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A note on Turán-type inequalities for polynomials

by

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Abstract

The goal of this paper is to establish some results for the polar derivative of a polynomial in the plane that are inspired by a classical result of Turán that relates the sup-norm of the derivative on the unit circle to that of the polynomial itself (on the unit circle) under some conditions. The obtained results sharpen as well as generalize some known estimates that relate the sup-norm of the polar derivative and the polynomial.

Key Words: Minimum modulus principle, Turán's inequality, zeros.
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1 Introduction and statement of results

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n in the complex plane and $P'(z)$ its derivative. A classical result due to Bernstein is that: for two polynomials $P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P^{'}(z)| \leq |Q^{'}(z)|$ on $|z| = 1$. In particular, for $Q(z) = z^n \max_{|z|=1} |P(z)|$, this classical result allows one to establish the famous Bernstein-inequality [3] for the sup-norm on the unit circle: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin. On the other hand, Turán's classical inequality [14] provides a lower bound estimate to the size of the derivative of a polynomial on the unit circle in the complex plane, relative to the size of the polynomial itself when there is a restriction on its zeros. It states that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [1] in the form

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \tag{1.3}$$

Equality in (1.2) and (1.3) holds for any polynomial which has all its zeros on $|z| = 1$. Over the years, the inequalities (1.2) and (1.3) have been generalized and extended in several directions. For a polynomial $P(z)$ of degree n having all its zeros in $|z| \le k,\ k \ge 1$, Govil [5] proved that

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.4}$$

As is easy to see that (1.4) becomes equality if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (1.4). In this direction, it was shown by Govil [4] that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.5}$$

Different versions of these Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. The one such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative. Before proceeding to our main results, let us remind that the polar derivative a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$ (see [9]), is defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Many of the generalizations of above mentioned inequalities involve the comparison of the polar derivative $D_\alpha P(z)$ with various choices of $P(z)$, α and other parameters. For more information on the polar derivative of polynomials, one can consult the comprehensive books of Marden [9], Milovanonić et al. [10] or Rahman and Schmeisser [13]. In 1998, Aziz and Rather [2] established the polar derivative analogue of (1.4) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k}{1+k^n} \right) \max_{|z|=1} |P(z)|. \tag{1.6}$$

The corresponding polar derivative analogue of (1.5) and a refinement of (1.6) was given by Govil and McTume [7]. They proved that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1 + k + k^n$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge n \left(\frac{|\alpha| - k}{1+k^n} \right) \max_{|z|=1} |P(z)| + n \left(\frac{|\alpha| - (1+k+k^n)}{1+k^n} \right) \min_{|z|=k} |P(z)|. \tag{1.7}$$

If we divide both sides of the above inequalities (1.6) and (1.7) by $|\alpha|$ and make $|\alpha| \to \infty$, we obtain the inequalities (1.4) and (1.5), respectively. One can see in the literature (for example, refer [6], [8], [11], [12], [15], [16]), the latest research and development in this direction. Recently, Govil and Kumar [6] established the following result from which several other results follow as special cases.

Theorem A. *If* $P(z) = z^s\left(\sum_{v=0}^{n-s} a_v z^v\right),\ 0 \le s \le n$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge (|\alpha| - k)\left\{\frac{n+s}{1+k^n} + \frac{k^{n-s}|a_{n-s}| - |a_0|}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0|)}\right\}\max_{|z|=1}|P(z)|. \tag{1.8}$$

Dividing both sides of (1.8) by $|\alpha|$ and let $|\alpha| \to \infty$, we have the following refinement and generalization of (1.4).

Theorem B. *If* $P(z) = z^s\left(\sum_{v=0}^{n-s} a_v z^v\right),\ 0 \le s \le n$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then*

$$\max_{|z|=1} |P'(z)| \ge \left\{\frac{n+s}{1+k^n} + \frac{k^{n-s}|a_{n-s}| - |a_0|}{(1+k^n)(k^{n-s}|a_{n-s}| + |a_0|)}\right\}\max_{|z|=1}|P(z)|. \tag{1.9}$$

If we take $s = 0$ in (1.8) and (1.9), we get as special cases from Theorems A and B, the following improvements of (1.6) and (1.4) respectively.

Theorem C. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha P(z)| \ge (|\alpha| - k)\left\{\frac{n}{1+k^n} + \frac{k^n|a_n| - |a_0|}{(1+k^n)(k^n|a_n| + |a_0|)}\right\}\max_{|z|=1}|P(z)|. \tag{1.10}$$

Theorem D. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then*

$$\max_{|z|=1} |P'(z)| \ge \left\{\frac{n}{1+k^n} + \frac{k^n|a_n| - |a_0|}{(1+k^n)(k^n|a_n| + |a_0|)}\right\}\max_{|z|=1}|P(z)|. \tag{1.11}$$

Now, we state the main result of our paper. The obtained result generalizes and sharpens (1.10) and yields strengthening of (1.7) and (1.11) as well.

Theorem 1. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *having all zeros in* $|z| \le k,\ k \ge 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge k$ *and* $0 \le t \le 1$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \ge{}& n\left(\frac{|\alpha| - k}{1+k^n}\right)\max_{|z|=1}|P(z)| + n\left(\frac{|\alpha| - (1+k+k^n)}{1+k^n}\right)tm \\ &+ \left(\frac{|\alpha| - k}{1+k^n}\right)\left(\frac{k^n|a_n| - |a_0| - tm}{k^n|a_n| + |a_0| + tm}\right)\left\{\max_{|z|=1}|P(z)| + tm\right\},\end{aligned} \tag{1.12}$$

where $m = \min_{|z|=k}|P(z)|$.

Remark 1. While going through the proof of the inequality (1.7) given by Govil and McTume [7], we see that this inequality still holds for $|\alpha| \ge k$. However to get a more refined bound by making the second term on the right hand side of (1.7) non negative, the authors have taken $|\alpha| \geq 1+k+k^n$. Since $k \leq 1+k+k^n$ and hence, Theorem 1 holds for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1+k+k^n$ as well. Therefore, if we take $|\alpha| \geq 1+k+k^n$ in Theorem 1, we can easily see that the bound obtained in Theorem 1 is much better than the bound obtained from (1.7).

Dividing both sides of (1.12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 1. *If $P(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for $0 \leq t \leq 1$ and $m = \min_{|z|=k} |P(z)|$, we have*

$$\begin{aligned}\max_{|z|=1} |P'(z)| \geq & \frac{n}{1+k^n}\left\{\max_{|z|=1} |P(z)| + tm\right\} \\ & + \frac{1}{1+k^n}\left(\frac{k^n|a_n| - |a_0| - tm}{k^n|a_n| + |a_0| + tm}\right)\left\{\max_{|z|=1} |P(z)| + tm\right\}. \end{aligned} \tag{1.13}$$

Equality in (1.13) *holds for $P(z) = z^n + k^n$.*

Remark 2. Since $P(z) = \sum_{v=0}^{n} a_v z^v \neq 0$ in $|z| > k,\ k \geq 1$ and, if $z_1, z_2, ..., z_n$, are the zeros of $P(z)$, then $|\frac{a_0}{a_n}| = |z_1 z_2 ... z_n| = |z_1||z_2|...|z_n| \leq k^n$. Also, as in the proof of Theorem 1 (given in the next section), we have for every λ with $|\lambda| \leq 1$ the polynomial $P(z) + \lambda m$ has all its zeros in $|z| \leq k,\ k \geq 1$, hence

$$\left|\frac{a_0 + \lambda m}{a_n}\right| \leq k^n. \tag{1.14}$$

If in (1.14), we choose the argument of λ suitably, we get

$$|a_0| + |\lambda| m \leq k^n |a_n|. \tag{1.15}$$

If we take $|\lambda| = t$ in (1.15), so that $0 \leq t \leq 1$, we get $|a_0| + tm \leq k^n |a_n|$.

Remark 3. In fact, excepting the case when the polynomial $P(z)$ has a zero on $|z| = k$ or $t = 0$, the bounds obtained in Theorem 1 and Corollary 1 are always sharp than the bounds obtained in Theorem C and Theorem D respectively. One can also observe that the inequality (1.12) also improves the inequality (1.7) considerably when $k^n|a_n| - |a_0| - tm \neq 0$. If we take $k = 1$ in Corollary 1, we get the following refinement of a result due to Govil and Kumar ([6], Corollary 1.6).

Corollary 2. *If $P(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $0 \leq t \leq 1$ and $m = \min_{|z|=1} |P(z)|$, we have*

$$\begin{aligned}\max_{|z|=1} |P'(z)| \geq & \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + tm\right\} \\ & + \frac{1}{2}\left(\frac{|a_n| - |a_0| - tm}{|a_n| + |a_0| + tm}\right)\left\{\max_{|z|=1} |P(z)| + tm\right\}. \end{aligned} \tag{1.16}$$

Equality in (1.16) *holds for $P(z) = z^n + 1$.*

Clearly, Corollary 2 sharpens inequality (1.3) due to Aziz and Dawood [1] in all cases excepting when $P(z)$ has all its zeros on $|z| = 1$.

2 Proof of the Theorem

Proof of Theorem 1. If $P(z) = \sum_{v=0}^{n} a_v z^v$ has some zeros on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and the result follows from Theorem C in this case. So, henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, $k \geq 1$. Let $H(z) = P(kz)$ and $G(z) = z^n \overline{H(\frac{1}{\bar{z}})} = z^n \overline{P(\frac{k}{\bar{z}})}$. Then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$. This gives

$$\left| z^n \overline{P\left(\frac{k}{\bar{z}}\right)} \right| = |P(kz)| \geq m \text{ for } |z| = 1.$$

It follows by the Minimum Modulus Principle, that

$$\left| z^n \overline{P\left(\frac{k}{\bar{z}}\right)} \right| \geq m \quad \text{for} \quad |z| \leq 1.$$

Replacing z by $\dfrac{1}{\bar{z}}$, it implies that

$$|P(kz)| \geq m|z|^n \text{ for } |z| \geq 1,$$

or

$$|P(z)| \geq m \left| \frac{z}{k} \right|^n \quad \text{for} \quad |z| \geq k. \tag{2.1}$$

Now, consider the polynomial $F(z) = P(z) + \lambda m$, where $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then all the zeros of $F(z)$ lie in $|z| \leq k$. Because, if for some $z = z_0$ with $|z_0| > k$, we have

$$F(z_0) = P(z_0) + \lambda m = 0,$$

then

$$|P(z_0)| = |\lambda m| \leq m < m \left| \frac{z_0}{k} \right|^n,$$

which contradicts (2.1). Hence for every complex number λ with $|\lambda| \leq 1$, the polynomial $F(z) = P(z) + \lambda m = (a_0 + \lambda m) + \sum_{v=1}^{n} a_v z^v$, has all its zeros in $|z| \leq k$, where $k \geq 1$. Applying Theorem C to the polynomial $F(z)$, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and $|z| = 1$,

$$\max_{|z|=1} \left| D_\alpha \big(P(z) + \lambda m \big) \right| \geq \left(\frac{|\alpha| - k}{1 + k^n} \right) \left(n + \frac{k^n |a_n| - |a_0 + \lambda m|}{k^n |a_n| + |a_0 + \lambda m|} \right) |P(z) + \lambda m|. \tag{2.2}$$

For every $\lambda \in \mathbb{C}$, we have

$$|a_0 + \lambda m| \leq |a_0| + |\lambda| m,$$

and since the function

$$x \mapsto \frac{k^n |a_n| - x}{k^n |a_n| + x}, \ (x \geq 0)$$

is non-increasing on

$$\Big\{x : x > -k^n|a_n|\Big\} \cup \Big\{x : x < -k^n|a_n|\Big\},$$

for every k, it follows from (2.2) that for every λ with $|\lambda| \leq 1$ and $|z| = 1$,

$$\max_{|z|=1} |D_\alpha P(z) + \lambda mn| \geq \left(\frac{|\alpha| - k}{1 + k^n}\right) \left(n + \frac{k^n|a_n| - |a_0| - |\lambda|m}{k^n|a_n| + |a_0| + |\lambda|m}\right) |P(z) + \lambda m|. \tag{2.3}$$

Choosing the argument of λ on the right hand side of (2.3) such that

$$|P(z) + \lambda m| = |P(z)| + |\lambda|m,$$

we obtain from (2.3) that

$$\max_{|z|=1} |D_\alpha P(z)| + |\lambda|mn \geq \left(\frac{|\alpha| - k}{1 + k^n}\right) \left(n + \frac{k^n|a_n| - |a_0| - |\lambda|m}{k^n|a_n| + |a_0| + |\lambda|m}\right) \big(|P(z)| + |\lambda|m\big),$$

which on taking $|\lambda| = t$, so that $0 \leq t \leq 1$, gives in particular (1.12) and this completes the proof of Theorem 1.

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