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On the derivative of a self-reciprocal polynomial

by

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Abstract

Similar to a polynomial $P(z)$ with zeros $w_1, w_2, \ldots, w_n$, being called a self-inversive polynomial (see [8]) if

$$\{w_1, w_2, \ldots, w_n\} = \{1/\overline{w_1}, 1/\overline{w_2}, \ldots, 1/\overline{w_n}\},$$

we have called a polynomial $p(z)$ with zeros $z_1, z_2, \ldots, z_n$, a self-reciprocal polynomial if

$$\{z_1, z_2, \ldots, z_n\} = \{1/z_1, 1/z_2, \ldots, 1/z_n\}$$

and have obtained for a self-reciprocal polynomial $p(z)$ of degree n

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) = n \max_{|z|=1} |p(z)|,$$

similar to

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} \max_{|z|=1} |P(z)|,$$

for a self-inversive polynomial $P(z)$ of degree n ([8]).

Key Words: Self-reciprocal polynomial, derivative, unit circle, maximum, self-inversive polynomial.
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1 Introduction

O'Hara and Rodriguez [8] called a polynomial $P(z)$ with zeros $w_1, w_2, \ldots, w_n$, a self-inversive polynomial if

$$\{w_1, w_2, \ldots, w_n\} = \{1/\overline{w_1}, 1/\overline{w_2}, \ldots, 1/\overline{w_n}\}$$

and they obtained

Theorem 1. *If $P(z)$ is a self-inversive polynomial of degree n then*

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} \max_{|z|=1} |P(z)|.$$

We have similarly called a polynomial $p(z)$ with zeros $z_1, z_2, \ldots, z_n$, a self-reciprocal polynomial if

$$\{z_1, z_2, \ldots, z_n\} = \{1/z_1, 1/z_2, \ldots, 1/z_n\}.$$

For a self-reciprocal polynomial of special type there exist many results ([5], [2], [3], [1], [4], [9], [7], [10], [11]) in literature. In this paper we obtain a result similar to Theorem 1 for a self-reciprocal polynomial. More precisely we prove

Theorem 2. *If $p(z)$ is a self-reciprocal polynomial of degree n then*

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) = n \max_{|z|=1} |p(z)| \tag{1.1}$$

and on unit circle, $|p'(z)| + |p'(\overline{z})|$ and $|p(z)|$ attain maximum at same point.

Using (1.1) we obtain the following interesting sharp result.

Corollary 1. *If $p(z)$ is a self-reciprocal polynomial of degree n then*

$$\min_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)| \leq \max_{|z|=1} |p'(z)|, \tag{1.2}$$

with equality holding in (1.2) for $p(z) = z^n + 1$.

2 Lemmas

For the proof of Theorem 2 we require the following lemmas.

Lemma 1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n then the following are equivalent:*

(i) $p(z)$ *is a self-reciprocal polynomial.*

(ii) $a_n p(z) = a_0 z^n p(1/z)$ *for each complex number z.*

(iii) $a_0 a_j = a_n a_{n-j}; j = 0, 1, \ldots, n.$

Proof of Lemma 1. It follows easily from the definition of a self-reciprocal polynomial.

Lemma 2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a self-reciprocal polynomial of degree n then*

$$a_n(np(z) - zp'(z)) = a_0 z^{n-1} p'(1/z), \text{ for each } z \tag{2.1}$$

and

$$|p'(\overline{z})| = |np(z) - zp'(z)| \text{ for each } z \text{ on } |z| = 1. \tag{2.2}$$

Proof of Lemma 2. (2.1) follows from (ii) of Lemma 1 by differentiation and (2.2) follows from (2.1) and the fact $|a_0| = |a_n|$, (by (iii) of Lemma 1).

Lemma 3. *Let $p(z)$ be a polynomial of degree n and*

$$q(z) = z^n \overline{p(1/\overline{z})}. \tag{2.3}$$

Then

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \ |z| = 1.$$

This lemma is a special case of a result due to Govil and Rahman [6, Lemma 10].

3 Proof of Theorem 2

Using (2.2) of Lemma 2 we obtain

$$|p'(z)| + |p'(\overline{z})| \geq n|p(z)|, \ |z| = 1. \tag{3.1}$$

Further, let

$$\max_{|z|=1} |p(z)| = |p(z_1)| \text{ for certain } z_1 \text{ with } |z_1| = 1. \tag{3.2}$$

Then using (3.1) we obtain

$$|p'(z_1)| + |p'(\overline{z_1})| \geq n \max_{|z|=1} |p(z)| \tag{3.3}$$

and

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) \geq n \max_{|z|=1} |p(z)|. \tag{3.4}$$

Now on using Lemma 3, along with parts (ii) and (iii) of Lemma 1 (which, by (2.3) (of Lemma 3), give $|q'(z)| = |p'(\overline{z})|$), we obtain

$$|p'(z)| + |p'(\overline{z})| \leq n \max_{|z|=1} |p(z)|, \ |z| = 1,$$

thereby implying

$$|p'(z_1)| + |p'(\overline{z_1})| \leq n \max_{|z|=1} |p(z)| \tag{3.5}$$

and

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) \leq n \max_{|z|=1} |p(z)|. \tag{3.6}$$

On combining (3.4) and (3.6) we get

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) = n \max_{|z|=1} |p(z)|, \tag{3.7}$$

thereby proving (1.1). Further on using (3.3) and (3.5) together in (3.7) we obtain

$$\max_{|z|=1}(|p'(z)| + |p'(\overline{z})|) = |p'(z_1)| + |p'(\overline{z_1})|,$$

which, by (3.2), helps us to assert that on unit circle, $|p'(z)| + |p'(\overline{z})|$ and $|p(z)|$ attain maximum at same point, thereby completing the proof of Theorem 2.

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