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On the weak ideal property and some related properties

by
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Abstract

Let (P) be any of the following properties: the weak ideal property, the topological dimension zero, the combination of pure infiniteness and the ideal property, residual (SP), pure infiniteness, strong pure infiniteness, provided that the zero C*-algebra is included, and D-stability for a separable unital strongly self-absorbing C*-algebra D. Let A be a separable unital C*-algebra with unit 1_A, and assume that there exist non-zero projections e_i, $1 \leqq i \leqq n$, in A such that $\sum_{i=1}^{n} e_i = 1_A$. We show that A has (P) $\Leftrightarrow$ e_iAe_i has (P) for every $1 \leqq i \leqq n \Leftrightarrow$ $c_0(A)$ has (P) (in fact, we prove a much more general result). We also show that, somehow surprisingly, for two large classes $\mathcal{C}$ of non-zero C*-algebras, if $A, B \in \mathcal{C}$, then the fact that $A \otimes B$ has the weak ideal property implies (or, is equivalent to) the fact that A and B have the weak ideal property. We prove that one of these two results still holds if we replace the weak ideal property by some related properties.

Key Words: Weak ideal property, topological dimension zero, ideal property, tensor product C*-algebra, type I C*-algebra.
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1 Introduction

The weak ideal property was introduced in [17].

Definition 1. *(Definition 8.1 of [17]) Let A be a C^*-algebra. We say that A has the* weak ideal property *if whenever $I \subseteq J \subseteq \mathcal{K} \otimes A$ are ideals in $\mathcal{K} \otimes A$ such that $I \neq J$, then J/I contains a non-zero projection.*

The topological dimension zero, was introduced in [7], and it is a non-Hausdorff version of total disconnectedness of the primitive spectrum of a C*-algebra. A C*-algebra A has topological dimension zero if $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets (Remark 2.5(vi) of [7]). Note that a separable purely infinite C*-algebra A has real rank zero if and only if A has topological dimension zero and it satisfies a certain K-theoretical condition (see Theorem 4.2 of [20].)

The weak ideal property and the topological dimension zero have good permanence properties (see, e.g., [16], [17], [18], and [7]) and are closely related. They are also close to the older concept of ideal property (a C*-algebra A has the ideal property if any ideal of A is generated, as an ideal, by its projections), but these properties are not identical (see [18]). However, it was shown in [18] that, in many interesting cases, these three concepts coincide. It is known that the ideal property $\Rightarrow$ the weak ideal property $\Rightarrow$ the topological dimension zero (the first implication is obvious, the second one is Theorem 2.8 of [18].) A good understanding of the weak ideal property and of other similar properties (like, e.g., the topological dimension zero, the ideal property, residual (SP) (see Definition 7.1 of [17]), and, the combination of pure infiniteness (see Definition 4.1 of [9]) and the ideal property) is important in identifying and studying regularity properties for non-simple C*-algebras, in an attempt to extend Elliott's Classification Program beyond the class of simple C*-algebras.

In this paper we continue our investigation of the weak ideal property (and also, of some related properties) (see [16], [17], [18], [19], [14], and [15]), trying to better understand its behavior in tensor products. Our motivation comes, in part, from Question 4.12 of [18]. Extending a result in [18], we prove, in particular, that the ideal property, the weak ideal property, and the topological dimension zero coincide in the case of type I C*-algebras. We also characterize the weak ideal property and several other important properties, for (a large class of) unital separable C*-algebras.

It was shown in Remark 1.9 of [15] that there are non-zero separable C*-algebras A and B with A or B exact, such that $A \otimes B$ has the weak ideal property but A or B does not have the weak ideal property. However, in this paper we show that, somehow surprisingly, for two large classes $\mathcal{C}$ of non-zero C*-algebras, if $A, B \in \mathcal{C}$, then the fact that $A \otimes B$ has the weak ideal property implies (or, is equivalent to) the fact that A and B have the weak ideal property. We prove that one of these two results still holds if we replace the weak ideal property by several other related properties. We show that if A and B are non-zero C*-algebras which are not antiliminary and such that $A \otimes B$ has the weak ideal property, then A and B have the weak ideal property (see Theorem 2.) Let (P) be any of the following properties: being an AF algebra, the real rank zero, the projection property, the ideal property, the weak ideal property, and the topological dimension zero. Let A and B be non-zero type I C*-algebras. Then $A \otimes B$ has (P) if and only if A and B have (P) (see Theorem 3.) We also answer some natural questions involving the weak ideal property and some ideals constructed in [15], the center of a C*-algebra or relative commutants in tensor products (see Proposition 1, Proposition 2, the paragraph after Question 1 and Corollary 1.)

We also prove a characterization of the weak ideal property (and of several other properties) for (a large class of) unital separable C*-algebras. Let A be a separable unital C*-algebra with unit 1_A, and assume that there are non-zero projections e_i, $1 \leqq i \leqq n$, in A such that $\sum_{i=1}^{n} e_i = 1_A$. We show that A has the weak ideal property $\Leftrightarrow$ $e_i A e_i$ has the weak ideal property for every $1 \leqq i \leqq n$ $\Leftrightarrow$ $c_0(A)$ has the weak ideal property (see Corollary 2 for a more general result.) (In fact, we prove in Theorem 4 that this holds for properties which are much more general than the weak ideal property.)

The definitions of the type I C*-algebras and of the antiliminary C*-algebras, together with some interesting results involving these concepts, could be found, e.g., in Section 6.1 of Chapter 6 of [21]. The definition of the real rank zero could be found in Section 1 of [6].

Ideals in C*-algebras are assumed to be closed and two sided. If A is a C*-algebra, then $\mathcal{P}(A)$ will denote the set of all projections of A ($\mathcal{P}(A) := \{p \in A \colon p = p^* = p^2\}$), $\mathrm{Prim}(A)$ will denote the primitive spectrum of A, $\mathcal{Z}(A)$ will denote the center of A, and $I \lhd A$ will denote the fact that I is an ideal of A. If A and B are C*-algebras, then $A \otimes B$ denotes the minimal tensor product of A with B. The C*-algebra of all linear bounded operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$, and the C*-algebra of all compact linear bounded operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$. When $\mathcal{H}$ is a separable infinite dimensional Hilbert space, we denote $\mathcal{K} := \mathcal{K}(\mathcal{H})$.

2 The results

We begin by recalling some notations from [15] (see also [1]).

Notation 1. *(see Notation 2.1 of [15]; see also Lemma 2.13(i) of [1]) Let A and B be C*-algebras and let $I \lhd A \otimes B$. Denote $I_A := \{a \in A : a \otimes B \subseteq I\}$ and $I_B := \{b \in B : A \otimes b \subseteq I\}$. When I is a prime ideal of $A \otimes B$, this notation was introduced in Lemma 2.13(i) of [1].*

Notation 2. *(see Notation 2.2 of [15]) Let A and B be non-zero C*-algebras and let $I \lhd A \otimes B$. Assume first that $I \neq 0$. Denote by $I(A)$ the ideal of A generated by all $a \in A$ with the property that there is $b \in B$ such that $a \otimes b$ is a non-zero element of I. Similarly, denote by $I(B)$ the ideal of B generated by all $b \in B$ with the property that there is $a \in A$ such that $a \otimes b$ is a non-zero element of I. If $I = 0$ is the zero ideal of $A \otimes B$, denote $I(A) := 0$ and $I(B) := 0$.*

Let A and B be non-zero C*-algebras such that A or B is exact. Let $0 \neq I \lhd A \otimes B$ and assume that I has the weak ideal property. Do I_A and I_B have the weak ideal property? Do $I(A)$ and $I(B)$ have the weak ideal property? The next two propositions show that these two natural questions have negative answers, even in the separable case.

Proposition 1. *There are separable non-zero C*-algebras A and B such that A or B is exact, and there is an ideal $0 \neq I \lhd A \otimes B$ such that I has the weak ideal property but I_A or I_B does not have the weak ideal property.*

Proof. By the proof of Remark 1.9 of [15] there are non-zero separable C*-algebras A and B such that A or B is exact, and such that $A \otimes B$ and A have the weak ideal property but B does not have the weak ideal property. Let $I := A \otimes B$. Then $0 \neq I \lhd A \otimes B$, I has the weak ideal property but $I_B = B$ does not have the weak ideal property. This ends the proof. □

Proposition 2. *There are separable non-zero C*-algebras A and B such that A or B is exact, and there is an ideal $0 \neq I \lhd A \otimes B$ such that I has the weak ideal property but $I(A)$ or $I(B)$ does not have the weak ideal property.*

Proof. Let A, B and I be as in the proof of Proposition 1. Note that $I_B = B$ implies that $I(B) = B$ (since $I_B \subseteq I(B) \subseteq B$), and hence $I(B) = B$ does not have the weak ideal property. The proof is over. □

Remark 1. *(1) If we replace the weak ideal property by the ideal property in Proposition 1 and Proposition 2, the results still hold. The proofs are similar with the ones of Proposition 1 and Proposition 2 (see the proof of Remark 1.9 of [15].)*

(2) Let A and B be non-zero C-algebras such that A or B is exact. If for every ideal $I \lhd A \otimes B$ that has the weak ideal property we have that $I(A) \otimes I(B)$ has the weak ideal property, then, if we denote by J the largest ideal of $A \otimes B$ which has the weak ideal*

property (whose existence is proved in [19]), we have that $J = J(A) \otimes J(B)$. Indeed, note first that $J \lhd A \otimes B$ and that J has the weak ideal property. We know from Theorem 2.3(2) of [15] that $J \subseteq J(A) \otimes J(B)$. Since also $J(A) \otimes J(B)$ has the weak ideal property (by the hypothesis), the definition of J implies that $J = J(A) \otimes J(B)$.

We next recall that an AF algebra is a C*-algebra which can be locally approximated by finite dimensional sub-C*-algebras (see [2].)

Question 1. *Let A be a C*-algebra. If A has the weak ideal property, does $\mathcal{Z}(A)$ have the weak ideal property?*

The answer to this question is "no". Indeed, let B be a commutative separable C*-algebra such that Prim(B) is not totally disconnected, i.e., such that B does not have the weak ideal property (e.g., we may take $B = C([0,1])$.) On the other hand, by the corollary in Section 4 of [3], there exists a separable AF algebra A such that $\mathcal{Z}(A) = B$. But (separable) AF algebras have the weak ideal property (e.g., since they have the ideal property) and hence, in particular, A has the weak ideal property while $\mathcal{Z}(A) = B$ does not have the weak ideal property.

Let A and B be C*-algebras. If B is unital with unit 1, we define the relative commutant of A in $A \otimes B$ by $A' \cap (A \otimes B) := \{x \in A \otimes B : xa = ax, \forall\ a \in A \otimes \mathbb{C}1\}$. If A is unital, we define $B' \cap (A \otimes B)$ in a similar way.

Proposition 3. *Let A and B be C*-algebras and assume that B is unital.*

(1) If $\mathcal{Z}(A)$ and B have the weak ideal property, then $A' \cap (A \otimes B)$ has the weak ideal property.

(2) If in addition A and B are separable, then $A' \cap (A \otimes B)$ has the weak ideal property $\Leftrightarrow$ $\mathcal{Z}(A)$ and B have the weak ideal property.

Proof. We first prove (1).

By Theorem 1 of [8] we have that $A' \cap (A \otimes B) = \mathcal{Z}(A) \otimes B$. Since also Prim$(\mathcal{Z}(A))$ is totally disconnected (since $\mathcal{Z}(A)$ has the weak ideal property) and B has the weak ideal property, Theorem 5.14(iii) of [18] implies that $\mathcal{Z}(A) \otimes B = A' \cap (A \otimes B)$ has the weak ideal property.

We now prove (2).

The equivalence can be proved combining the relation $A' \cap (A \otimes B) = \mathcal{Z}(A) \otimes B$ (mentioned in the proof of (1)) with Theorem 5.17(iii) of [18] (we also used that $B \neq 0$, A and B are separable, Prim$(\mathcal{Z}(A))$ is second countable and the fact that the commutative C*-algebra $\mathcal{Z}(A)$ has the weak ideal property if and only if Prim$(\mathcal{Z}(A))$ is totally disconnected.)
□

Corollary 1. *Let A and B be unital separable C*-algebras. The following are equivalent:*

(1) $A' \cap (A \otimes B)$ and $B' \cap (A \otimes B)$ have the weak ideal property.

(2) A, B and $\mathcal{Z}(A \otimes B)$ have the weak ideal property.

Proof. By Corollary 1 of [8] it follows that $\mathcal{Z}(A \otimes B) = \mathcal{Z}(A) \otimes \mathcal{Z}(B)$. Also, it is an easy topology exercise to see that the non-zero commutative C*-algebra $\mathcal{Z}(A) \otimes \mathcal{Z}(B)$ has the weak ideal property if and only if the non-zero commutative C*-algebras $\mathcal{Z}(A)$ and $\mathcal{Z}(B)$ have the weak ideal property (since for any commutative C*-algebra D, we have that D has the weak ideal property if and only if $\mathrm{Prim}(D)$ is totally disconnected.) Now, the proof ends using also Proposition 3(2). ∎

The proof of the next result is inspired by the proof of Theorem 2.24 of [12].

Theorem 1. *Let A be a C*-algebra and let B be a non-zero C*-algebra which is not antiliminary. Suppose that $A \otimes B$ has the weak ideal property. Then A has the weak ideal property.*

Proof. By Proposition 6.2.7 of [21], B has a largest ideal I of type I and B/I is antiliminary. Since $B \neq 0$ and B is not antiliminary, it follows that $I \neq 0$. Since $A \otimes I \lhd A \otimes B$, $A \otimes B$ has the weak ideal property and the weak ideal property passes to ideals by Theorem 8.5(5) of [17], it follows that $A \otimes I$ has the weak ideal property. Now, if $\pi \colon I \to \mathcal{B}(\mathcal{H})$ is a (non-zero) irreducible representation of I on a Hilbert space $\mathcal{H}$, then Theorem 6.1.5(i) of [21] implies that:

$$\mathcal{K}(\mathcal{H}) \subseteq \pi(I). \tag{2.1}$$

Since $A \otimes \pi(I)$ is isomorphic to a quotient of $A \otimes I$, $A \otimes I$ has the weak ideal property and the weak ideal property is preserved under isomorphism and passes to quotients by Theorem 8.5.(5) of [17], it follows that $A \otimes \pi(I)$ has the weak ideal property. Using this, the fact that $A \otimes \mathcal{K}(\mathcal{H}) \lhd A \otimes \pi(I)$ (see (2.1)) and the fact that the weak ideal property passes to ideals (by Theorem 8.5(5) of [17]), it follows that $A \otimes \mathcal{K}(\mathcal{H})$ has the weak ideal property. Let $0 \neq e$ be a minimai projection of $\mathcal{K}(\mathcal{H})$. Since $\mathcal{K}(\mathcal{H})$ has real rank zero (as an inductive limit of matrix algebras, which have real rank zero), it follows by Corollary 2.8 of [6] that $e\mathcal{K}(\mathcal{H})e$ has real rank zero. In particular, it follows that $e\mathcal{K}(\mathcal{H})e$ is the closed linear span of its projections. But, since $0 \neq e$ is a minimal projection of $\mathcal{K}(\mathcal{H})$, the only projections of $e\mathcal{K}(\mathcal{H})e$ are e and 0. This implies that $e\mathcal{K}(\mathcal{H})e = \mathbb{C}e \cong \mathbb{C}$. Since $e\mathcal{K}(\mathcal{H})e$ is a hereditary subalgebra of $\mathcal{K}(\mathcal{H})$, it follows that $A \otimes e\mathcal{K}(\mathcal{H})e$ is a hereditary subalgebra of $A \otimes \mathcal{K}(\mathcal{H})$, which was proved to have the weak ideal property. Using this, the fact that, by Theorem 8.5(3) of [17], the weak ideal property passes to hereditary subalgebras, and the fact that the weak ideal property is preserved under isomorphism, it follows that A ($\cong A \otimes \mathbb{C} \cong A \otimes e\mathcal{K}(\mathcal{H})e$) has the weak ideal property. ∎

Theorem 2. *Let A and B be non-zero C*-algebras which are not antiliminary (in particular, A and B could be type I C*-algebras.) Suppose that $A \otimes B$ has the weak ideal property. Then A and B have the weak ideal property.*

Proof. The proof uses Theorem 1 and the fact that $A \otimes B \cong B \otimes A$. ∎

The next result uses the above definition of AF algebras (given before Question 1) and removes the separability condition in Proposition 7.14 of [18] (using essentially the same proof as there; see also Remark 2.12 of [13].)

Proposition 4. *Let A be a type I C*-algebra. Then the following are equivalent:*

(1) A has topological dimension zero.

(2) A has the weak ideal property.

(3) A has the ideal property.

(4) A has the projection property (every ideal in A has an increasing approximate identity consisting of projections; Definition 1 of [11]).

(5) A has real rank zero.

(6) A is an AF algebra.

Proof. It is clear that any condition on the above list implies the previous one. We need only to show that (1) implies (6). If A has topological dimension zero, $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets. Then, the proof ends using the fact that the theorem in Section 7 of [4] remains true without the separability condition, but using the above definition of the AF algebras, where separability is not required (see the comment at the end of the proof of the theorem in Section 7 of [4].) ∎

Theorem 3. *Let (P) be any of the following properties: being an AF algebra, the real rank zero, the projection property, the ideal property, the weak ideal property, and the topological dimension zero. Let A and B be non-zero type I C*-algebras. The following are equivalent:*

(1) A and B have (P).

(2) $A \otimes B$ has (P).

Proof. Note that since A and B are type I C*-algebras, it follows that $A \otimes B$ is a type I C*-algebra (see [24].)

We first prove that (1) $\Rightarrow$ (2).

Assume that A and B have (P). Then, by Proposition 4 it follows that A and B are AF algebras, and hence A and B are locally approximated by the families of their finite dimensional sub-C*-algebras. Therefore, $A \otimes B$ is locally approximated by a family of finite dimensional sub-C*-algebras (namely, the tensor product of the families of the finite dimensional sub-C*-algebras of A and B), i.e., $A \otimes B$ is an AF algebra. Then (using again Proposition 4), it easily follows that $A \otimes B$ has (P).

We now prove that (2) $\Rightarrow$ (1).

Assume that $A \otimes B$ has (P). Then, Proposition 4 implies that $A \otimes B$ has the weak ideal property. Using now Theorem 2, we deduce that A and B have the weak ideal property. Finally, using again Proposition 4, it follows that A and B have (P). ∎

We recall a definition from [19].

Definition 2. *(see Definition 1.1 of [19]) Let (P) be a property that a C*-algebra may or may not have (such as real rank zero, satisfying the Universal Coefficient Theorem, or the weak ideal property.) Then, we make the following definitions:*

(1) We say that (P) *is* isomorphism invariant *if whenever* A *and* B *are C*-algebras,* A *has the property* (P)*, and* $B \cong A$*, then* B *has the property* (P)*.*

(2) We say that (P) *is* (separably) stable *if whenever* A *and* B *are (separable) C*-algebras,* A *has the property* (P)*, and* $\mathcal{K} \otimes B \cong \mathcal{K} \otimes A$*, then* B *has the property* (P)*.*

(3) We say that (P) (separably) passes to ideals *if whenever* A *is a (separable) C*-algebra with* (P) *and* $I \lhd A$*, then* I *has the property* (P)*.*

(4) We say that (P) (separably) admits largest ideals *if for every (separable) C*-algebra* A *there is a largest ideal in* A *which has the property* (P)*.*

We give now a definition, which together with Definition 2, will be used in Theorem 4:

Definition 3. *Let* (P) *be a property that a C*-algebra may or may not have.*

(1) We say that (P) *is* separably isomorphism invariant *if whenever* A *and* B *are separable C*-algebras,* A *has* (P)*, and* $B \cong A$*, then* B *has* (P)*.*

(2) We say that (P) separably passes to finite direct sums *if whenever* $n \in \mathbb{N}$ *and* $A_1, A_2, \ldots, A_n$ *are separable C*-algebras with* (P)*, then* $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *has* (P)*.*

(3) We say that (P) separably passes to quotients *if whenever* A *is a separable C*-algebra with* (P) *and* $I \lhd A$*, then* A/I *has* (P)*.*

For each C*-algebra A let $l^\infty(A)$ denote the C*-algebra of all bounded functions from $\mathbb{N}$ to A with entry-wise algebraic operations, and let $c_0(A)$ be the ideal in $l^\infty(A)$ consisting of those sequences (a_n) for which $\lim \|a_n\| = 0$.

We now state the main result of this paper:

Theorem 4. *Let* (P) *be a property that a C*-algebra may or may not have. Assume that* (P) *is separably stable, separably passes to ideals, separably admits largest ideals, separably passes to finite direct sums, and separably passes to quotients (see Definition 2 and Definition 3.) Let* A *be a separable unital C*-algebra with unit* 1_A*, and assume that there exist* $0 \neq e_i \in \mathcal{P}(A)$*,* $1 \leqq i \leqq n$*, such that* $\sum_{i=1}^n e_i = 1_A$*. The following are equivalent:*

(1) A *has* (P)*.*

(2) $e_i A e_i$ *has* (P) *for every* $1 \leqq i \leqq n$*.*

(3) $c_0(A)$ *has* (P)*.*

Proof. Let B be the sub-C*-algebra of A generated (as a C*-algebra) by $e_i, 1 \leqq i \leqq n$. Note that since $e_i, \sum_{j=1}^n e_j \in \mathcal{P}(A)$ for every $1 \leqq i \leqq n$, it follows that $e_i e_j = 0$ $(= e_j e_i)$, for every $i \neq j, 1 \leqq i, j \leqq n$. Hence:

$$B = \mathbb{C}e_1 + \mathbb{C}e_2 + \cdots + \mathbb{C}e_n.$$

Denote by G the unitary group of the C*-algebra B, with the induced norm topology. Then $G = \mathbb{T}e_1 + \mathbb{T}e_2 + \cdots + \mathbb{T}e_n$ $(\cong \mathbb{T}^n)$ is a second countable compact abelian group (here, of course, $\mathbb{T}$ is the torus, i.e., $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$.) We define now an action $\alpha \colon G \to \mathrm{Aut}(A)$ of G on A by:

$$\alpha_g(a) := gag^*, \ \forall\, g \in G, \ \forall\, a \in A.$$

Then, by Theorem 3.2 of [19] it follows that:

$$A^{\alpha} \ has \ (P) \Leftrightarrow C^*(G,A,\alpha) \ has \ (P). \tag{2.2}$$

But:

$$A^{\alpha} = A \cap B' = \sum_{i=1}^{n} e_i A e_i \cong \oplus_{i=1}^{n} e_i A e_i \tag{2.3}$$

where, of course, $A \cap B'$ denotes the relative commutant of B in A. To prove the second equality in (2.3), fix an arbitrary $x \in A$. Then $x = \sum_{k,l=1}^{n} e_k x e_l$ and hence: $x \in A \cap B' \Leftrightarrow (\sum_{k,l=1}^{n} e_k x e_l) e_i = e_i (\sum_{k,l=1}^{n} e_k x e_l), \forall\ 1 \leqq i \leqq n \Leftrightarrow \sum_{k=1}^{n} e_k x e_i = \sum_{l=1}^{n} e_i x e_l, \forall\ 1 \leqq i \leqq n \Leftrightarrow (e_k x e_i = 0, \forall\ k \neq i, 1 \leqq k, i \leqq n)$ and $(e_i x e_l = 0, \forall\ l \neq i, 1 \leqq l, i \leqq n) \Leftrightarrow x = \sum_{i=1}^{n} e_i x e_i$.

Observe that (2.3), the fact that (P) separably passes to ideals, the fact that (P) is separably isomorphism invariant (since (P) is separably stable) and the fact that (P) separably passes to finite direct sums imply that:

$$A^{\alpha} \ has \ (P) \Leftrightarrow \oplus_{i=1}^{n} e_i A e_i \ has \ (P) \Leftrightarrow e_i A e_i \ has \ (P),\ \forall\ 1 \leqq i \leqq n. \tag{2.4}$$

On the other hand, we have:

$$C^*(G,A,\alpha) \cong C^*(G) \otimes A \cong C_0(\hat{G}, A) \cong c_0(A) \tag{2.5}$$

since α is an inner action of G on A, G is abelian and compact and hence $\hat{G}$ is discrete, $\hat{G} \cong (\mathbb{T}^n)\hat{} \cong \mathbb{Z}^n$ (isomorphisms of topological groups), and hence the infinite countable discrete topological spaces $\hat{G}$ and $\mathbb{N}$ are homeomorphic.

Using (2.2), (2.4), (2.5), and the fact that (P) is separably isomorphism invariant (since (P) is separably stable), we obtain that:

$$e_i A e_i \ has \ (P),\ \forall\ 1 \leqq i \leqq n \Leftrightarrow c_0(A) \ has \ (P)$$

which means that we proved the equivalence (2) $\Leftrightarrow$ (3).

We now prove that (1) $\Rightarrow$ (2).

Assume that A has (P). Fix some arbitrary $1 \leq i \leq n$. Let I be the ideal of A generated by $e_i A e_i$. Since A is separable, Theorem 2.8 of [5] implies that $e_i A e_i$ is stably isomorphic to I. Since A has (P), $I \lhd A$, (P) separably passes to ideals and (P) is separably stable, it follows that $e_i A e_i$ has (P). Since $1 \leq i \leq n$ was arbitrary, this ends the proof of (1) $\Rightarrow$ (2).

We finally prove that (3) $\Rightarrow$ (1).

Assume that $c_0(A)$ has (P). Fix some $k \in \mathbb{N}$ and denote by $ev\colon c_0(A) \to A$ the evaluation at k. Then ev is a surjective $*$-homomorphism, and hence A is isomorphic to a quotient of $c_0(A)$. Since (P) is separably isomorphism invariant (since (P) is separably stable) and (P) separably passes to quotients, it follows that A has (P). This ends the proof of (3) $\Rightarrow$ (1).

The proof of the theorem is over. ∎

Remark 2. *(1) If $n = 1$ in the hypothesis of Theorem 4 (case in which the corresponding condition for the projections e_i is automatically satisfied, because A is unital with unit 1_A), then Theorem 4(1) and Theorem 4(2) become identical, and the requirements that (P) separably passes to finite direct sums and separably passes to quotients need not be mentioned in Theorem 4.*

(2) *We recall from Definition 5.1 of [17] that a class $\mathcal{C}$ of C*-algebras is* upwards directed *if whenever A is a C*-algebra which contains a subalgebra isomorphic to an algebra in $\mathcal{C}$, then $A \in \mathcal{C}$, and from Definition 5.2 of [17] that a C*-algebra A is* residually hereditarily *in such a class $\mathcal{C}$ if, for every $I \lhd A$ and every non-zero hereditary subalgebra $B \subseteq A/I$, we have $B \in \mathcal{C}$. Note that the weak ideal property, residual (SP), the combination of pure infiniteness for non-simple C*-algebras and the ideal property, and, for separable C*-algebras, the topological dimension zero are examples of properties of this form (see [17] and [18].)*

Corollary 2. *Let (P) be any of the following properties:*

(1) *Being residually hereditarily in a fixed upwards directed class of C*-algebras (e.g., the weak ideal property, the combination of pure infiniteness for non-simple C*-algebras and the ideal property, and residual (SP) (see Remark 2 and [17]).)*

(2) *Topological dimension zero.*

(3) *D-stability for a separable unital strongly self-absorbing C*-algebra D (see [22] and [23].)*

(4) *Pure infiniteness for non-simple C*-algebras.*

(5) *Strong pure infiniteness for non-simple C*-algebras, provided that the zero C*-algebra is included (see [10].)*

Let A be a separable unital C-algebra with unit 1_A, and assume that there exist $0 \neq e_i \in \mathcal{P}(A)$, $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} e_i = 1_A$. The following are equivalent:*

(a) *A has (P).*

(b) *$e_i A e_i$ has (P) for every $1 \leqq i \leqq n$.*

(c) *$c_0(A)$ has (P).*

Proof. The proof follows from Theorem 4, using Proposition 2.1 of [19], Remark 2.2 of [19], Lemma 2.9 of [19], Remark 2(2), [23], Lemma 5.7 of [17], Proposition 5.8 of [17], Corollary 3.3 of [22], Theorem 4.19 of [9], and Proposition 5.11(i) of [10]. (The other verifications follow immediately from definitions.) □

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