when being deformed, such indexes, which are not decreased, might be bounded from above in an open and clear fashion. For (analytic or geometric) information about relevant upper estimates, see Yamamoto [5, 6]; we note that the estimates are not obtained through the immediate application of the distance decreasing principle for holomorphic mappings between hyperbolic manifolds.

As such the original motivation only requires Fuchsian groups (in the absence of elliptic transformations), for which the problem studied here has a simpler formulation, being compared with the way the statements are now written. At the same time, it could be said that our procedures for solving the proposed problem are related, for instance, to an important step in the generic or typical problem of describing conjugacy classes in a discrete group. Although the current article consists of a series of rather elementary observations, we therefore hope that those allow us to provide factual frameworks for further research.

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References

[1] A. F. Beardon, *The geometry of discrete groups*, Corrected reprint of the 1983 original, Graduate Texts in Mathematics, Volume 91, Springer-Verlag, New York (1995).

[2] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Reprint of the 1992 edition, Modern Birkhäuser Classics, Birkhäuser Boston, Ltd., Boston, MA (2010).

[3] Y. Imayoshi, M. Ito, H. Yamamoto, On the number of holomorphic mappings between Riemann surfaces of finite analytic type, *Proc. Edinb. Math. Soc.*, **54**, 711–730 (2011).

[4] M. Ito, H. Yamamoto, Holomorphic mappings between compact Riemann surfaces, *Proc. Edinb. Math. Soc.*, **52**, 109–126 (2009).

[5] H. Yamamoto, On the multiplicity of the image of simple closed curves via holomorphic maps between compact Riemann surfaces, *Kodai Math. J.*, **26**, 69–84 (2003).

[6] H. Yamamoto, An estimate for the hyperbolic length of closed geodesics on Riemann surfaces, *Complex Var. Elliptic Equ.*, **64**, 1582–1607 (2019).

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