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Strong and shifted stability for the cohomology of configuration spaces

by
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Abstract

Homological stability for unordered configuration spaces of connected manifolds was discovered by Th. Church and extended by O. Randal-Williams and B. Knudsen: $H_i(C_k(M);\mathbb{Q})$ is constant for $k \geq f(i)$. We characterize the manifolds satisfying strong stability: $H^*(C_k(M);\mathbb{Q})$ is constant for $k \gg 0$. We give few examples of closed oriented manifolds with even cohomology, whose top Betti numbers are stable after a shift of degree.

Key Words: Unordered configuration spaces, homological stability, Knudsen model, Félix-Thomas model.
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Contents

1 Introduction and statement of results

For a topological space X we consider the k-points *ordered configuration space* $F_k(X)$ and the *unordered configuration space* $C_k(X)$ defined by

$$F_k(X) = \{(x_1, \ldots, x_k) \in X^k | x_i \neq x_j \, for \, i \neq j\}, \quad C_k(X) = F_k(X)/S_k,$$

with the induced topology and quotient topology respectively.

One of the first results in the study of configuration spaces was the cohomological strong stability theorem of V. I. Arnold [1]: for $k \geq 2$

$$H^i(C_k(\mathbb{R}^2);\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } i = 0,1 \\ 0, & \text{if } i \geq 2. \end{cases}$$

The abelianization of Artin braid group is $\mathbb{Z}$; Arnold proved that higher cohomology groups are finite groups (they are trivial for $i \geq k$) and also he proved cohomological stability for the torsion part:

$$H^i(C_{2i-2}(\mathbb{R}^2);\mathbb{Z}) \cong H^i(C_{2i-1}(\mathbb{R}^2);\mathbb{Z}) \cong H^i(C_{2i}(\mathbb{R}^2);\mathbb{Z}) \cong \ldots$$

The isomorphisms (for k large depending on i)

$$H^i(C_k(M);\mathbb{Q}) \cong H^i(C_{k+1}(M);\mathbb{Q}) \cong H^i(C_{k+2}(M);\mathbb{Q}) \cong \ldots$$

were generalized for open manifolds by D. McDuff [24] and G. Segal [31]. Using representation stability, Th. Church [7] proved that

$$H^i(C_k(M);\mathbb{Q}) \cong H^i(C_{k+1}(M);\mathbb{Q}) \cong H^i(C_{k+2}(M);\mathbb{Q}) \cong \ldots$$

for $k > i$ and M a connected oriented manifold of finite type. This result was extended by O. Randal-Williams [27] and B. Kundsen [21].

We will define and study other stability properties of the rational cohomology of unordered configuration spaces of connected manifolds of finite type. Without a special mention, the (co)homology groups will have coefficients in $\mathbb{Q}$. For a manifold M of dimension n, its Betti numbers, its Poincaré polynomial and its *total Betti number* are defined by

$$\beta_i(M) = \dim_{\mathbb{Q}} H^i(M), \quad P_M(t) = \sum_{i=0}^{n} \beta_i(M)t^i, \quad \beta(M) = P_M(1).$$

The *top Betti number* $\beta_\tau(M)$ is the last non-zero Betti number of M, its cohomological dimension is $\mathrm{cd}(M) = \tau(M) = \tau$ and its *q-truncated Poincaré polynomial* contains the last q-Betti numbers:

$$P_M^{[q]}(t) = \beta_{\tau-q+1}(M)t^{\tau-q+1} + \ldots + \beta_\tau(M)t^\tau.$$

A space X has *even cohomology* if all its odd Betti numbers are zero, and a space Y has *odd cohomology* if all its positive even Betti numbers are zero (and it is path connected):

$$H^*(X) = H^{even}(X), \text{ respectively } \tilde{H}^*(Y) = H^{odd}(Y).$$

We say that a manifold M^{4m} is a *homology projective plane* if its Poincaré polynomial is $1 + t^{2m} + t^{4m}$.

Remark 1. There are classical results on topological spaces with three nonzero integral Betti numbers; see many example in the paper of J. Eells and N. Kuiper "Manifolds which are like projective planes" [12]. In all of them m takes values 1, 2, 4. More rational projective planes are described in [26], [17], [20] and [33].

The computation of $H^*(C_k(M))$, using cohomology algebra of M, is easy in the odd dimensional case (see [5] and [16]):

Theorem 1. *(C-F. Bödigheimer, F. Cohen, L. Taylor – Y. Félix, D. Tanré)*
For a manifold M^{2m+1} we have

$$H^*(C_k(M)) = Sym^k(H^*(M)).$$

In the even dimensional case, the cohomology groups $H^*(C_*(M))$ are given by the cohomology of a differential bigraded algebra $(\Omega^*(*)(V^*, W^*), \partial)$ introduced by Y. Félix and J. C. Thomas [15] and extended by B. Kundsen [21] (the two graded vector spaces V^*, and W^* and the differential ∂ depend on various cohomology groups of M and cohomology product):

Theorem 2. *(Y. Félix, J. C. Thomas – B. Knudsen)*
For a manifold M^{2m} we have

$$H^*(C_k(M)) \cong H^*(\Omega^*(k)(V^*, W^*), \partial).$$

We recall the definition of V^*, W^* and ∂ in Section 4, for a closed oriented manifold M^{2m}, and in Section 5, for an arbitrary even dimensional manifold; the model Ω^* introduces a bigrading on the cohomology of $C_k(M)$:

$$H^*(C_k(M)) = \bigoplus_{i\geq 0} H^i(C_k(M)), \quad H^i(C_k(M)) = \bigoplus_{j\geq 0} H^{i,j}(C_k(M))$$

and we can use the two-variables Poincaré polynomial

$$P_{C_k(M)}(t,s) = \sum_{i,j\geq 0} \dim_{\mathbb{Q}} H^{i,j}(C_k(M)) t^i s^j = \sum_{i,j\geq 0} \beta_{i,j} t^i s^j$$

(of course we have $P_{C_k(M)}(t) = P_{C_k(M)}(t,1)$).

We will prove a bigraded version of classical stability:

Theorem 3. *For a manifold M^{2m} we have:*
a) if $i \leq k$

$$H^{i,0}(C_k(M)) \cong H^{i,0}(C_{k+1}(M)) \cong H^{i,0}(C_{k+2}(M)) \cong \dots$$

b) if $j \geq 1$ and $i \leq k + (2m-2)j - 1$

$$H^{i,j}(C_k(M)) \cong H^{i,j}(C_{k+1}(M)) \cong H^{i,j}(C_{k+2}(M)) \cong \dots.$$

Here is our first definition:

Definition 1. *A connected manifold satisfies the* strong stability condition *for its unordered configuration spaces $\{C_k(M)\}_{k\geq 1}$, with* range r, *if and only if the cohomology groups are eventually constant:*

$$H^*(C_r(M)) \cong H^*(C_{r+1}(M)) \cong H^*(C_{r+2}(M)) \cong \dots.$$

In the literature there are few examples of manifolds satisfying this condition: $\mathbb{R}^2$-V. I. Arnold [1], $\mathbb{R}^n$-F. Cohen [9] (see also [28]), S^2-M. B. Sevryuk [32], S^n-P. Salvatore [29] (see also [28]), $\mathbb{C}\mathrm{P}^2$-Y. Félix and D. Tanré [16] (see also [22]), $\mathbb{R}\mathrm{P}^n$-B. Knudsen [21].

Remark 2. The (strong) stability property is missing in the torsion part of homology: E. Fadell and J. Van Buskirk [13] computed the first homology group of $C_k(S^2)$: $\mathbb{Z}/(2k-2)\mathbb{Z}$. Also D. B. Fuchs [18] proved that, for an arbitrary degree i, one can find a large k such that $H^{\geq i}(C_k(\mathbb{R}^2);\mathbb{Z}_2)$ is non-zero, hence S^2 and $\mathbb{R}^2$ have not the strong stability property with integral cohomology.

The first results say the previous examples are essentially all manifolds with the strong stability property.

Theorem 4. *A manifold of odd dimension has the strong stability property if and only if M has odd cohomology. In this case the range of stability is:*

$$r = \begin{cases} 1, & \text{if } M \text{ is rationaly acyclic,} \\ \beta(M) - 1, & \text{otherwise.} \end{cases}$$

Theorem 5. *A closed oriented manifold of even dimension has the strong stability property if and only if M is a homology sphere or a homology projective plane and the ranges of stability are* 3 *and* 4 *respectively.*

Corollary 1. *A closed oriented manifold M has the strong stability property if and only if M is a homology sphere or a homology projective plane.*

Various results and conjectures on stability of the top Betti number could be found in the literature: J. Miller and J. Wilson [25], Th. Church, B. Farb and A. Putman [8] or M. Maguire [23] and, recently, S. Galatius, A. Kupers and O. R. Williams [19]. Here is our second definition:

Definition 2. *A connected manifold M satisfies the* shifted stability condition *for its unordered configuration spaces $\{C_k(M)\}_{k\geq 1}$, with* range r, shift σ *and* length q $(r, \sigma, q \geq 1)$, *if and only if the q-truncated Poincaré polynomial is stable after a shift: for any $k \geq r$ we have*

$$P^{[q]}_{C_{k+1}(M)}(t) = t^{\sigma} P^{[q]}_{C_k(M)}(t).$$

We give two examples, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^3$, where classical stability and shifted stability properties combined give the entire two variable Poincaré polynomials:

Proposition 1. *The product of two projective lines,* $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, *has the shifted stability property with range 8, shift 2 and length 5:*

$$P^{[5]}_{C_{k+1}(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) = t^2 P^{[5]}_{C_k(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) \quad \text{for } k \geq 8.$$

More precisely, for $k \geq 8$, we have:

$$\begin{aligned} P_{C_k(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) = {} & 1 + 2t^2 + 3t^4 + 2t^6 + 2t^8 + \ldots + 2t^{2k} + \\ & + s(2t^7 + 4t^9 + 5t^{11} + 4t^{13} + 4t^{15} + \ldots + 4t^{2k+1} + 2t^{2k+3}) + \\ & + s^2(t^{14} + 2t^{16} + 2t^{18} + \ldots + 2t^{2k+4}). \end{aligned}$$

Proposition 2. *The complex projective space, $\mathbb{CP}^3$, has the shifted stability property with range 8, shift 2 and length 6:*

$$P^{[6]}_{C_{k+1}(\mathbb{CP}^3)}(t,s) = t^2 P^{[6]}_{C_k(\mathbb{CP}^3)}(t,s) \quad \textit{for } k \geq 8.$$

More precisely, for $k \geq 8$, we have:

$$\begin{aligned} P_{C_{k+1}(\mathbb{CP}^3)}(t,s) = {} & 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12} + \ldots + t^{2k} + \\ & + s(t^{11} + 2t^{13} + 3t^{15} + 3t^{17} + 3t^{19} + 2t^{21} + 2t^{23} + \ldots + 2t^{2k+5} + t^{2k+7}) + \\ & + s^2(t^{24} + t^{26} + \ldots + t^{2k+12}). \end{aligned}$$

More examples will be given in [4].

In Section 2 we introduce the algebraic tool to analyze Félix-Thomas model and Knudsen model, a sequence of *weighted spectral sequences*. As a first application we give the proof of Theorem 3 and an improved version of it. The proof of Theorem 4 is given in Section 3 and the proof of Theorem 5 in Section 5. Partial results for even dimensional manifolds, open or non-orientable, are presented in Section 5. In Section 6 we introduce three new notions of shifted stability and we describe their relations. Two necessary conditions for these shifted stability conditions are given. For large k, combining the classical stability with the shifted stability, we obtain the whole Poincaré polynomial $P_{C_k(M)}(t)$. Section 7 contains stability properties of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^3$ and the proofs of Propositions 1 and 2.

2 Weighted spectral sequences

In this section we analyze algebraic properties of the differential algebra $(\Omega^*(*)(V^*, W^*), \partial)$ introduced by Y. Félix and J. C. Thomas [15] and extended by B. Knudsen [21].

Let us introduced some notation. For a graded $\mathbb{Q}$-vector space $A^* = \oplus_{i\in\mathbb{Z}} A^i$ we will use the notation

$$A^{\geq q} = \bigoplus_{i \geq q} A^i, \quad A^{even} = \bigoplus_{i\in\mathbb{Z}} A^{2i}, \quad \tilde{A}^* = \bigoplus_{i\neq 0} A^i,$$

and similarly $A^{\leq q}$ and A^{odd}; the degree i component of the shifted graded space $A^*[r]$ is A^{i+r}. We suppose that A^* is connected: if $A^0 \neq 0$, then $A^0 \cong \mathbb{Q}$. The symmetric algebra $Sym(A^*)$ is the tensor product of a polynomial algebra and an exterior algebra:

$$Sym(A^*) = \bigoplus_{k\geq 0} Sym^k(A^*) = Polynomial(A^{even}) \bigotimes Exterior(A^{odd}),$$

where Sym^k is generated by the monomials of length k (without any other convention, the elements in A^* have length 1).

Fix a positive even number $2m$, the "geometric " dimension, and consider two graded vector spaces V^*, W^*, and a degree 1 linear map ∂_W :

$$V^* = \bigoplus_{i=0}^{2m} V^*, \; W^* = \bigoplus_{j=2m-1}^{4m-1} W^j, \; \partial_W : W^* \longrightarrow Sym^2 V^*.$$

By definition, the elements in V^* have length 1 and weight 0 and the elements in W^* have length 2 and weight 1. We choose bases in V^i and W^j as

$$V^i = \mathbb{Q}\langle v_{i,1}, v_{i,2}, \dots\rangle, \quad W^j = \mathbb{Q}\langle w_{j,1}, w_{j,2}, \dots\rangle$$

(the degree of an element is marked by the first lower index; x_i^q stands for the product $x_i \wedge x_i \wedge \dots \wedge x_i$ of q-factors). Always we take $V^0 = \mathbb{Q}\langle v_0\rangle$. The graded vector space V^* is $(h-1)$-connected if $V^* = V^0 \oplus V^{\geq h}$.

The definition of the bigraded differential algebra $\Omega^*(k)$ is

$$\Omega^*(*)(V^*, W^*) = \bigoplus_{k\geq 1} \Omega^*(k)(V^*, W^*),$$

$$\Omega^*(k)(V^*, W^*) = \bigoplus_{i\geq 0} \Omega^i(k)(V^*, W^*) = Sym^k(V^* \oplus W^*),$$

where the total degree i is given by the grading of V^* and W^* and the length degree k is the multiplicative extension of length on V^* and W^*. The differential is defined by $\partial|_{V^*} = 0$, $\partial|_{W^*} = \partial_W$ and it has bidegree $(1,0)$. For instance,

$$H^*(\Omega^*(1)(V^*, W^*), \partial) = H^*(Sym^1(V^*), \partial = 0) = V^*.$$

We are interested in the stability properties of the sequence $\{H^*(\Omega^*(k)(V^*, W^*), \partial)\}_{k\geq 1}$ i.e. we have to compare $H^*(\Omega^*(k-1)(V^*, W^*), \partial)$ with $H^*(\Omega^*(k)(V^*, W^*), \partial)$, and for this we introduce a sequence of weighted spectral sequences.

The subspace of $\Omega^*(k)$ containing the elements of weight ω is denoted ${}^{\omega}\Omega^*(k)$ and we have

$$\Omega^*(k)(V^*, W^*) = \bigoplus_{\omega=0}^{\lfloor \frac{k}{2} \rfloor} {}^{\omega}\Omega^*(k), \quad {}^{0}\Omega(k) = Sym^k(V^*),$$

$$\partial : {}^{\omega}\Omega^*(k) \longrightarrow {}^{\omega-1}\Omega^{*+1}(k).$$

We define an increasing filtration of subcomplexes $\{F^i\Omega^*(k)(V^*, W^*)\}_{i=0,\dots,2m}$:

$$F^i\Omega^*(k) = [V^{\leq i} \otimes \Omega^*(k-1)(V^*, W^*)] + [W^{\leq 2i} \otimes \Omega^*(k-2)(V^*, W^*)].$$

Obviously we have

$$\begin{array}{rcl} \partial(V^{\leq i} \otimes \Omega^*(k-1)) & \subset & V^{\leq i} \otimes \Omega^*(k-1) \text{ and} \\ \partial(W^{\leq 2i} \otimes \Omega^*(k-2)) & \subset & V^{\leq i} \otimes \Omega^*(k-1) + W^{\leq 2i} \otimes \Omega^*(k-2). \end{array}$$

The filtration $\{F^i\}_{i=0,\dots,2m}$ and the weight decomposition $\{{}^{\omega}\Omega^*(k)\}_{\omega=0,\dots,\lfloor \frac{k}{2} \rfloor}$ are compatible:

$$F^i\Omega^*(k) = F^i \cap {}^{0}\Omega^*(k) \oplus F^i \cap {}^{1}\Omega^*(k) \oplus \dots \oplus F^i \cap {}^{\lfloor \frac{k}{2} \rfloor}\Omega^*(k) = \bigoplus_{\omega=0}^{\lfloor \frac{k}{2} \rfloor} F^i\Omega^*(k),$$

hence the spectral sequence $E_*^{*,*}(k)$ associated with the filtration $\{F^i\Omega^*(k)\}_{i=0,\dots,2m}$ is weight-splitted at any page:

$$E_*^{*,*}(k) = \bigoplus_{\omega=0}^{\lfloor \frac{k}{2} \rfloor} {}^{\omega}E_*^{*,*}(k),$$

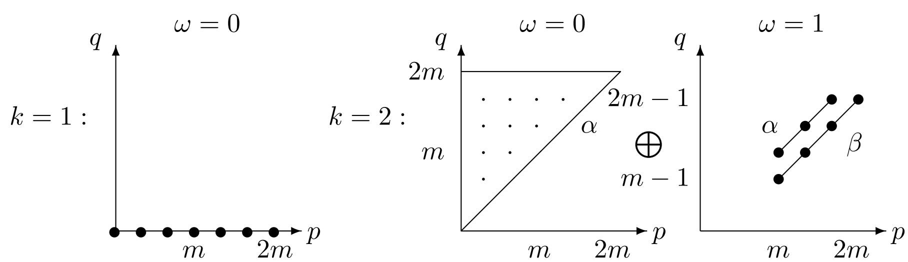
with differential

$$d_r^{i,j} : {}^{\omega}E_r^{i,j}(k) \longrightarrow {}^{\omega-1}E_r^{i-r,j+r+1}(k).$$

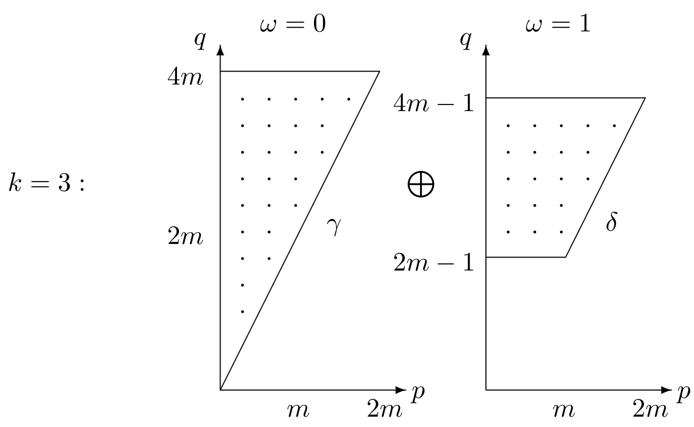
Some general properties of these spectral sequences are obvious:

Proposition 3. *Every $E_*^{*,*}(k)$ is a first quadrant spectral sequence; as $E_r^{\geq 2m+1,q}(k) = 0$, the spectral sequence degenerate at $2m+1$.*

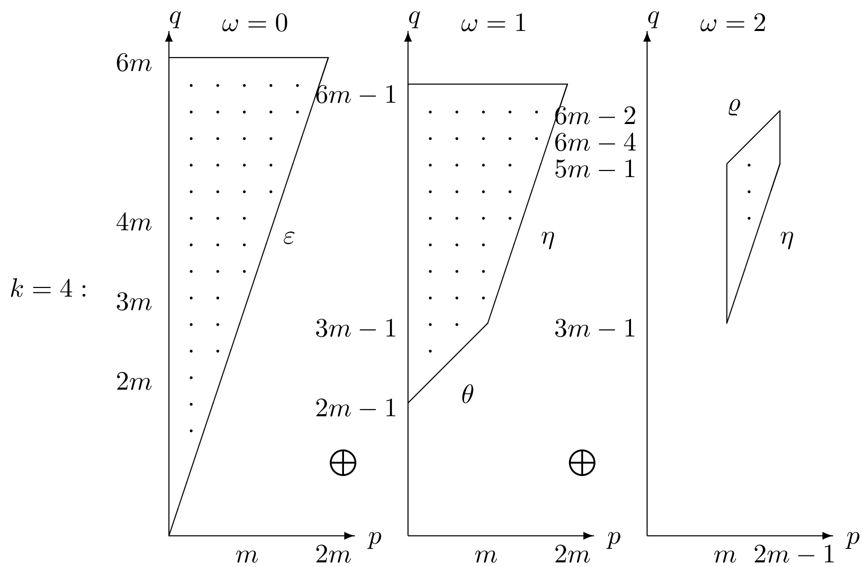
Here are few pictures of the polygons containing the support of the weighted components of the first page of the spectral sequences $E_*^{*,*}(k)$:



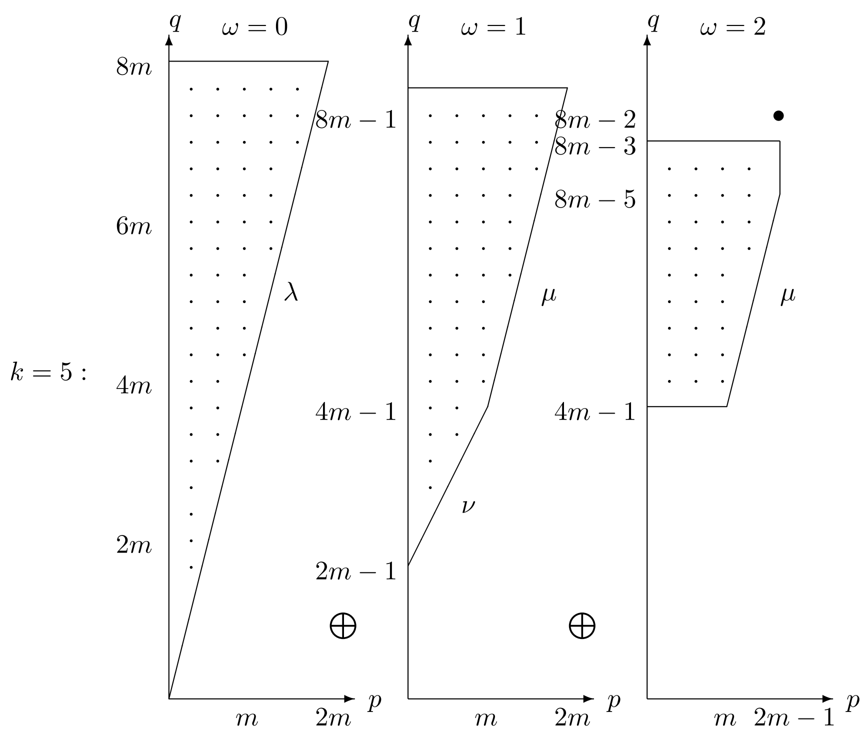
The equations of the lines are $\alpha: \ q = p$ and $\beta: \ q = p - 1$.



The equations of the lines are $\gamma: \ q = 2p$ and $\delta: \ q = p - 1$.



The equations of the lines are $\varepsilon : q = 3p$, $\theta : q = p + 2m - 1$, $\eta : q = 3p - 1$ and $\varrho : q = p + 4m - 1$.



The equations of the lines are $\lambda : q = 4p$, $\mu : q = 4p - 1$ and $\nu : q = 4p - 1$. Using the definition of the filtration F^i, one can describe the support of ${}^{\omega}E_*^{*,*}(k)$, in general:

Proposition 4. *a) If $2\omega > k$, then ${}^{\omega}E_0^{*,*}(k) = 0$.*

b) The support of the weighted componenets of ${}^{\omega}E_0^{*,*}(k)$ are contained in the following regions:

$\omega = 0:$ *the triangle defined by* $0 \leq (k-1)p \leq q \leq 2(k-1)m$;
$\omega = 1:$ *if* $k = 2$, *the trapezoid defined by* $m-1 \leq p-1 \leq q \leq \min(p, 2m-1)$;
if $k \geq 3$, *the quaddrilateral defined by*
$\max((k-3)p + 2m - 1, (k-1)p - 1) \leq q \leq 2(k-1)m - 1$;
$\omega \geq 2:$ *if* $k = 2\omega$, *the trapezoid defined by* $m \leq p \leq 2m-1$ *and*
$(k-1)p - 1 \leq q \leq p + (2k-4)m - k + 3$;
if $k \geq 2\omega + 1$, *the pentagon defined by*
$\max((k-2\omega-1)p + 2\omega m - 1, (k-1)p - 1) \leq q \leq 2(k-1)m - 2\omega + 1$
and the exterior point $(p, q) = (2m-1, 2(k-1)m-2)$.

Proof. In the table there is a list of elements of minimal degree (in bottom position) and elements of maximal degree (in top position) in the column F^p/F^{p-1} of the spectral sequence ${}^{\omega}E_0^{*,*}(k)$:

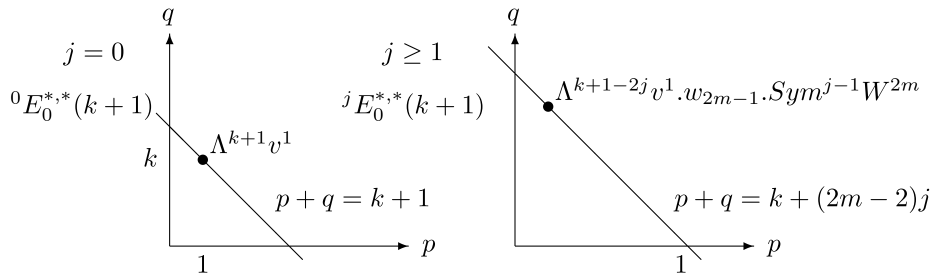
(ω, k)	$0 \leq p \leq m-1$	$m \leq p \leq 2m-1$	$p = 2m$
$\omega = 0$	$v_p v_{4m}^{k-1}$ v_p^k	$v_p v_{4m}^{k-1}$ v_p^k	v_{4m}^k v_{2m}^k
$\omega = 1$ $k = 2$	—	w_{2p} w_{2p-1}	w_{4p-1} w_{4p-1}
$\omega = 1$ $k > 2$	$v_p v_{2m}^{k-3} w_{4m-1}$ $v_p^{k-2} w_{2m-1}$	$v_p v_{2m}^{k-3} w_{4m-1}$ $v_p^{k-2} w_{2p-1}$	$v_{2m}^{k-2} w_{4m-1}$ $v_{2m}^{k-2} w_{4m-1}$
$\omega \geq 2$ $k = 2\omega$	—	$w_{2p} w_{4m-2}^{\omega-2} w_{4m-1}$ $w_{2p-1} w_{2p}^{\omega-1}$	—
$\omega \geq 2$ $k > 2\omega$	$v_p v_{2m}^{k-2\omega-1} w_{4m-2}^{\omega-1} w_{4m-1}$ $v_p^{k-2\omega} w_{2m-1} w_{2m}^{\omega-1}$	$v_p v_{2m}^{k-2\omega-1} w_{4m-2}^{\omega-1} w_{4m-1}$ $v_p^{k-2\omega} w_{2p-1} w_{2p}^{\omega-1}$	—

There is a unique exception: if $p = 2m-1$, $\omega \geq 2$ and $k \geq 2\omega + 1$, the element of maximal degree is $v_{2m}^{k-2\omega} w_{4m-2}^{\omega-1} w_{4m-1}$. □

Proof of Theorem 3. Let us define

$$k(j) = \begin{cases} k, & \text{if } j = 0 \\ k + (2m-2)j - 1 & \text{if } j \geq 1. \end{cases}$$

On the 0-th page of the spectral sequence ${}^{*}E_*^{*,*}(k+1)$ we find that ${}^{j}E_0^{\geq 1,*}(k+1)$ has no element under the line $p+q = k+1$ for $j = 0$ and nothing under the line $p+q = k+(2m-2)j$ for $j \geq 1$.



On the column 0 we have ${}^jE_1^{0,*}(k+1) = H^{*,j}(C_*(M))$ and also

$$H^{\le k(j),j}(C_k(M)) \cong {}^jE_1^{0,\le k(j)}(k+1) \cong {}^jE_\infty^{0,\le k(j)}(k+1) \cong H^{\le k(j),j}(C_{k+1}(M)).$$

□

Theorem 6. *For a $(h-1)$-connected closed orientable manifold M^{2m} we have:*
a) if $i \le h(k+1) - 1$

$$H^{i,0}(C_k(M)) \cong H^{i,0}(C_{k+1}(M)) \cong H^{i,0}(C_{k+2}(M)) \cong \dots$$

b) if $j \ge 1$ and $i \le hk + (2m - h - 1)j - 1$

$$H^{i,j}(C_k(M)) \cong H^{i,j}(C_{k+1}(M)) \cong H^{i,j}(C_{k+2}(M)) \cong \dots .$$

Proof. In this case the two graded spaces V^* and W^* are given by

$$V^* = V^0 \oplus V^h \oplus V^{h+1} \oplus \dots V^{2m-h} \oplus V^{2m},$$

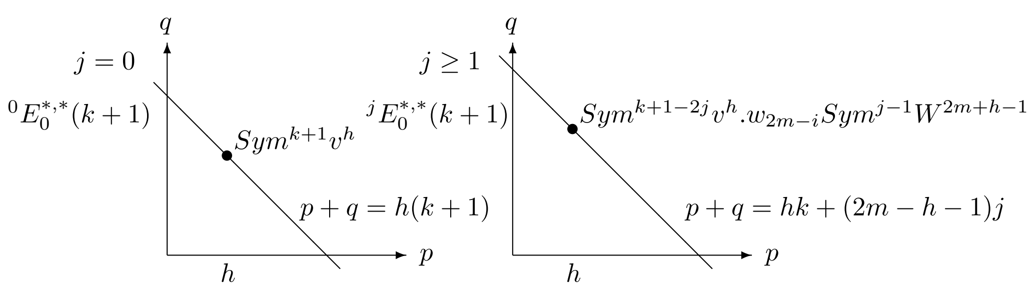
$$W^* = W^{2m-1} \oplus W^{2m+h-1} \oplus W^{2m+h} \oplus \dots \oplus W^{4m-h-1} \oplus W^{4m-1},$$

where the first (and last) components are one dimensional:

$$V^0 = \langle v_0 \rangle, \quad W^{2m-1} = \langle w_{2m-1} \rangle \quad \text{(see [15] or Section 4).}$$

As in the previous proof we find out the lowest lines:

$$p + q = h(k+1) \text{ for } j = 0 \text{ and } p + q = hk + (2m - h - 1)j \text{ for } j \ge 1.$$



□

Corollary 2. *(Th. Church) For a $(h-1)$-connected closed oriented manifold M^{2m} we have*

$$H^i(C_k(M)) \cong H^i(C_{k+1}(M)) \cong H^i(C_{k+2}(M)) \cong \dots$$

for $i \leq hk + h - 2$

Proof. As M^{2m} is not a homology sphere, we have the relation $m \geq h$ and the Theorem 6 gives the inequality $\min\{h(k+1) - 1, hk + (2m - h - 1)j - 1\}_{j \geq 1} \geq hk + h - 2$. □

3 Strong stability: odd dimensional case

In this section the manifold M has odd dimension.

Proof of Theorem 4. If there is a non-zero cohomology class (of positive degree) $x \in H^{2i}(M)$, then $x \wedge x \wedge \ldots \wedge x = x^k$ will give a non-zero cohomology class in $H^{2ki}(C_k(M))$, with arbitrary high degree, hence $H^*(C_k(M))$ cannot be stable.

If M has odd cohomology, with total Betti number $\beta(M) = \beta$, and a basis $\{1 = x_1, x_2, \ldots, x_\beta\}$ of $H^*(M)$, then the highest degree of a product of length $\beta + q - 1$ is $\sum_{i=0}^{\beta_\tau} i\beta_i(M)$, the degree of the product $x_1^{q-1} \wedge (\wedge_{i=2}^{\beta} x_i)$. We have the sequence of isomorphisms:

$$\begin{array}{ccccccc}
H^*(C_{\beta-1}(M)) & & H^*(C_\beta(M)) & & H^*(C_{\beta+1}(M)) & & \dots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
Sym^{\beta-1}(H^*(M)) & \xrightarrow[\cong]{x_1 \wedge} & Sym^{\beta}(H^*(M)) & \xrightarrow[\cong]{x_1 \wedge} & Sym^{\beta+1}(H^*(M)) & \xrightarrow[\cong]{x_1 \wedge} &
\end{array}$$

□

Proof of Corollary 1. If M^{2m+1} is a closed oriented manifold, by Poincaré duality we find that $\beta_{2i+1}(M) \neq 0$ implies $\beta_{2m-2i}(M) \neq 0$; if M has the strong stability property, this implies $m = i$.

If M has even dimension, the statement is a direct consequence of Theorem 5. □

4 Strong stability: closed orientable even dimensional manifolds

First we give a necessary restriction for the strong stability property.

Proposition 5. *If M^{2m} has negative Euler-Poincaré characteristics, then M^{2m} cannot have the strong stability property.*

Proof. From [15] and [14] we have

$$1+\sum_{k=1}^{\infty}\chi(C_k(M))t^k=(1+t)^{\chi(M)},$$

hence, if $\chi(M)$ is negative, the sequence $\{\chi(C_k(M))\}_{\geq 1}$ is not eventually constant. ∎

In this section we analyze the strong stability property for a closed oriented manifold of even dimension M^{2m}. The DG-algebra introduced by Y. Félix and J. C. Thomas [15] is defined by

$$V^*=H_*(M),\quad W^*=H_*(M)[2m-1]$$

and the differential ∂ is dual to the cup product

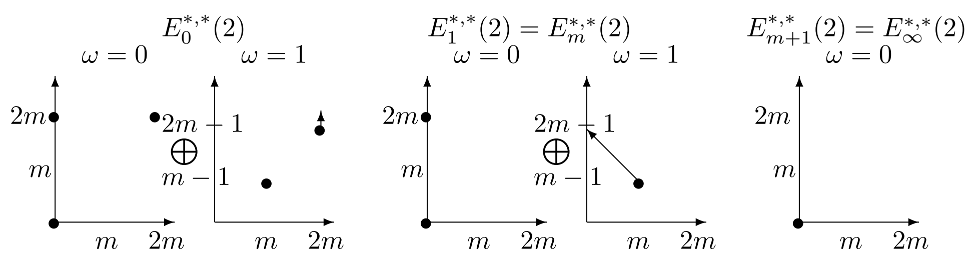
$$H^*(M)\bigotimes H^*(M)\xrightarrow{\cup}H^*(M).$$

Lemma 1. *If M^{2m} is a homology sphere, then M has the strong stability property with the range of stability 3.*

Proof. As $H^*(M)=\mathbb{Q}[x]/(x^2)$, the two graded vector spaces are $V^*=\mathbb{Q}\langle v_0,v_{2m}\rangle$, and $W^*=\mathbb{Q}\langle w_{2m-1},w_{4m-1}\rangle$ with differential

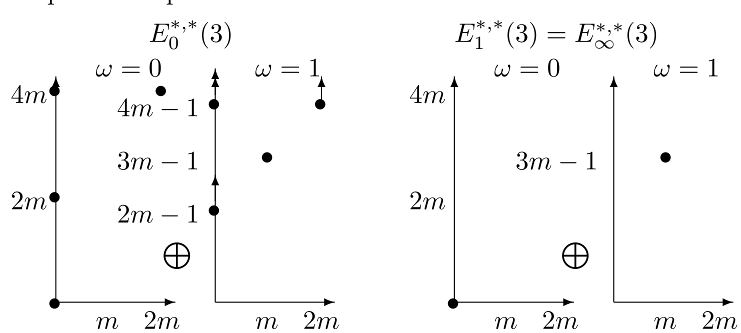
$$\partial w_{2m-1}=2v_0v_{2m},\quad \partial w_{4m-1}=v_{2m}^2.$$

The second spectral sequence is



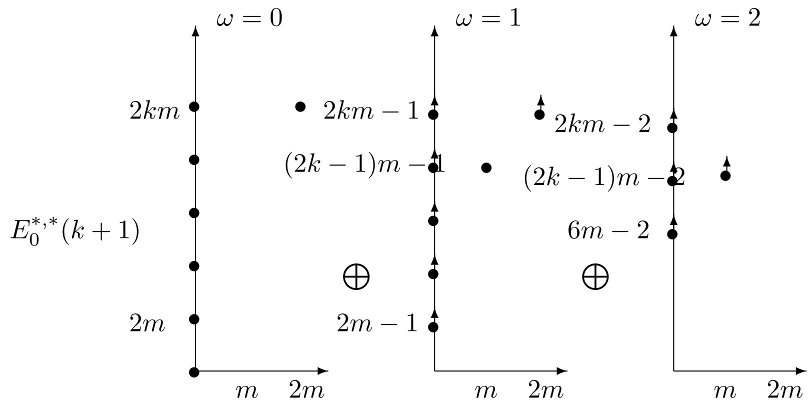
and this implies that $P_{C_2(M)}(t,s)=1$.

The third spectral sequence is

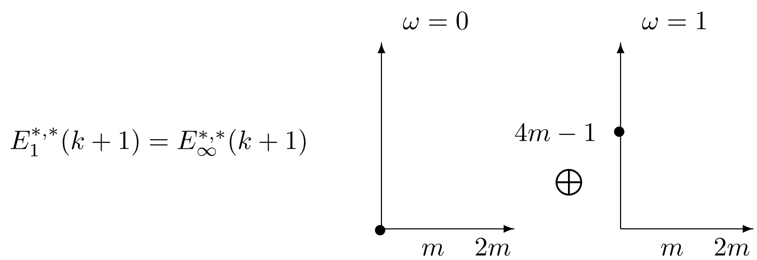


therefore $P_{C_3(M)}(t,s) = 1 + st^{4m-1}$.

By induction on k, we suppose that $P_{C_k}(M)(t,s) = 1 + st^{4m-1}$. In the $k+1$-th spectral sequence we have



The differential d_0 kills the $2m$-th column; the 0-th column has the cohomology of $C_k(M)$:



hence $P_{C_{k+1}}(t,s) = 1 + st^{4m-1}$. ∎

The case $m = 1$ in the following lemma, that is $M = \mathbb{C}\mathrm{P}^2$, was obtained by Y. Félix and D. Tanré [16].

Lemma 2. *If M^{4m} is a homology projective plane, then M has the strong stability property with the range of stability 4.*

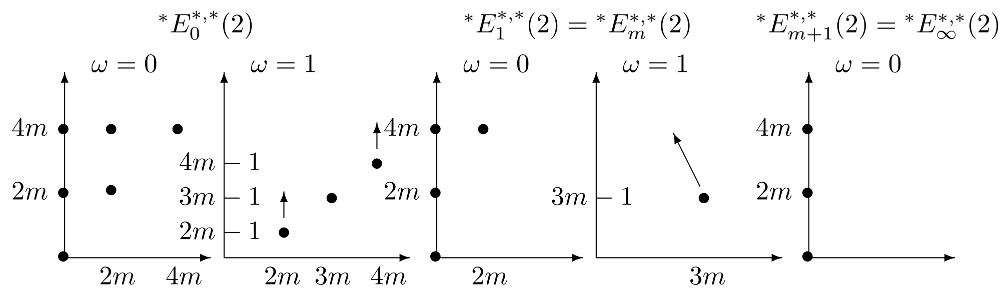
Proof. The two graded spaces are $V^* = \mathbb{Q}\langle v_0, v_{2m}, v_{4m}\rangle$, $W^* = \mathbb{Q}\langle w_{4m-1}, w_{6m-1}, w_{8m-1}\rangle$, with differential

$$\partial w_{4m-1} = 2v_0v_{4m} + v_{2m}^2, \quad \partial w_{6m-1} = 2v_{2m}v_{4m}, \quad \partial w_{8m-1} = v_{4m}^2.$$

The sequence of spectral sequences starts with:

$${}^*E_0^{*,*}(1) = {}^*E_\infty^{*,0}(1) \cong V^*$$

and



so $\mathrm{P}_{C_1(M)}(t,s) = \mathrm{P}_{C_2(M)}(t,s) = 1 + t^{2m} + t^{4m}$. The result for the spectral sequences ${}^*E_*^{*,*}(k)$, $k = 3, 4, \ldots$ are given in the following table

k	non-zero terms ${}^*E_r^{\geq 1,*}(k) = {}^*E_\infty^{\geq 1,*}(k)$
3	${}^1E_1^{2m,6m-1}(3) = \langle v_{4m} w_{4m-1} \rangle$ ${}^1E_1^{3m,7m-1}(3) = \langle v_{4m} w_{6m-1} \rangle$
4	${}^1E_1^{2m,10m-1}(4) = \langle 2v_{4m}^2 w_{4m-1} - v_{2m} v_{4m} w_{6m-1} \rangle$
5	-

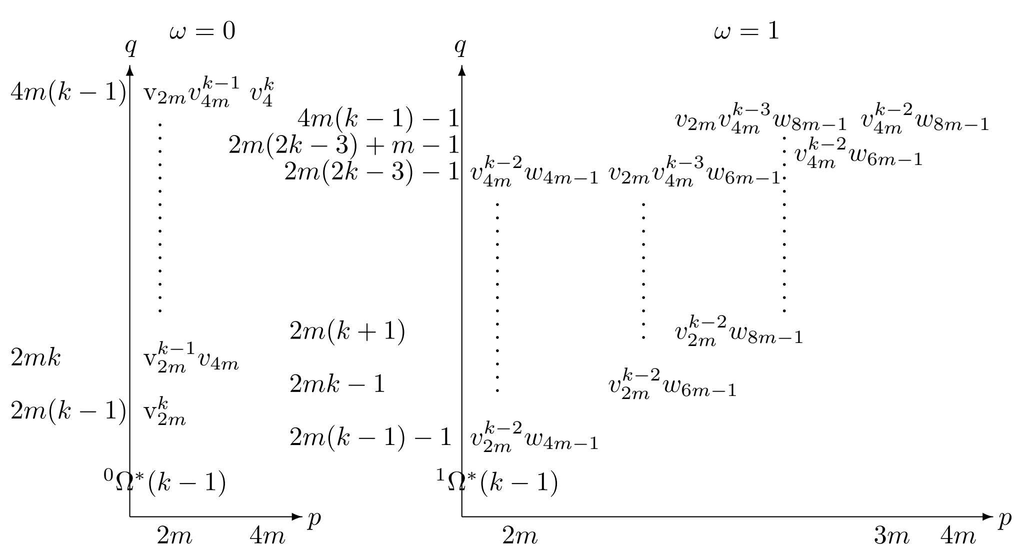
hence

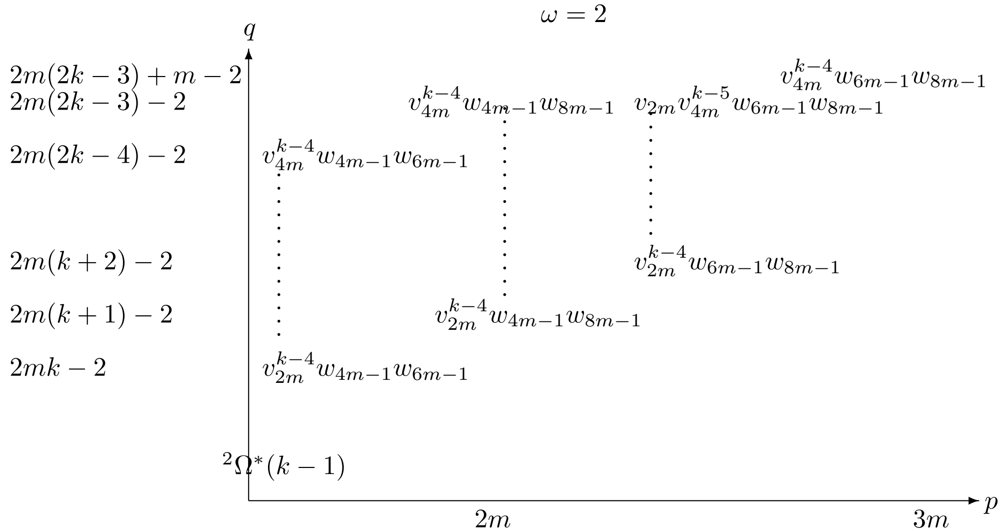
$$\mathrm{P}_{C_3(M)}(t,s) = 1 + t^{2m} + t^{4m} + s(t^{8m-1} + t^{10m-1})$$

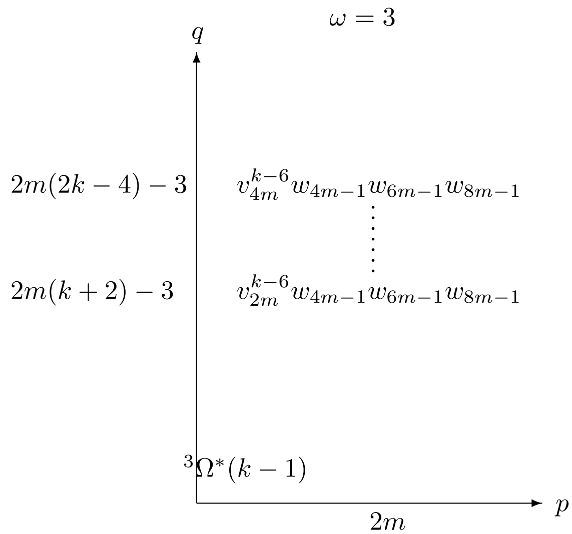
and

$$\mathrm{P}_{C_4(M)}(t,s) = 1 + t^{2m} + t^{4m} + s(t^{8m-1} + t^{10m-1} + t^{12m-1}).$$

From $k = 6$ the spectral sequences become stable at ${}^*E_1^{*,*}$:







The differential d_0 is given by $d_0(w_{4m-1}, w_{8m-1}) = (v_{2m}^2, v_{4m}^2)$ and

$$d_0(v_{2m}^{\alpha}v_{4m}^{\beta}w_{6m-1}) = \begin{cases} 0, & \text{if } \alpha = 0 \\ 2v_{2m}^{\alpha+1}v_{4m}^{\beta+1} & \text{if } \alpha \geq 1. \end{cases}$$

On the column $p = 0$ we get ${}^{\omega}(C_{k-1})$ and nothing on the last two columns, $p = 3m$ and $p = 4m$; the differential d_0 is also an isomorphism in the cases:

$${}^1E_0^{2m,2m(k-1)-1}(k) \longrightarrow {}^0E_0^{2m,2m(k-1)}(k)$$

$${}^1E_0^{2m,2m(2k-2)-1}(k) \longrightarrow {}^0E_0^{2m,2m(2k-2)}(k).$$

In general, we have the exact sequence ($j = k+2,\ k+3, \dots,\ 2k-4$)

$$\underset{[1]}{{}^3E_0^{2m,2mj-3}(k)} \rightarrowtail \underset{[3]}{{}^2E_0^{2m,2mj-2}(k)} \xrightarrow{d_0} \underset{[3]}{{}^1E_0^{2m,2mj-1}(k)} \twoheadrightarrow \underset{[1]}{{}^0E_0^{2m,2mj}(k)} :$$

in the square brackets are given the dimensions the first and last terms are of dimension one and the matrix of d_0 is

$$d_0 = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\underset{[1]}{{}^2E_0^{2m,2mk-2}(k)} \rightarrowtail \underset{[2]}{{}^1E_0^{2m,2mk-1}(k)} \twoheadrightarrow \underset{[1]}{{}^0E_0^{2m,2mk}(k)}$$

$$\underset{[1]}{{}^2E_0^{2m,2m(k+1)-2}(k)} \rightarrowtail \underset{[2]}{{}^1E_0^{2m,2m(k+1)-1}(k)} \twoheadrightarrow \underset{[1]}{{}^0E_0^{2m,2m(k+1)}(k)}$$

$$\underset{[1]}{{}^2E_0^{2m,2m(2k-3)-1}(k)} \rightarrowtail \underset{[2]}{{}^1E_0^{2m,2m(2k+1)-1}(k)} \twoheadrightarrow \underset{[1]}{{}^3E_0^{2m,2m(2k-3)}(k)}$$

In conclusion, we get

$${}^*E_1^{*,*}(k) = {}^*E_\infty^{*,*}(k) = {}^0E_\infty^{0,*}(k) \oplus {}^1E_1^{0,*}(k),$$

with the Poincaré polynomial ($k \geq 4$) :

$$\mathrm{P}_{C_k(M)}(t,s) = 1 + t^{2m} + t^{4m} + s(t^{8m-1} + t^{10m-1} + t^{12m-1}).$$

□

Proof of Theorem 5. Lemmas 1 and 2 give one implication of the theorem. For the opposite implication, we show in the next three lemmas that M cannot have strong stability property in the following cases:

Case 1) the Poincaré polynomial of M^{4m} is $1 + \beta_{2m}t^{2m} + t^{4m}$, $\beta_{2m} \geq 2$;

Case 2) there is a non-zero odd Betti number β_{2i+1};

Case 3) there is a non-zero even Betti number of M^{2m}, β_{2i}, $i \neq 0$, $\frac{m}{2}$, m. □

Lemma 3. *If M^{4m} has the Poincaré polynomial $1+bt^{2m}+t^{4m}$, with $b \geq 2$, then M cannot have the strong stability property.*

Proof. The associated graded spaces are $V^* = \mathbb{Q}\langle v_0; v_{2m,1}, v_{2m,2}, \dots v_{2m,b}; v_{4m}\rangle$ and $W^* = \mathbb{Q}\langle w_0; w_{2m,1}, w_{2m,2}, \dots w_{2m,b}; w_{4m}\rangle$ (although irrelevant for the argument, one can choose the basis such that $\partial w_{4m-1} = 2v_0v_{4m} + \sum_{i=1}^b v_{2m,i}^2$, $\partial w_{6m-1,i} = 2v_{2m,i}v_{4m}$, $\partial w_{8m-1} = v_{4m}^2$).

In the k-th spectral sequence, the domain and the range of the differential

$$d_0 :\ {}^1E_0^{2m,2m(k-1)-1}(k) \longrightarrow {}^0E_0^{2m,2m(k-1)}(k)$$

have dimensions $\binom{k+b-3}{b-1}$ and $\binom{k+b-1}{b-1}$ respectively. Obviously $d_0({}^0E_0^{*,*}(k)) = 0$ and $d(v_{4m}^{k-2} \otimes W^*) \subset v_{4m}^{k-2} \otimes \wedge^2 V^*$, therefore we have non-zero elements in ${}^0E_\infty^{2m,2m(k-1)}(k)$ of arbitrary large degree. □

Lemma 4. *If M^{2m} has a non-zero odd Betti number, then M cannot have the strong stability property.*

Proof. Choose the non-zero odd Betti number of the highest degree, $2i+1$. The graded spaces V^* and W^* are

$$V^* = \mathbb{Q}\langle v_0, \dots, v_{2i+1}, v'_{2i+1}, \dots, v_{2p}, \dots, v_{2q}, \dots, v_{2m}\rangle,$$

$$W^* = \mathbb{Q}\langle w_{2m-1}, \dots, w_{2m+2i}, w'_{2m+2i}, \dots, w_{2m+2p-1}, \dots, w_{2m+2q-1}, \dots, w_{4m-1}\rangle.$$

The differential of w_{2m+2i} contains a unique term, $2v_{2i+1}v_{2m}$, for degree reason (a quadratic product $v_s v_t$ with $2i+1 < s, t < 2m$ has even degree). The spectral sequence ${}^kE^{*,*}_*(2k+1)$ contains the product $z = v_{2i+1}w^k_{2m+2i}$, which is a permanent cocycle. Its is never a coboundary:

$$d(\wedge^* V^* \otimes \wedge^* W^*) \subset \wedge^{\geq 2} V \otimes \wedge^* W^*.$$

The degree of z is arbitrary large, therefore M has not strong stability. ∎

Remark 3. C. Schliessl [30] computed all Betti numbers of $C_k(\mathbb{T}^2)$. Its top Betti number is $\beta_{k+1}(C_k(\mathbb{T}^2)) = \dfrac{2k-1-3(-1)^k}{4}$ (see also [11] and [23]).

Lemma 5. *If M^{2m} has a non-zero even Betti number β_{2i}, (with $2i \neq 0$, m and $2m$), then M cannot have the strong stability property.*

Proof. Using Poincaré duality we can choose a positive i satisfying $0 < 2i < m$: $V_* = \mathbb{Q}\langle v_0, \dots, v_{2i}, \dots, v_{2m}\rangle$. In the spectral sequence ${}^0E^{*,*}_*(2k)$, the product v^{2k}_{2i} is a permanent cocycle and it is never a coboundary:

$$d(Sym^{k-2}V^{\geq 2i+1} \otimes W^*) \subset Sym^{k-2}V^{\geq 2i+1} \otimes Sym^2 V.$$

∎

Remark 4. M. Maguire [23] computed all Betti numbers of $C_k(\mathbb{CP}^3)$. Its top Betti number is $\beta_{k+12}(C_k(\mathbb{CP}^3)) = 1$ $(k \geq 11)$.

5 Strong stability: open or nonorientable even dimensional manifolds

In this section we use B. Knudsen model [21]: the differential graded algebra computing the cohomology of $C_k(M)$ for an even dimensional manifold M^{2m} is given by

$$H^*(\Omega^*(k)(V^*, W^*), \partial),$$

where the graded spaces V^* and W^* are

$$V^* = H_c^{-*}(M; \mathbb{Q}^w)[2m], \quad W^* = H_c^{-*}(M; \mathbb{Q})[4m-1],$$

and the differential is the shifted dual of the product

$$H_c^{-*}(M;\mathbb{Q}^w)\otimes H_c^{-*}(M;\mathbb{Q}^w)\longrightarrow H_c^{-*}(M;\mathbb{Q})$$

(here H_c^{-*} is cohomology with compact supports and $\mathbb{Q}^w$ is the orientation sheaf; as before $A^*[q]$ is the graded space A^* shifted by q).

In the same paper B. Kundsen computed the cohomology of $C_k(M)$ for three even dimensional manifolds with odd cohomology: Klein bottle $\mathbb{K}$, the punctured Euclidean space $\mathring{\mathbb{R}}^n=\mathbb{R}^n\setminus\{pt\}$ and the punctured torus $\mathring{\mathbb{T}}=\mathbb{T}\setminus\{pt\}$. He found that their top Betti numbers are

$$\beta_\tau(C_k(\mathbb{K}))=\beta_k(C_k(\mathbb{K}))=2,$$
$$\beta_\tau(C_k(\mathring{\mathbb{R}}^n))=\beta_k(C_k(\mathring{\mathbb{R}}^n))=1,$$
$$\beta_\tau(C_k(\mathring{\mathbb{T}}))=\beta_k(C_k(\mathring{\mathbb{T}}))=\frac{3+(-1)^{k+1}}{4}k+1,$$

so these three spaces does not have the strong stability.

We will describe few cases of manifolds of even dimensions with the strong stability property.

Proposition 6. *Let M^{2m} be closed nonoriented manifold with $\tau(M)\le\left\lfloor\frac{4m-2}{3}\right\rfloor$. Then M has the strong stability property if and only if M is acyclic.*

Proof. For a closed non-orientable manifold we have

$$H_c^{-*}(M)=H^{-*}(M)\text{ and }H^{-*}(M;\mathbb{Q})\cong H_{2m-*}(M)$$

and these imply that

$$V^*=V^0\oplus V^1\oplus\ldots\oplus V^{\tau(M)},\quad W^*=W^{4m-\tau(M)-1}\oplus W^{4m-\tau(M)}\oplus\ldots\oplus W^{4m-1}$$

The product $V^*\bigotimes V^*\to W^*$ is zero by degree relation:

$$2\tau(M)<4m-\tau(M)-1\quad\text{or}\quad\tau(M)\le\frac{4m-2}{3},$$

hence the differential ∂ is also zero. If M^{2m} is not acyclic, then there is a non-zero $x\in H^{\ge 1}(M)$. If the degree of x is even, then there is a corresponding non-zero $v\in V^{even}\ge 2$; otherwise, there is corresponding non-zero $w\in W^{even}$. Therefore v^k, respectively w^k are non-zero cohomology classes of arbitrary large degree, and this contradicts the strong stability property. ∎

Proposition 7. *Let M^{2m} be a closed nonorientable manifold with odd cohomology. Then M has the strong stability property if and only if M is acyclic.*

Proof. For a closed non-orientable manifold we have $H_c^{-*}(M)=H^{-*}(M)$, $H^{-2m}(M)=0$, $H^0(M)=\mathbb{Q}$, hence $W^*=W^{\ge 2m}$, $W^{4m-1}=\mathbb{Q}\langle w_{4m-1}\rangle$.

If M has odd cohomology, we have

$$W^*=W^{even}\oplus W^{4m-1}$$

by Poincaré duality (see [B] or [D]), $H^{-*}(M^{2m};\mathbb{Q}^w) \cong H_{2m-*}(M;\mathbb{Q})$, hence

$$V^* = V^{\leq 2m-1} = \mathbb{Q}\langle v_0\rangle \oplus V^{odd}.$$

A non-zero (odd) Betti number of M^{2m} will give a nonzero $w \in W^{even}$. Its differential ∂w is in $(\bigwedge^2 V)^{odd} = v_0 \wedge V^{odd}$; the degree of w is at least $2m$, and the degree of an element in $v_0 \wedge V^{odd}$ is at most $2m-1$, therefore $\partial w = 0$. The product w^k gives a permanent cocycle in $E_*^{*,*}(2k)$, and it is never a coboundary:

$$\partial(SymV^* \otimes SymW^*) \subset Sym^{\geq 2}V^* \otimes SymW^*.$$

The degree of w^k is arbitrary large, hence M cannot have strong stability property.

If M^{2m} is acyclic, the cohomology of $C_k(M)$ is reduced to

$$H^*(C_k(M)) \cong \mathbb{Q}\langle v_0^k, v_0^{k-2}w_{4m-1}\rangle$$

and this is stable. ∎

Proposition 8. *Let M^{2m} be an open orientable manifold with odd cohomology. Then M has the strong stability property if and only if M is acyclic.*

Proof. For an open oriented manifold M^{2m} we have, by Poincaré duality,

$$H_c^{-*}(M;\mathbb{Q}^w) = H_c^{-*}(M;\mathbb{Q}) \cong H_{2m-*}(M;\mathbb{Q}),\ H_{2m}(M) = 0,\ H_0(M) = \mathbb{Q},$$

hence $W^* = \mathbb{Q}\langle w_{2m-1}\rangle \oplus W^{\geq 2m}$. If M has odd cohomology, we also have

$$W^* = \mathbb{Q}\langle w_{2m-1}\rangle \oplus W^{even} \text{ and } V^* = \mathbb{Q}\langle v_0\rangle \oplus V^{odd\leq 2m-1}.$$

Now we can repeat the argument of the proof of Proposition 4. ∎

Remark 5. It seems that in general the sequence of Betti numbers $\{\beta_i(C_k(M))\}_{k\geq 1}$ is increasing for any $i \geq 0$ and for any manifold M, with the exception of S^{2m}. For other peculiar properties of the cohomology of configuration spaces of S^2, see [2] and [3].

6 Shifted stability

We start to analyse the odd dimensional case.

Proposition 9. *A manifold M^{2m+1} satisfies the shifted stability condition if and only if the top positive even Betti number is one.*

Proof. Let β_{2a} the top even Betti number ($a \geq 1$). For $k \geq \Sigma_{i\geq 2a}\,\beta_i + 1 = \beta + 1$ we have

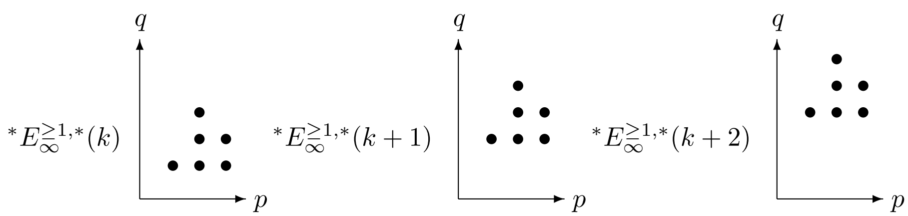
$$H^{top}(C_k(M)) = Sym^{k-\beta}V^{2a} \bigotimes \bigwedge^{\beta} V^{\geq 2a+1}$$

hence M has the shifted stability property if and only if dim$Sym^{k-\beta}V^{2a}$ does not depend on k therefore β_{2a} should be one. ∎

Now we give three new definitions for shifted stability of the sequence $\{C_k(M)\}_{k\geq 1}$ for an even dimensional manifold M. In the first definition, the spectral sequences $\{{}^*E_*^{*,*}(k)\}_{k\geq 1}$ are those defined in Section 2. We suppose that M is not an even dimensional homology sphere, that is ${}^\omega E_\infty^{0,*}(k+1) = {}^\omega H^*(C_k(M))$.

Definition 3. *The manifold M satisfies the* spectral shifted stability condition *with* range r *and* shift σ $(r, \sigma \geq 1)$ *if and only if, for any $k \geq r$, any $p \geq 1$ and any $\omega \geq 0$, we have*

$$ {}^\omega E_\infty^{p,q+\sigma}(k+1) = {}^\omega E_\infty^{p,q}(k) \text{ and this is non-zero.} $$



Definition 4. *The manifold M satisfies the* Poincaré polynomial shifted stability condition *with* range r, shift σ *and* ratio $R(s,t) \neq 0$ $(r, \sigma \geq 1)$ *if and only if, for any $k \geq r$, we have*

$$ P_{C_{k+1}(M)}(t,s) = P_{C_k(M)}(t,s) + t^{(k+1-r)\sigma} R(t,s). $$

Definition 5. *The manifold M satisfies the* extended shifted stability condition *with* range r *and* shift σ $(r, \sigma \geq 1)$ *if and only if, for any $k \geq r$, we have*

$$ P_{C_{k+1}(M)}^{[(k-r+1)\sigma]}(t,s) = t^\sigma P_{C_k(M)}^{[(k-r+1)\sigma]}(t,s). $$

The relation between these shifted stability conditions are given by:

Proposition 10. *Spectral shifted stability $\Rightarrow$ Poincaré polynomial shifted stability $\Rightarrow$ extended shifted stability $\Rightarrow$ shifted stability.*

Proof. First implication: Let us define the polynomial $R(s,t)$, the ratio of an arithmetical sequence, as the double Poincaré of ${}^\omega E_\infty^{\geq 1,*}(r)$:

$$ R(s,t) = \sum_{\omega \geq 0} \sum_{\substack{p+q=i \\ p \geq 1}} \dim {}^\omega E_\infty^{p,q}(r) t^i s^\omega. $$

By induction we get

$$ {}^\omega E_\infty^{p,q}(r) = {}^\omega E_\infty^{p,q+(k-r)\sigma}(k) \qquad \text{(for a positive } p \text{ and } k \geq r) $$

and this implies that the Poincaré polynomial of ${}^\omega E_\infty^{\geq 1,*}(k)$ is constant, for $k \geq r$, up to a shift with t^σ. Therefore we have:

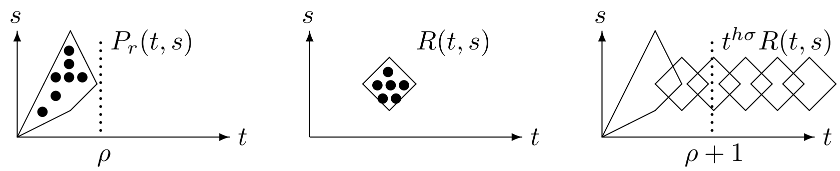
$$ \begin{aligned} P_{C_{k+1}(M)}(t,s) &= P_{{}^*E_\infty^{0,*}(k+1)}(t,s) + P_{{}^*E_\infty^{\geq 1,*}(k+1)}(t,s) \\ &= P_{C_k(M)}(t,s) + t^{(k+1-r)\sigma} R(t,s), \end{aligned} $$

hence the spectral shifted stability condition with range r and shift σ gives the Poincaré polynomial shifted stability condition with the same range r and shift σ.

Second implication: The recurrence formula $P_{C_{k+1}} = P_{C_k} + t^{(k+1-r)\sigma}R$ gives

$$P_{C_k(M)}(t,s) = P_{C_r(M)}(t,s) + (t^{\sigma} + t^{2\sigma} + \ldots + t^{(k-r)\sigma})R(t,s).$$

Take ρ such that the strip $[0,\rho]\times\mathbb{R}$ contains the support of $P_{C_r(M)}(t,s)$ and h big enough such that support of $t^{h\sigma}R(s,t)$ is contained in $[\rho+1,\infty)\times\mathbb{R}$.



For $\rho = r+h-1$ we have $P^{[\sigma]}_{\rho+1}(t,s) = t^{\sigma}P^{[\sigma]}_{\rho}(t,s)$, next we have $P^{[2\sigma]}_{\rho+2}(t,s) = t^{\sigma}P^{[2\sigma]}_{\rho+1}(t,s)$, and in general, for $k\geq\rho$,

$$P^{[(k+1-\rho)\sigma]}_{k+1}(t,s) = t^{\sigma}P^{[(k+1-\rho)\sigma]}_{k}(t,s).$$

Third implication: This is obvious. □

Remark 6. In order to have “weight stability at 0” in the sequence of spectral sequences $\{{}^{*}E^{*,*}_{*}(k)\}_{k\geq1}$ (i.e, there is a range r and a weight ω_{max} such that ${}^{\omega}E^{*,*}_{0}(k) = 0$ for any $k\geq r$ and any $\omega > \omega_{max}$), we have to consider only manifolds with even cohomology: a non-zero odd cohomology class $x\in H^{odd}(M)$ will give a non-zero $\omega\in W^{even}$ and infinitely many non-zero terms $\omega^{s}\in{}^{s}E^{*,*}_{0}(2s)$ of arbitrary large weights.

In fact, if the manifold M has the spectral shifted stability condition, then M should have even cohomology.

Proposition 11. *If there is a nonzero cohomology class $x\in H^{odd}(M)$, then M cannot have the spectral sequence shifted stability property.*

Proof. Take a maximal odd degree element $v_{2i+1}\in V^{*} = \langle v_0,\ldots,v_{2m}\rangle$ and the coresponding $w_{2m+2i}\in W^{*}$. The relations

$$d(w_{2m+2i}) = 2v_{2i+1}v_{2m} \quad\text{and}\quad d(w_{4m-1}) = v^{2}_{2m}$$

gives the infinite (non-zero) cocycle

$$2hv_{2m+1}w^{h-1}_{2m+2i}w_{4m-1} + v_{2m}w^{h}_{2m+2i}\in{}^{h}E^{*,*}_{\infty}(2h+1)$$

of arbitrary large weight. Definitely, the spectral sequence shifted stability condition implies the “weight stability condition at ∞”: there is a range r and a weight such that ${}^{\omega}E^{*,*}_{\infty}(k) = 0$ for $k\geq r$ and $\omega>\omega_{max}$. □

For the Poincaré polynomial shifted stability condition, a weaker condition is needed.

Proposition 12. *If $\chi(M) \leq -2$, then the manifold M does not satisfy the Poincaré polynomial shifted stability condition.*

Proof. The recurrence relations ($k \geq r$)

$$P_{C_{k+1}(M)}(t,s) = P_{C_k(M)}(t,s) + t^{(k+1-r)\sigma} R(t,s),$$
$$P_{C_{k+2}(M)}(t,s) = P_{C_{k+1}(M)}(t,s) + t^{(k+2-r)\sigma} R(t,s)$$

imply that, for large k, $\chi(C_k(M))$ is an arithmetic sequence (if σ is even) or $\chi(C_k(M)) = \chi(C_{k+2}(M)) = \chi(C_{k+4}(M)) = \ldots$ (if σ is odd).

If $\chi(M) \leq -2$, the Euler characteristics $\{\chi(C_k(M))\}_{k\geq 1}$, that is the coefficients in the expansion of $(1+t)^{\chi(M)}$, have a polynomial growth (at least quadratic) for $\chi(M) \leq -3$ and, for $\chi(M) = -2$, $\chi(C_k(M)) = (-1)^{k+1}(k+1)$. □

In the case of a manifold M^{2m} with Poincaré polynomial shifted stability, Propositions 3 and 6 give some restriction for the shift σ and ratio $R(t,s)$. For instance, we have:

Proposition 13. *For a $(h-1)$-connected closed orientable manifold M^{2m} satisfying Poincaré polynomial shifted condition with shift σ, we have the inequality $h \leq \sigma$.*

Proof. Choose $j \geq 0$ such that there is a non-zero coefficient $r^{i,j}$ of the ratio polynomial $R(s,t)$. From Proposition 6

$$(k+1-r)\sigma + i \geq \begin{cases} h(k+1) & \text{if } i = 0 \\ hk + (2m-h-1)j & \text{if } i \geq 1, \end{cases}$$

and, for large k, this implies $\sigma \geq h$. □

Remark 7. In the following examples, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^3$, we have $h = \sigma = 2$; the same is true for $\mathbb{C}\mathrm{P}^4$. For $\mathbb{C}\mathrm{P}^5$ and $\mathbb{C}\mathrm{P}^6$ we have $h = 2$, $\sigma = 4$.

The shifted stability property gives a formula for the $cd(k)$, the cohomological dimension of $C_k(M)$.

Proposition 14. *For a manifold M satisfying the shifted stability condition with range r and shift σ we have, for any $k \geq r$,*

$$cd(k) = cd(r) + (k-r)\sigma.$$

Proof. This is clear from the definition. □

Example 1. The cohomological dimension of $C_k(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1)$ is given by

$$cd(1) = cd(2) = 4,$$

$$cd(3) = 9,$$

$$cd(4) = 11,$$

$$cd(k) = 2k + 4 \,\text{if}\; k \geq 5.$$

For large k, the classical stability property and the extended shifted stability property give all the Betti numbers of $C_k(M)$.

Proposition 15. *Let M^{2m} be a $(h-1)$-connected closed orientable manifold satisfying the extended shifted stability condition with the range r and shift σ. Then, for any k satisfying the inequality $max\{r, cd(r)\} \leq hk+h-2$, we have the recurrence relation*

$$H^*(C_{k+1}(M)) = H^{\leq cd(r)}(C_k(M)) \bigoplus H^{\geq cd(r)-\sigma+1}(C_K(M))[\sigma].$$

Proof. If $cd(r) \leq hk+h-2$, from Corollary 2, we have the initial equality

$$H^{\leq cd(r)}(C_k(M)) = H^{\leq cd(r)}(C_k(M))$$

and, if $r \leq k$, using the extended shifted stability property, we have final equality

$$H^{\geq cd(r)+1}(C_{k+1}(M)) = H^{\geq cd(r)-\sigma+1}(C_k(M))[\sigma].$$

□

7 Shifted stability: examples

The first example is the product $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. In this case the graded spaces V^*, W^* and the differential ∂_ω are

$$V^* = \langle v_0, v_2, \bar{v}_2, v_4\rangle, \quad W^* = \langle w_3, w_5, \bar{w}_5, w_7\rangle$$

$$\partial_\omega(w_3, w_5, \bar{w}_5, w_7) = (2v_0v_4 + v_2^2, 2v_2v_4, 2\bar{v}_2v_4, v_4^2).$$

New cocycles in ${}^*E_0^{*,*}(k)$, $k \geq 3$, are generated by

$$\begin{aligned} &\gamma = v_2\bar{w}_5 - v_4w_3, \quad \bar{\gamma} = \bar{v}_2w_5 - v_4w_3,\\ &\varepsilon = v_4w_5 - v_2w_7, \quad \bar{\varepsilon} = v_4\bar{w}_5 - \bar{v}_2w_7,\\ &\eta = v_4w_5\bar{w}_5 - \bar{v}_2w_5w_7 + v_2\bar{w}_5w_7 \quad (\bar{\eta} = -\eta). \end{aligned}$$

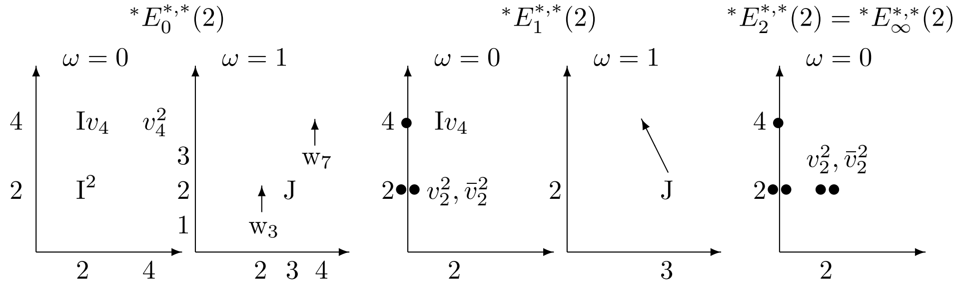
Proposition 16. *The product of two projective lines, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, satisfies the spectral shifted stability condition with range 6 and shift 2. More precisely, the nonzero pieces ${}^*E_\infty^{\geq 1,*}(k)$ (for $k \geq 6$) are:*

$$\begin{aligned} &{}^0E_\infty^{2,2k-2}(k) = \langle v_2^k, \bar{v}_2^k\rangle,\\ &{}^1E_\infty^{2,2k-1}(k) = \langle v_2^{k-3}\gamma, \bar{v}_2^{k-3}\bar{\gamma}\rangle, {}^1E_\infty^{2,2k+1}(k) = \langle v_2^{k-3}\varepsilon, \bar{v}_2^{k-3}\bar{\varepsilon}\rangle,\\ &{}^2E_\infty^{2,2k+2}(k) = \langle v_2^{k-5}\eta, \bar{v}_2^{k-5}\bar{\eta}\rangle. \end{aligned}$$

Proof. The sequence of spectral sequences starts with

$${}^*E_0^{*,*}(1) = {}^*E_\infty^{*,0}(1) \cong V^*$$

and



Here and in the following computations we use the notation

$$\mathrm{I} = \langle v_2, \bar{v}_2 \rangle,\ \ \mathrm{J} = \langle w_5, \bar{w}_5 \rangle \text{ and also } \mathrm{I}^k = \langle v_2^k, v_2^{k-1}\bar{v}_2, \ldots, \bar{v}_2^k \rangle,$$

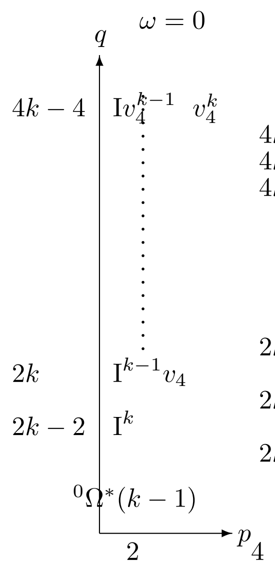
$$\mathrm{J}^2 = \langle w_5, \bar{w}_5, \rangle,\ \ \mathrm{IJ} = \langle v_2 w_5, \bar{v}_2 w_5, v_2 \bar{w}_5, \bar{v}_2 \bar{w}_5 \rangle \text{ and so on.}$$

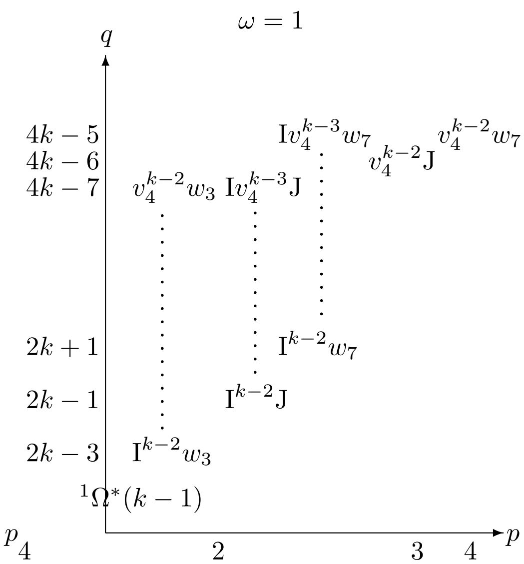
The results for the spectral sequences ${}^*E_*^{*,*}(k)$, $k = 3, 4, \ldots, 7$, the "weight unstable part," are given in the table ($\triangle_k = P_{C_k}(t,s) - P_{C_{k-1}}(t,s)$):

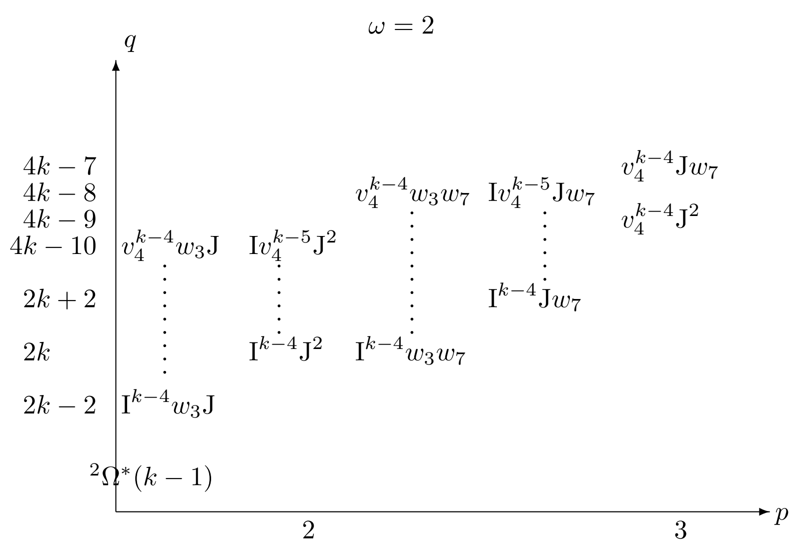
Table 1

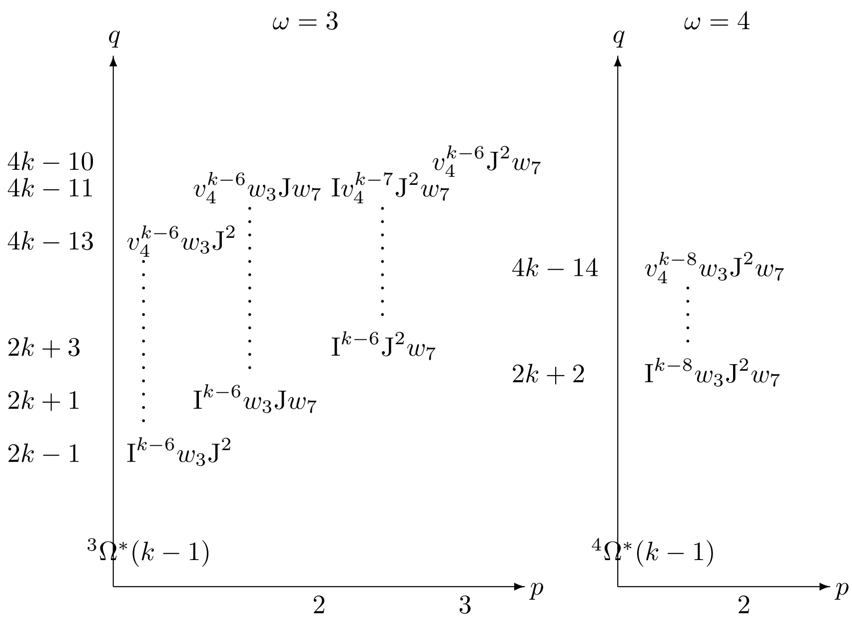
k	non-zero terms ${}^*E_r^{\geq 1,*}(k) = {}^*E_\infty^{\geq 1,*}(k)$	$\triangle_k$
3	${}^0E_1^{2,4} = \langle v_2^3, \bar{v}_2^3 \rangle$, ${}^1E_1^{2,5} = \langle \gamma, \bar{\gamma} \rangle, {}^1E_1^{2,7} = \langle \varepsilon, \bar{\varepsilon} \rangle$	$2t^6 + s(2t^7 + 2t^9)$
4	${}^0E_1^{2,6} = \langle v_2^4, \bar{v}_2^4 \rangle$, ${}^1E_1^{2,7} = \langle v_2\gamma, \bar{v}_2\bar{\gamma} \rangle$, ${}^1E_2^{2,9} = \langle v_2\varepsilon, \bar{v}_2\bar{\varepsilon}, v_2\bar{\varepsilon}\,(= -\bar{v}_2\varepsilon) \rangle$	$2t^8 + s(2t^9 + 3t^{11})$
5	${}^0E_1^{2,8} = \langle v_2^5, \bar{v}_2^5 \rangle$, ${}^1E_1^{2,9} = \langle v_2^2\gamma, \bar{v}_2^2\bar{\gamma} \rangle, {}^1E_1^{2,11} = \langle v_2^2\varepsilon, \bar{v}_2\bar{\varepsilon} \rangle$, ${}^2E_1^{3,11} = \langle v_4 \mathrm{J}^2 \rangle$	$2t^{10} + s(2t^{11} + 2t^{13}) + s^2t^{14}$
6	${}^0E_1^{2,10} = \langle v_2^6, \bar{v}_2^6 \rangle$, ${}^1E_1^{2,11} = \langle v_2^3\gamma, \bar{v}_2^3\bar{\gamma} \rangle, {}^1E_1^{2,11} = \langle v_2^3\varepsilon, \bar{v}_2^3\bar{\varepsilon} \rangle$, ${}^2E_1^{2,14} = \langle v_2\eta, \bar{v}_2\bar{\eta} \rangle$	$2t^{12} + s(2t^{13} + 2t^{15}) + 2s^2t^{16}$
7	${}^0E_1^{2,12} = \langle v_2^7, \bar{v}_2^7 \rangle$, ${}^1E_1^{2,13} = \langle v_2^4\gamma, \bar{v}_2^4\bar{\gamma} \rangle, {}^1E_1^{2,15} = \langle v_2^4\varepsilon, \bar{v}_2^4\bar{\varepsilon} \rangle$, ${}^2E_1^{2,16} = \langle v_2^2\eta, \bar{v}_2^2\bar{\eta} \rangle$	$2t^{14} + s(2t^{15} + 2t^{17}) + 2s^2t^{18}$

For $k \geq 8$ the sequence $\{{}^*E_*^{*,*}(k)\}$ is "weight stable at 0" (${}^{\geq 5}E_0^{*,*}(k) = 0$) and we have the following picture of the first page of the k-th term ${}^*E_0^{*,*}(k)$:


$\omega = 0$
q
$4k-4$
$\mathrm{I}v_4^{k-1}$
v_4^k
$2k$
$\mathrm{I}^{k-1}v_4$
$2k-2$
I^k
${}^0\Omega^*(k-1)$
p
2
4


$\omega = 1$
q
$4k-5$
$4k-6$
$4k-7$
$\mathrm{I}v_4^{k-3}w_7$
$v_4^{k-2}\mathrm{J}$
$v_4^{k-2}w_7$
$v_4^{k-2}w_3$
$\mathrm{I}v_4^{k-3}\mathrm{J}$
$2k+1$
$\mathrm{I}^{k-2}w_7$
$2k-1$
$\mathrm{I}^{k-2}\mathrm{J}$
$2k-3$
$\mathrm{I}^{k-2}w_3$
${}^1\Omega^*(k-1)$
p
2
3
4


$\omega = 2$
q
$4k-7$
$4k-8$
$4k-9$
$4k-10$
$v_4^{k-4}\mathrm{J}w_7$
$v_4^{k-4}w_3w_7$
$\mathrm{I}v_4^{k-5}\mathrm{J}w_7$
$v_4^{k-4}\mathrm{J}^2$
$v_4^{k-4}w_3\mathrm{J}$
$\mathrm{I}v_4^{k-5}\mathrm{J}^2$
$2k+2$
$\mathrm{I}^{k-4}\mathrm{J}w_7$
$2k$
$\mathrm{I}^{k-4}\mathrm{J}^2$
$\mathrm{I}^{k-4}w_3w_7$
$2k-2$
$\mathrm{I}^{k-4}w_3\mathrm{J}$
${}^2\Omega^*(k-1)$
p
2
3



The differential d_0 is $d_0(w_3, w_7) = (2v_2\bar{v}_2, v_4^2)$ and

$$d_0(v_2^\alpha \bar{v}_2^\beta v_4^\gamma w_5, v_2^\alpha \bar{v}_2^\beta v_4^\gamma \bar{w}_5) = \begin{cases} (0,0) & \text{if } \alpha = \beta = 0 \\ (2v_2^{\alpha+1}\bar{v}_2^\beta v_4^{\gamma+1}, 2v_2^\alpha \bar{v}_2^{\beta+1} v_4^{\gamma+1}) & \text{if } \alpha + \beta \geq 1. \end{cases}$$

On the column $p = 0$ we get ${}^\omega H^*(C_{k-1})$ and nothing on the columns $p = 3$ and $p = 4$: the differential d_0 is an isomorphism in the following case:

$$\begin{aligned} {}^2E_0^{3,4k-7}(k) = v_4^{k-4}\mathrm{J}w_7 &\longrightarrow {}^1E_0^{3,4k-6}(k) = v_4^{k-2}\mathrm{J}, \\ {}^3E_0^{3,4k-10}(k) = v_4^{k-6}\mathrm{J}^2 w_7 &\longrightarrow {}^2E_0^{3,4k-9}(k) = v_4^{k-4}\mathrm{J}^2, \\ {}^1E_0^{4,4k-5}(k) = v_4^{k-2}w_7 &\longrightarrow {}^2E_0^{4,4k-4}(k) = v_4^{k}. \end{aligned}$$

On the column $p = 2$ we have a five components cochain complex $e(q)$, where q takes values in the interval $[k-2, 2k-3]$:

$$
\begin{array}{ccccccccc}
{}^4E_0^{2,2q-2}(k) & \xrightarrow{a} & {}^3E_0^{2,2q-1}(k) & \xrightarrow{b} & {}^2E_0^{2,2q}(k) & \xrightarrow{c} & {}^1E_0^{2,2q+1}(k) & \xrightarrow{d} & {}^0E_0^{2,2q+2}(k) \\
\| & & \| & & \| & & \| & & \| \\
\mathrm{I}^{2k-q-6}v_4^{q-k-2}w_3\mathrm{J}^2w_7 & & \mathrm{I}^{2k-q-6}v_4^{q-k}w_3\mathrm{J}^2 & & \mathrm{I}^{2k-q-5}v_4^{q-k+1}w_3\mathrm{J} & & \mathrm{I}^{2k-q-4}v_4^{q-k+2}w_3 & & \mathrm{I}^{2k-q-2}v_4^{q-k+2} \\
 & & \bigoplus & & \bigoplus & & \bigoplus & & \\
 & & \mathrm{I}^{2k-q-5}v_4^{q-k-1}w_3\mathrm{J}w_7 & & \mathrm{I}^{2k-q-4}v_4^{q-k}\mathrm{J}^2 & & \mathrm{I}^{2k-q-3}v_4^{q-k+1}\mathrm{J} & & \\
 & & \bigoplus & & \bigoplus & & \bigoplus & & \\
 & & \mathrm{I}^{2k-q-4}v_4^{q-k-2}\mathrm{J}^2w_7 & & \mathrm{I}^{2k-q-4}v_4^{q-k}w_3w_7 & & \mathrm{I}^{2k-q-2}v_4^{q-k}w_7 & & \\
 & & & & \bigoplus & & & & \\
 & & & & \mathrm{I}^{2k-q-3}v_4^{q-k-1}\mathrm{J}w_7 & & & &
\end{array}
$$

In the generic case, $q \in [k+2, 2k-6]$, all the five components are non-zero and $e(q)$ is acyclic; the matrices of the differentials are

$$
a = \begin{pmatrix} \mathrm{id} & * & * \end{pmatrix} \quad b = \begin{pmatrix} * & \mathrm{id} & 0 \\ * & 0 & \mathrm{id} \\ 0 & * & * \\ 0 & * & * \end{pmatrix}
$$

$$
c = \begin{pmatrix} * & 0 & \mathrm{id} & 0 \\ * & * & 0 & \mathrm{id} \\ 0 & 0 & * & * \end{pmatrix} \quad d = \begin{pmatrix} * & * & \mathrm{id} \end{pmatrix}.
$$

For the last values of q the cochain complex $e(q)$ is shorter and still acyclic:

$q = 2k-5$

$$
\begin{array}{ccccccc}
{}^3E_0^{2,4k-11}(k) & \xrightarrow{b} & {}^2E_0^{2,4k-10}(k) & \xrightarrow{c} & {}^1E_0^{2,4k-9}(k) & \xrightarrow{d} & {}^0E_0^{2,4k-8}(k) \\
\| & & \| & & \| & & \| \\
v_4^{k-6}w_3\mathrm{J}w_7 & & v_4^{k-4}w_3\mathrm{J} & & \mathrm{I}v_4^{k-3}w_3 & & \mathrm{I}^3v_4^{k-3} \\
\bigoplus & \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \\ * & * \\ * & * \end{pmatrix} & \bigoplus & \begin{pmatrix} * & 0 & \mathrm{id} & 0 \\ * & * & 0 & \mathrm{id} \\ 0 & 0 & * & * \end{pmatrix} & \bigoplus & \begin{pmatrix} * & * & \mathrm{id} \end{pmatrix} & \\
\mathrm{I}v_4^{k-7}\mathrm{J}^2w_7 & & \mathrm{I}v_4^{k-5}\mathrm{J}^2 & & \mathrm{I}^2v_4^{k-4}\mathrm{J} & & \\
 & & \bigoplus & & \bigoplus & & \\
 & & \mathrm{I}v_4^{k-5}w_3w_7 & & \mathrm{I}^3v_4^{k-5}w_7 & & \\
 & & \bigoplus & & & & \\
 & & \mathrm{I}^2v_4^{k-6}\mathrm{J}w_7 & & & &
\end{array}
$$

$q = 2k-4$

$$
\begin{array}{ccccc}
{}^{2}E_0^{2,4k-8}(k) & \xrightarrow{\quad c \quad} & {}^{1}E_0^{2,4k-7}(k) & \xrightarrow{\quad d \quad} & {}^{0}E_0^{2,4k-6}(k) \\
\parallel & & \parallel & & \parallel \\
v_4^{k-4}w_3w_7 & \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \\ * & * \end{pmatrix} & v_4^{k-2}w_3 & \begin{pmatrix} * & * & \mathrm{id} \end{pmatrix} & \mathrm{I}^2 v_4^{k-2} \\
\bigoplus & & \bigoplus & & \\
\mathrm{I}v_4^{k-5}\mathrm{J}w_7 & & \mathrm{I}v_4^{k-3}\mathrm{J} & & \\
 & & \bigoplus & & \\
 & & \mathrm{J}^2 v_4^{k-4} w_7 & &
\end{array}
$$

$q = 2k-3$

$$
\begin{array}{ccc}
{}^{1}E_0^{2,4k-5}(k) & \xrightarrow{\quad d \quad} & {}^{0}E_0^{2,4k-4}(k) \\
\parallel & & \parallel \\
\mathrm{I}v_4^{k-3}w_7 & (\mathrm{id}) & \mathrm{I}v_4^{k-1}
\end{array}
$$

For the first values of q we obtain non-zero cohomology classes.

$q = k-2$

$$
\begin{array}{ccc}
{}^{1}E_0^{2,2k-3}(k) & \xrightarrow{\quad d \quad} & {}^{0}E_0^{2,2k-2}(k) \\
\parallel & & \parallel \\
\mathrm{I}^{k-2}w_3 & \begin{pmatrix} 0 \\ \mathrm{id} \\ 0 \end{pmatrix} & \mathrm{I}^{k}
\end{array}
$$

and this gives ${}^{0}E_1^{2,2k-2}(k) = \langle v_2^k, \bar{v}_2^k \rangle$.

$q = k-1$

$$
\begin{array}{ccccc}
{}^{2}E_0^{2,2k-2}(k) & \overset{c}{\rightarrowtail} & {}^{1}E_0^{2,2k-1}(k) & \twoheadrightarrow & {}^{0}E_0^{2,2k}(k) \\
\parallel & & \parallel & & \parallel \\
\mathrm{I}^{k-4}w_3\mathrm{J} & & \mathrm{I}^{k-3}v_4w_3 & & \mathrm{I}^{k-1}v_4 \\
 & & \bigoplus & & \\
 & & \mathrm{I}^{k-2}\mathrm{J} & &
\end{array}
$$

Obviously the first differential is injective and the second is surjective, the Euler characteristic is $(2k-6)-(3k-4)+k = -2$ and $\langle v_2^{k-3}\gamma, \bar{v}_2^{k-3}\bar{\gamma} \rangle$ is a complement for the image of c, hence ${}^{1}E_1^{2,2k-1}(k) = \langle v_2^{k-3}\gamma, \bar{v}_2^{k-3}\bar{\gamma} \rangle$.

$q = k$

$$\begin{array}{ccccccc}
{}^3E_0^{2,2k-1}(k) & \overset{b}{\rightarrowtail} & {}^2E_0^{2,2k}(k) & \overset{c}{\longrightarrow} & {}^1E_0^{2,2k+1}(k) & \overset{d}{\twoheadrightarrow} & {}^0E_0^{2,2k+2}(k) \\
\parallel & & \parallel & & \parallel & & \parallel \\
\mathrm{I}^{k-6}w_3\mathrm{J}^2 & & \mathrm{I}^{k-5}v_4w_3\mathrm{J} & & \mathrm{I}^{k-4}v_4^2w_3 & & \mathrm{I}^{k-2}v_4^2 \\
& & \bigoplus & & \bigoplus & & \\
& & \mathrm{I}^{k-4}\mathrm{J}^2 & & \mathrm{I}^{k-3}v_4\mathrm{J} & & \\
& & \bigoplus & & \bigoplus & & \\
& & \mathrm{I}^{k-4}w_3w_7 & & \mathrm{I}^{k-2}w_7 & &
\end{array}$$

Definitely, $\langle v_2^{k-3}\varepsilon, \bar{v}_2^{k-3}\bar{\varepsilon}\rangle \subset \ker(d)$ and its intersection with $\mathrm{Im}(e)$ is 0. The subcomplex $f(k) \subset e(k)$ generated by v_4^2 and w_7 is acyclic ($\alpha v_4^2 \mapsto \alpha w_7$ gives a homotopy $\mathrm{id}_{f(k)} \simeq 0$) and the quotient complex $e(k)/f(k)$ is

$$\mathrm{I}^{k-6}w_3\mathrm{J}^2 \rightarrowtail \mathrm{I}^{k-5}v_4w_3\mathrm{J} \oplus \mathrm{I}^{k-4}\mathrm{J}^2 \to \mathrm{I}^{k-3}v_4\mathrm{J}$$

with dimensions $k-5$, $3k-11$ and $2k-4$ respectively. Therefore ${}^1E_1^{2,2k+1}(k)$ has dimension 2 and it is equal to $\langle v_2^{k-3}\varepsilon, \bar{v}_2^{k-3}\bar{\varepsilon}\rangle$.

$q = k+1$

$$\begin{array}{ccccccc}
{}^3E_0^{2,2k+1}(k) & \rightarrowtail & {}^2E_0^{2,2k+2}(k) & \longrightarrow & {}^1E_0^{2,2k+3}(k) & \twoheadrightarrow & {}^0E_0^{2,4k+4}(k) \\
\parallel & & \parallel & & \parallel & & \parallel \\
\mathrm{I}^{k-7}v_4w_3\mathrm{J}^2 & & \mathrm{I}^{k-6}v_4^2w_3\mathrm{J} & & \mathrm{I}^{k-5}v_4^3w_3 & & \mathrm{I}^{k-3}v_4^3 \\
\bigoplus & & \bigoplus & & \bigoplus & & \\
\mathrm{I}^{k-6}w_3\mathrm{J}w_7 & & \mathrm{I}^{k-5}v_4\mathrm{J}^2 & & \mathrm{I}^{k-4}v_4^2\mathrm{J} & & \\
& & \bigoplus & & \bigoplus & & \\
& & \mathrm{I}^{k-5}v_4w_3w_7 & & \mathrm{I}^{k-3}v_4w_7 & & \\
& & \bigoplus & & & & \\
& & \mathrm{I}^{k-4}\mathrm{J}w_7 & & & &
\end{array}$$

As in the previous case, $\langle v_2^{k-5}\eta, \bar{v}_2^{k-5}\bar{\eta}\rangle \subset \ker(e)$ and its intersection with $\mathrm{Im}(b)$ is 0. The same subcomplex $f(k+1) \subset e(k+1)$ is acyclic and in the quotient subcomplex

$$\mathrm{I}^{k-3}v_4w_3\mathrm{J}^2 \rightarrowtail \mathrm{I}^{k-5}v_4\mathrm{J}^2$$

the dimensions are $k-6$ and $k-4$. Hence ${}^2E_1^{2,2k+2}(k) = \langle v_2^{k-5}\eta, \bar{v}_2^{k-5}\bar{\eta}\rangle$.

In conclusion, the spectral sequences $\{{}^*E_*^{*,*}(k)\}_{k\geq 8}$ degenerate at ${}^*E_1^{*,*}$ with the described eight cohomology classes in ${}^*E_*^{\geq 1,*}(k)$. $\square$

As a consequence of the computation we have the table of the double variables Poincaré polynomials (for each k, the first line contains the coefficients corresponding to $s = 0$, the second line those with $s = 1$ and the third line corresponds to $s = 2$) and Corollary 3, 4 and the proof of Proposition 1.

Table 2

k	0	2	4	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	2	1													
2	1	2	3													
3	1	2	3	2												
					2		2									
4	1	2	3	2		2										
					2		4		3							
5	1	2	3	2		2		2								
					2		4		5		2					
												1				
6	1	2	3	2		2		2		2						
					2		4		5		4		2			
												1		2		
7	1	2	3	2		2		2		2		2				
					2		4		5		4		4		2	
												1		2		2

Corollary 3. *The space* $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ *satisfies the Poincaré polynomial shifted stability condition with range 6, shift 2 and recurrence relation*

$$P_{C_{k+1}(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) = P_{C_k(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) + 2t^{2k+2}[1+s(t+t^2)+s^2t^4] \quad (k \geq 6).$$

Corollary 4. *The space* $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ *satisfies the extended shifted stability condition with range 6 and shift 2. For any* $k \geq 6$ *we have:*

$$P^{[(k-5)2]}_{C_{k+1}(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s) = t^2 P^{[(k-5)2]}_{C_k(\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1)}(t,s).$$

With a different terminology, that of "stable instability," M. Maguire proved in [23] the shifted stability property for the complex projective space $\mathbb{C}\mathrm{P}^3$. Using our method one can obtain M. Maguire's result as Proposition 17 and Corollaries 5, 6

Proposition 17. *The complex projective space* $\mathbb{C}P^3$ *satisfies the spectral sequence shifted stability condition with range 6 and shift 2. More precisely, the nonzero pieces* ${}^*E_\infty^{\geq 1,*}(k)$ *(for* $k \geq 6$*) are:*

$${}^0E_\infty^{2,2k-2}(k), \quad {}^1E_\infty^{2,2k+3}(k), \quad {}^1E_\infty^{2,2k+5}(k) \quad \textit{and} \quad {}^2E_\infty^{2,2k+10}(k)$$

and all these spaces have dimension one.

The next table contains the double variable Poincaré polynomials of the first configuration spaces $C_k(\mathbb{C}\mathrm{P}^3)$ (we use the convention of Table 2):

Table 3

k	0	2	4	6	8	10	11	12	13	14	15	17	19	21	24	26
1	1	1	1	1												
2	1	1	2	1	1											
3	1	1	2	2	1											
							1		1		1					
4	1	1	2	2	2											
							1		2		2	1	1			
5	1	1	2	2	2	1										
							1		2		3	2	1			
6	1	1	2	2	2	1		1								
							1		2		3	3	2			
															1	
7	1	1	2	2	2	1		1		1						
							1		2		3	3	3	1		
															1	1

Corollary 5. *The space $\mathbb{C}P^3$ satisfies the Poincaré polynomial shifted stability condition with range 5, shift 2 and recurrence relation*

$$P_{C_{k+1}(\mathbb{C}\mathrm{P}^3)}(t,s) = P_{C_k(\mathbb{C}\mathrm{P}^3)}(t,s) + t^{2k+2}[1 + s(t^5 + t^7) + s^2t^{12}] \quad (k \geq 5).$$

Corollary 6. *The spaces $\mathbb{C}P^3$ satisfies the extended shifted stability condition with range 6 and shift 2. For any $k \geq 6$ we have:*

$$P^{[(k-5)2]}_{C_{k+1}(\mathbb{C}\mathrm{P}^3)}(t,s) = t^2 P^{[(k-5)2]}_{C_k(\mathbb{C}\mathrm{P}^3)}(t,s).$$

The complete details of the proofs of these and other results for unordered configuration spaces of $\mathbb{C}\mathrm{P}^n$ will be given in [4].

Proof of Proposition 2 Obvious from Corollary 6. □

Remark 8. In these two examples, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^3$, the sequence of odd Betti numbers is unimodal for any k. This is not true for the sequence of even Betti numbers, but the sequences of double Betti numbers, for each $s \in \{0, 1, 2\}$, are unimodal too.

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