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**On vertex 3-colourable disc triangulations**
by
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**Abstract**

A vertex 3-colouring theorem for a suitably triangulated 2-disc, rooting back in the late years of the nineteenth century, is considered and discussed in careful detail. A whole string of corollaries are derived, two particular classes of triangulations are singled out, and the boundary even vertex set is enquired into in further detail.

**Key Words**: Vertex 3-colourable disc triangulation.
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# 1 Introduction

The problem of characterising vertex 3-colourable planar triangulations dates back to [8]. Roughly speaking, the vertices of a triangulation of the 2-sphere are 3-colourable if and only if they are all even. The first complete proof of this fact, by induction on the number of vertices, seems to be that in [5]. Since then, several different proofs came out of various settings [3, 12, 16, 18, 19], and the problem triggered a whole string of related topics [3, 6, 9, 10, 15, 16].

Our purpose here is to prove and discuss in careful detail a version of the 3-colour theorem for suitable disc triangulations. Section 2 introduces the setting and includes a proof by induction on the number of triangles. Several corollaries are then derived, alternative proofs are included, and related facts and examples are considered.

Sections 3 and 4 single out two classes of triangulations and describe some of their particular features. Although most statements are corollaries to the 3-colour theorem, alternative proofs are again included. An outline of an alternative proof of the 3-colour theorem is sketched at the end of Section 3.

Finally, Section 5 deals with the boundary even vertex set of a suitably triangulated disc. The combinatorial location of such vertices along the boundary is enquired into.

# 2 Suitable Triangulations

For the topological and graph theoretic notions and results in the first few paragraphs below, standard references include [1, 2, 13].

A *triangulation* of the closed 2-disc consists of a simplicial complex $K$ of dimension 2 and a homeomorphism of the polyhedron $|K|$ onto the disc. A closed 2-disc with a triangulation will be referred to as a *triangulated* 2-disc. The particular homeomorphism involved is irrelevant here, and so $K$ alone will loosely be referred to as a triangulation of the disc. The polyhedron $|K|$ will always be assumed embedded in the Euclidean plane or, if need be, in the 2-sphere $S^2$, both with an orientation fixed once and for all. The Jordan curve theorem, the Schoenflies theorem and the $\Theta$-space separation theorem are assumed and referred to more or less explicitly, more or less loosely.

All vertices and all edges on the boundary $\partial|K|$ of $|K|$ form a subcomplex $\partial K$ of $K$, loosely referred to as the *boundary* of $K$; it is, of course, the outer cycle of the 1-skeleton $K^1$, and $|\partial K| = \partial|K|$ is a simple closed curve in the plane or in $S^2$. Notice that $\partial K$ has at least three vertices. A *chord* is an edge in $K \setminus \partial K$ with both end points in $\partial K$.

A vertex of $K$ is *interior* if it lies in the open disc; otherwise, it is a *boundary* vertex. The *parity* of a vertex of $K$ is, of course, the parity of its degree in the 1-skeleton $K^1$ (the number of edges having an end point at that vertex). Similarly, the *parity* of an edge path in $K^1$ is the parity of its combinatorial length (the number of edges the path consists of).

A set $S$ of vertices of $K$ is *$k$-colourable* if each vertex in $S$ may be assigned one of $k$ colours so that no two $K$-neighbours in $S$ share colour; alternatively, but equivalently, the subgraph $S$ induces in $K^1$ is $k$-colourable, i. e., there exists a function $f$ from $S$ to a $k$-element set such that two $K^1$-neighbours in $S$ have distinct images under $f$.

**1. Theorem.** *The vertices of a triangulation of the* 2*-disc are* 3*-colourable if and only if each interior vertex is even, in which case the colouring is unique up to a colour permutation.*

Necessity is clear: Simply consider the *simplicial neighbourhood* of an interior vertex, i. e., the set of all simplices of the triangulation that contain that vertex. If the latter were odd, then any valid vertex colouring of its simplicial neighbourhood would require (at least) four colours.

For convenience, a triangulation satisfying the condition in the above theorem will be referred to as a *suitable* triangulation, and the polyhedron it triangulates — as a *suitably* triangulated polyhedron. A straightforward degree argument shows that a suitable triangulation has an even number of (boundary) odd vertices.

A suitable triangulation whose boundary vertices are all odd, respectively even, will be referred to as an *ideal*, respectively *even*, triangulation; and the polyhedron it triangulates — as an *ideally*, respectively *evenly*, triangulated polyhedron. Such triangulations will be considered in Sections 3 and 4, respectively.

Theorem 2.1 may fail for a suitable triangulation whose polyhedron is not a disc. For instance, subdivide an annulus into three interior disjoint quadrangles by means of three 'radial' chords, each of which has an end point on the inner circle and the other on the outer circle. Drawing both diagonals in each quadrangle then yields a suitable triangulation whose vertices are not 3-colourable; they are, of course, 4-colourable.

Unless otherwise stated, only triangulated 2-discs are considered in the sequel. One of the simplest examples of a suitable triangulation is the *even spoke umbrella*: It is obtained by joining an interior point 'radially' to each of an even number of pairwise distinct points on the boundary of the disc. This is a chordless ideal triangulation. More elaborate examples of this kind will be considered and discussed later on. The 1-skeleton of an umbrella is a *wheel*.

A less trivial example is the triangulation obtained by barycentric subdivision of a triangulation of the disc.

Another example is the standard 'diagonal' triangulation of a polygon — it has no interior vertices, so it is suitable; all edges, but those on the boundary, are chords. It can be shown that such a triangulation always has at least two even vertices on the boundary; moreover, only even sided polygons have a diagonal triangulation with exactly two even vertices on the boundary. For convenience (and for obvious reasons), if all diagonals emanate from one single vertex $a$, the triangulation will be referred to as a *fan* centred at $a$; it is the *cone* of apex $a$ on the chain of edges not containing $a$.

To complete the proof of Theorem 2.1, we show that suitable triangulations of the 2-disc are 3-colourable in a rather special way.

**2. Theorem.** *Any vertex* 3*-colouring of an arbitrarily chosen triangle of a suitable triangulation of the* 2*-disc extends uniquely to a* 3*-colouring of all vertices. Consequently, overall vertex* 3*-colourings differ from one another by a colour permutation.*

**Proof.** Let $K$ be a suitable triangulation of the 2-disc, and let $\Delta$ be the triangle of $K$ whose vertex 3-colouring $f_\Delta$ is to be extended.

Induct on the number of triangles in $K$. The base case is clear, so let $K$ contain more than one triangle.

If $K$ has chords, let $ab$ be one such, and notice that it 'splits' $K$ into two 'interior disjoint' suitable non-empty subtriangulations, $L$ and $M$, whose boundaries share vertices $a$ and $b$ and the edge $ab$ alone (the two are glued along the chord). Both $L$ and $M$ have fewer triangles than $K$, and exactly one of them, say $L$, contains $\Delta$. By the induction hypothesis, $f_\Delta$ extends to a vertex 3-colouring in $L$. Now, let $\Delta'$ be the unique triangle $ab$ faces in $M$. The colours $a$ and $b$ inherit from $L$ then fix a vertex 3-colouring in $\Delta'$. Refer again to the induction hypothesis to extend this vertex 3-colouring in $\Delta'$ to one in $M$, and complete thereby a vertex 3-colouring $f$ in $K$ that extends $f_\Delta$.

To prove uniqueness, let $g$ be any extension of $f_\Delta$ to $K$. Since $f$ and $g$ both extend $f_\Delta$ to colourings in $L$, uniqueness in the induction hypothesis forces the two to agree in $L$. Then they both agree in $\Delta'$, and uniqueness in the induction hypothesis now forces them both to agree in $M$ as well. Consequently, $f$ and $g$ agree in $K$.

Henceforth, assume $K$ chordless. Then every triangle in $K$ has at most two vertices in $\partial K$; and if it has two such, then the edge joining the two is also in $\partial K$.

Consider first the case where $\Delta$ has one single edge in $\partial K$, say $ab$. Let $c$ be the third vertex of $\Delta$ and let $\Delta'$ be the other triangle the edge $bc$ faces in $K$. The colours $f_\Delta(b)$ and $f_\Delta(c)$ fix a vertex 3-colouring $f_{\Delta'}$ in $\Delta'$. Remove $\Delta$ and $ab$ from $K$ to obtain a suitable subtriangulation $K'$ containing $\Delta'$, with the same vertex set as $K$, but fewer triangles. By the induction hypothesis, $f_{\Delta'}$ extends to a vertex 3-colouring $f$ in $K'$. Since $c$ is a boundary even vertex of $K'$, the colourings $f$ and $f_\Delta$ agree at $a$; and since $K$ and $K'$ have the same vertex set, $f$ is an extension of $f_\Delta$ to $K$.

To establish uniqueness, let again $g$ be any extension of $f_\Delta$ to $K$. Since $f$ and $g$ both extend $f_{\Delta'}$ in $K'$, uniqueness in the induction hypothesis forces the two to agree in $K'$; and since $K$ and $K'$ have the same vertex set, $f$ and $g$ agree in $K$ as well.

We are left with the case where $\Delta$ has no edge in $\partial K$. Let $e$ be an edge in $\partial K$, let $\Delta'$ be the unique triangle $e$ faces in $K$, and let $a$ be the vertex of $\Delta'$ opposite $e$. Remove $\Delta'$ and $e$ from $K$ to obtain a suitable subtriangulation $K'$ containing $\Delta$, with the same vertex set as $K$, but fewer triangles. By the induction hypothesis, $f_\Delta$ extends to a vertex 3-colouring $f$ in $K'$. Since $K$ and $K'$ have the same vertex set, $f$ is an extension of $f_\Delta$ to $K$.

As before, uniqueness is proved by considering an arbitrary extension $g$ of $f_\Delta$ to $K$. Since $f$ and $g$ both extend $f_\Delta$ in $K'$, uniqueness in the induction hypothesis forces the two to agree in $K'$; and since $K$ and $K'$ have the same vertex set, $f$ and $g$ agree in $K$ as well.

The following two colour theorem is intimately related to the three colour theorem above. It can be established mutatis mutandis along the same lines. However, we supply two more proofs of the fact that such a colouring exists.

**3. Theorem.** *Let $K$ be a suitable triangulation of the 2-disc, and let $\Delta_0$ be a triangle in $K$. Assigning $\Delta_0$ one of two colours extends uniquely to a 2-colouring of all triangles in $K$, i. e., each can be assigned one of those two colours so that no two triangles that share an edge have the same colour.*

As mentioned earlier, the two proofs below deal with existence of such an extension alone. Uniqueness can then be established inductively as in the proof of Theorem 2.2.

The first proof refers to vertex 3-colouring.

**1st Proof.** Let $f$ be a vertex 3-colouring in $K$. Recall the orientation fixed once and for all, and view $f$ as a simplicial extension $f\colon |K| \to \Delta_0$ of the identity of $\Delta_0$. Then $\deg(f|\partial\Delta) = \pm 1$ for each triangle $\Delta$ in $K$, and $\deg(f|\partial\Delta) = -\deg(f|\partial\Delta')$ whenever triangles $\Delta$ and $\Delta'$ share an edge. Assigning a triangle $\Delta$ of $K$ the colour of $\Delta_0$ if $\deg(f|\partial\Delta) = 1$, and the other colour if $\deg(f|\partial\Delta) = -1$, defines an overall 2-colour extension satisfying the required condition.

**Remark.** The 2-colour extension can equally well be obtained by pulling the overall orientation along $f$ back to the boundary of each triangle.

The second proof hinges on a duality argument.

**2nd Proof.** Extend $K$ to a cell decomposition $\hat{K}$ of the 2-sphere $S^2$ as follows: If $K$ is even, include the outer disc (Schoenflies); otherwise, draw edges from an extra vertex in the open outer disc to all odd vertices of $K$ (the latter all lie on the boundary) and include all resulting cells. In either case, each vertex of $\hat{K}$ is even. The 2-cells of $\hat{K}$ are then 2-colourable, by duality: Each can be assigned one of two colours so that no two adjacent 2-cells share colour. (The 2-cocells are all even sided, so 1-cocycles are all even, and hence the 1-skeleton of the dual is bipartite.) In particular, the triangles in $K$ are 2-colourable. If necessary, swap colours to let $\Delta_0$ have the preassigned colour.

Before moving on any further, let us take time out for a brief digression into a consequence of Theorem 2.3, that goes, in the 3-colour setting, back to the combinatorial Stokes formula [4, 7, 11, 14, 17]. The 2-colour approach seems, however, a bit easier to deal with. Let $n$ be the number of boundary even vertices of a suitable triangulation $K$; Corollary 2.4 in the sequel rules out the case $n = 1$. Consider the $n$ pairwise edge disjoint boundary paths from one even vertex to the next such, and let $k$ be the number of odd such paths. Measure path length by the number of inner vertices, and recall that the number of boundary odd vertices of $K$ is even, to infer that $k \equiv n \pmod 2$. By 2.3, the triangles of $K$ are

2-colourable; let $t_0$ be the number of triangles of one colour, and let $t_1$ be the number of triangles of the other colour. Count triangles by simplicial neighbourhoods of vertices to infer that $3(t_0+t_1) \equiv n \pmod 2$, so $t_0+t_1 \equiv n \pmod 2$. Consequently, $t_0-t_1 \equiv t_0+t_1 \equiv n \equiv k \pmod 2$. We end this digression by mentioning that, if $K$ is ideal ($n = 0$), then an overall 3-colouring assigns every other boundary vertex the same colour, and the combinatorial Stokes formula yields $t_0 = t_1$, as expected; in this case, colours (colour orientations) of peripheral triangles alternate along the boundary.

The following corollary can be proved by using an overall vertex 3-colouring or directly, as in the second proof above. Recall that there are at least three boundary vertices.

**4. Corollary.** *The number of even vertices on the boundary of a suitably triangulated 2-disc is always different from* 1*.*

**1st Proof.** Suppose, if possible, that some suitable triangulation has one single boundary even vertex. Then the number of boundary vertices is odd. To reach a contradiction, consider an overall vertex 3-colouring and notice that vertex colours alternate along the boundary.

**2nd Proof.** Suppose again that $K$ is a suitable triangulation with one single boundary even vertex. Recall the cell extension $\hat{K}$ in the second proof of Theorem 2.3. In the case at hand, $\hat{K}$ has one single non-triangular 2-cell which is quadrangular. Recall that the 2-cells of $\hat{K}$ are 2-colourable. To reach a contradiction, count edges by 2-cells of each colour separately, to infer that the total number of edges of $\hat{K}$ is divisible by 3, on one hand, and congruent to 1 modulo 3, on the other.

The example below shows that, given any non-negative integer $n \neq 1$, there exist suitable triangulations of the 2-disc with exactly $n$ even vertices on the boundary; existence of odd vertices on the boundary is not ruled out. This example will also be referred to several times later on in connection with the location of even vertices along the boundary.

**5. Example.** The even spoke umbrella settles the case $n = 0$. If $n \geq 2$, write $p = n + 2\lfloor n/2 \rfloor$, and let $a_0, \ldots, a_{p-1}$ be pairwise distinct points on the boundary, labelled in cyclic order. Draw chords $a_{2i}a_{2i+2}$, $i = 0, \ldots, \lfloor (p-3)/2 \rfloor$. Consider an interior point $x$ outside all triangles $a_{2i}a_{2i+1}a_{2i+2}$ and join it 'radially' to all $a_{2i}$ and to both $a_1$ and $a_{p-1}$. Finally, let $a_0a_2$ cross $xa_1$ at $y$ to obtain a suitable triangulation whose boundary even vertices are $a_0$ and $a_{2\lfloor (p-1)/2 \rfloor}$, along with all $a_{2i+1}$, $i = 1, \ldots, \lfloor (p-3)/2 \rfloor$, if $n \geq 3$; a total score of exactly $n$ boundary even vertices in all cases.

The next corollary is a first hint at the configuration of even vertices on the boundary of a suitable triangulation. It can be proved with reference to Theorem 2.2 or Corollary 2.4, or directly, along the lines in the second proof of Corollary 2.4. Recall again that there are at least three boundary vertices.

**6. Corollary.** *If a suitable triangulation has exactly two boundary even vertices, then the boundary edge paths joining the two are both even.*

**1st Proof.** Consider a vertex 3-colouring in a suitable triangulation with exactly two boundary even vertices. Then vertex colours alternate along each boundary edge path joining these two vertices. Since the number of boundary vertices is even, if one of those paths is odd, then so is the other, in which case the boundary neighbours of either boundary even vertex share colour. This is a contradiction.

**2nd Proof.** Let $K$ be a suitable triangulation with exactly two boundary even vertices, $a$ and $b$. Suppose one of the boundary edge paths joining $a$ and $b$ is odd. Let $a = x_0, x_1, \ldots, x_m = b$ be the vertices this boundary edge path passes through in order; thus, $m$ is odd, $\deg_K x_0$ and $\deg_K x_m$ are both even, and $\deg_K x_i$ is odd for all other $i$'s. Let $c$ be an extra vertex inside the outer disc (Schoenflies), and extend $K$ to a triangulation $K'$ by including $c$, along with all edges $cx_i$, $i = 0, \ldots, m$, and all triangles $cx_ix_{i+1}$, $i = 0, \ldots, m-1$. Thus, $K'$ has one more boundary vertex at $c$ and $m-1$ more interior vertices $x_1, \ldots, x_{m-1}$. Since $\deg_{K'} x_i = \deg_K x_i + 1$, $i = 0, \ldots, m$, and $\deg_{K'} c = m + 1$, the extension $K'$ is a suitable triangulation with exactly one boundary even vertex at $c$. This contradicts Corollary 2.4 and concludes the proof.

The extension in the second proof above is the outcome of glueing two (suitable) triangulations. Let $K_1$ and $K_2$ be two triangulations. Identifying an edge $e_1$ in $\partial K_1$ and an edge $e_2$ in $\partial K_2$ glues $K_1$ and $K_2$ together along $e_1 \equiv e_2$ to form a new triangulation $K_1 \sqcup_{e_1 \equiv e_2} K_2$ in which $e_1 \equiv e_2$ is a chord. If $K_1$ and $K_2$ are both suitable, then so is $K_1 \sqcup_{e_1 \equiv e_2} K_2$. This extends in an obvious way to glueing two suitable triangulations along boundary edge paths (one from each triangulation) of equal lengths whose corresponding interior vertices are of like parity. The resulting triangulation is again suitable.

The glueing operation briefly described above turns out to be useful and will be referred to in the subsequent sections.

## 3 Ideal Triangulations

Recall that an ideal triangulation is a suitable triangulations whose boundary vertices are all odd. Alternatively, but equivalently, each vertex whatsoever, interior or on the boundary, is incident with an even number of triangles. An ideal triangulation has an even number of vertices on the boundary.

The first proof of Corollary 2.4 shows that a suitable triangulation is ideal if and only if an overall vertex 3-colouring (and hence any such) assigns alternate boundary vertices the same colour. The boundary vertex set of an ideal triangulation is hence 2-colourable. However, a suitable triangulation whose boundary vertex set is 2-colourable is not necessarily ideal. For $n = 2$, Example 2.5 exhibits a four boundary vertex suitable triangulation that is not ideal. Since it is chordless, the four vertices on the boundary are 2-colourable. Yet, a 2-colouring of the boundary vertex set requires two more colours, one for each of the two interior vertices, to complete an overall vertex colouring that be valid.

The following proposition is part of an alternative proof of vertex 3-colourability of suitable triangulations — see the outline at the end of this section.

**1. Proposition.** *Any suitable triangulation extends to an ideal triangulation.*

**Proof.** Let $K$ be a suitable triangulation that is not ideal and recall the construction in the second proof of Corollary 2.6: It extends $K$ to a suitable triangulation with fewer boundary even vertices by glueing an extra cone (a fan centred at an extra vertex) along a boundary edge path from one even vertex on to the next such — the number of boundary even vertices drops by 1 if that path is odd, and by 2 otherwise. Iteration eventually embeds $K$ into an ideal triangulation.

**Remark.** Here is a more effective way of dropping the number $n$ of boundary even vertices of a suitable triangulation. Consider $\lfloor n/2 \rfloor$ alternate boundary edge paths, each from one even vertex on to the next such. Glueing extra cones with pairwise distinct apices simultaneously, one along each of these vertex disjoint paths, almost halves the number of boundary even vertices at a stroke: The resulting suitable triangulation has at most $\lceil n/2 \rceil$ even vertices on the boundary. For instance, if those paths are all even, then so is $n$, by Corollary 2.4, and the resulting triangulation is ideal.

Glueing two ideal triangulations along an edge yields an ideal triangulation with a chord along that edge. The next corollary to Theorem 2.2 shows that, for every chord $e$ in an ideal triangulation $K$, the configuration $(K, e)$ is the outcome of glueing the copies of two ideal subtriangulations of $K$ along $e$.

**2. Corollary.** *Let $K$ be an ideal triangulation of the 2-disc.*

**(a)** *No three pairwise distinct boundary vertices span a triangle in $K$. Alternatively, but equivalently, a triangle in $K$ has at most two vertices in $\partial K$, and so $K$ has at least one interior vertex.*

**(b)** *The two suitable (non-empty) subtriangulations a chord splits $K$ into are both ideal. Consequently, the two edge paths the end points of a chord split $\partial K$ into are both odd.*

**Proof.** Recall that a suitable triangulation is ideal if and only if an overall vertex 3-colouring assigns alternate boundary vertices the same colour.

To prove **(a)**, consider a vertex 3-colouring $f$ in $K$. Since $K$ is ideal, $f$ restricts to a 2-colouring of the boundary vertex set, and the conclusion follows.

To establish **(b)**, let $L$ be one of the subtriangulations and let $f$ be a vertex 3-colouring in $L$. By Theorem 2.2, $f$ extends (uniquely) to a vertex 3-colouring $\tilde{f}$ in $K$. Since $K$ is ideal, $\tilde{f}$ assigns alternate vertices along $\partial K$ the same colour. Then so does $f$ along $\partial L$. Consequently, $L$ is ideal. In particular, $\partial L$ has an even number of edges, one of which is the chord. The remaining edges are all in $\partial K$ and form an odd path joining the end points of the chord.

**2nd Proof of (b).** Let $ab$ be a chord in $K$, and let $L$ be one of the subtriangulations $ab$ splits $K$ into. Suppose, if possible, $L$ is not ideal. Corollary 2.4 then forces $a$ and $b$ to be the only boundary even vertices of $L$. Since $ab$ is a boundary edge of $L$, this contradicts Corollary 2.6. The second part follows as before.

We end this section by outlining an alternative proof of vertex 3-colourability of suitable triangulations. Recall that Corollaries 2.4, 2.6 and 3.2**(b)** can be proved with no reference to vertex 3-colouring whatsoever. Using these facts alone, it is not hard to show that, if

$K$ is an ideal triangulation, then the third vertex of the unique triangle a boundary edge faces in $K$ is an interior vertex, and the boundary vertex set is 2-colourable. Induction on the number of triangles then shows that a 2-colouring of the boundary vertices extends uniquely to a vertex 3-colouring in $K$. Proposition 3.1 then completes an alternative proof of vertex 3-colourability of suitable triangulations.

# 4 Even Triangulations

Recall that an even triangulation is one whose vertices are all even; it is a suitable triangulation whose boundary vertices are all even. Alternatively, but equivalently, each interior vertex is incident with an even number of triangles, while each boundary vertex is incident with an odd number of triangles.

A disc with three pairwise distinct points on the boundary is clearly evenly triangulated. The following example exhibits a less trivial even triangulation with three boundary vertices: Let $a_0$, $a_1$, $a_2$ be pairwise distinct points on the boundary of the disc. For each $i$ modulo 3, draw the chord $a_i a_{i+1}$ and let $x_{i+2}$ be an interior point of this chord. Joining each $x_i$ to $x_{i+1}$ yields a triangulation each vertex of which has degree 4.

Here is a consequence of the three colour theorem for even triangulations.

**1. Corollary.** *Given a positive integer $n$, there exists an even triangulation of the 2-disc with exactly $n$ boundary vertices if and only if $n$ is divisible by 3, in which case the minimal number of triangles is $n-2$.*

**Proof.** Sufficiency and minimality are easily established. Leaving the trivial case $n = 3$ aside, let $n \geq 6$, and let $a_0, \ldots, a_{n-1}$ be pairwise distinct points on the boundary of the disc, labelled in cyclic order. Drawing chords $a_0 a_{3i\pm1}$ and $a_{3i-1}a_{3i+1}$ for all positive $i \leq n/3-1$, yields an evenly triangulated disc with exactly $n$ boundary vertices; the number of triangles is $n-2$, and hence minimal.

To prove necessity, simply notice that an overall vertex 3-colouring assigns every third vertex on the boundary the same colour. Tracing the boundary completely then shows $n$ divisible by 3.

**2nd Proof of Necessity.** Recall the cell extension in the second proof of Theorem 2.3 whose 2-cells are 2-colourable. Counting edges by 2-cells of each colour separately, the total number of edges is divisible by 3, on one hand, and congruent to $n$ modulo 3, on the other. Consequently, $n$ is divisible by 3.

As a by-product of 4.1, the number of inner edges, and hence the number of edges, of an even triangulation are both divisible by 3. Indeed, let $e_{\mathrm{inn}}$ and $e_{\mathrm{out}}$ be the numbers of inner and outer (boundary) edges, respectively, and let $t$ be the number of triangles, to write $2e_{\mathrm{inn}} + e_{\mathrm{out}} = 3t$ and settle the case by 4.1.

The following corollary is a slight extension of 4.1. It can be proved directly, along the lines in either argument above, or derived from Corollary 4.1. Let us opt for the latter.

**2. Corollary.** *Let $b_0, \ldots, b_{n-1}$, $n \geq 2$, be the pairwise distinct boundary even vertices of a suitable triangulation, labelled in cyclic order. If the $n$ pairwise edge disjoint boundary* *paths $b_i \dots b_{i+1}$ are all odd, then $n$ is divisible by 3. Consequently, if $n$ is not divisible by 3, then at least two of these paths are even.*

**Proof.** To prove the first statement, glue $n$ extra cones with pairwise distinct apices, one along each boundary path $b_i \dots b_{i+1}$, to obtain a $2n$ boundary vertex even triangulation. Applied to this latter, Corollary 4.1 then shows that $n$ is indeed divisible by 3.

To establish the second statement, recall that the total number of odd vertices is even (they all lie on the boundary), and notice that the length of each boundary path $b_i \dots b_{i+1}$ exceeds by 1 the number of odd vertices it passes through. Consequently, the total number of even paths $b_i \dots b_{i+1}$ is even. If $n$ is not divisible by 3, then the first statement forces at least one such, and the conclusion follows.

The boundary even vertex set and the boundary edge paths considered above are dealt with in further detail in the next section.

# 5 The Boundary Even Vertex Set

Corollaries 2.4 and 2.6, the Remark following Proposition 3.1, and Corollaries 4.1 and 4.2 are first glimpses into what the boundary even vertex set of a suitable triangulation may look like. It seems reasonable to enquire further into the number of even vertices a suitably triangulated disc may have on the boundary and the way they are located. Recall that existence of odd vertices on the boundary is not ruled out, and the total number of such vertices is even.

Parity of edge disjoint boundary paths from one even vertex to the next such is of particular interest in locating even vertices along the boundary of a suitably triangulated disc.

A suitably triangulated disc with exactly $n$ even vertices on the boundary will be referred to as a suitably triangulated *$n$-gon*; its *sides* are the edge disjoint boundary paths just mentioned. In the setting of Corollary 4.2, the $b_i$ are the vertices of the $n$-gon, and the $b_i \dots b_{i+1}$ are its sides. The *length* and *parity* of a side are, of course, those of the corresponding boundary edge path.

Here are a few examples: An ideally triangulated disc is a suitably triangulated 0-gon. Corollary 2.4 states that there are no suitably triangulated 1-gons. Example 2.5 exhibits suitably triangulated $n$-gons for every integer $n \geq 2$: If $n = 2$, then both sides are even, as required by Corollary 2.6; if $n$ is even, $n \geq 4$, there are exactly two odd sides, $a_0a_3$ of length 3 and $a_{2n-3}a_{2n-2}$ of unit length; and if $n$ is odd, there are exactly three odd sides: $a_0a_3$ of length 3, and $a_{2n-3}a_{2n-2}$ and $a_{2n-2}a_0$, both of unit length. Corollary 2.6 states that the sides of a suitably triangulated 2-gon are both even. Proposition 3.1 states that every suitably triangulated $n$-gon extends to a suitably triangulated 0-gon; and the Remark that follows 3.1 shows that, if $\lfloor n/2 \rfloor$ alternate sides are even, then so is $n$. Corollary 4.1 states that evenly triangulated $n$-gons exist if and only if $n$ is divisible by 3. Corollary 4.2 states that, if the sides of a suitably triangulated $n$-gon are all odd, then $n$ is divisible by 3; thus, if $n$ is not divisible by 3, then at least two sides are even.

Consider a suitably triangulated $n$-gon, $n \geq 2$. The length of a side exceeds by one the number of odd vertices it contains. Since the total number of odd vertices is even, so is the number of even sides. Consequently, the number of odd sides and $n$ share parity. Thus, if there are no odd sides, then $n$ is even. For such an $n$, glueing $n$ caps with pairwise distinct apices, one along each boundary edge of an $n$ spoke umbrella, yields a suitably triangulated $n$-gon whose sides are all even (each of length 2). For convenience, this configuration will be referred to later on as the $n$ spoke *ruffled* umbrella.

The following proposition is intimately related and similar to Corollary 2.4.

**1. Proposition.** *No suitably triangulated $n$-gon has exactly one odd side. Consequently, if $n$ is odd, then any suitably triangulated $n$-gon has at least three odd sides*; *in particular, the sides of a suitably triangulated triangle are all odd.*

**Proof.** Suppose, if possible, that some suitably triangulated $n$-gon has exactly one odd side. Then $n$ is odd, and glueing $(n-1)/2$ cones with pairwise distinct apices, one along every other even side, yields a suitably triangulated 1-gon. This contradicts Corollary 2.4 and proves the first statement; the next two then follow at once.

**Remark.** Example 2.5 shows that the lower bound in the second statement above is achieved for every odd $n$.

By the preceding, a suitably triangulated $n$-gon whose sides are not all even, has at least two odd sides (and $n \geq 3$, by 2.6). The (possibly empty) stretch of consecutive even sides between two successive odd sides will be referred to as a *separating chain*; if there are no odd sides, there are no separating chains either — as mentioned earlier, this may very well be the case for every even $n$. The *length* of a separating chain is, of course, the number of consecutive even sides it consists of; and its *parity* is the parity of its length. Separating empty chains are, of course, even.

Let $n \geq 3$ and refer again to Example 2.5. If $n$ is even, there are exactly two separating chains and they are both odd: $a_3a_5 \ldots a_{2n-3}$ has length $n-3$, and $a_{2n-2}a_0$ has unit length. If $n$ is odd, there are exactly three separating chains and they are all even: two are empty, and the third, $a_3a_5 \ldots a_{2n-3}$, has length $n-3$.

The separating chains of an evenly triangulated $n$-gon (Corollary 4.1) are all empty; the same holds, of course, for any suitably triangulated $n$-gon whose sides are all odd, e. g., if $n = 3$, by 5.1. Corollary 4.2 shows that, if $n$ is not divisible by 3, then any suitably triangulated $n$-gon has at least one separating non-empty chain.

Suitably triangulated 0-gons and 2-gons have no separating chains; neither has the $2m$ spoke ruffled umbrella.

Clearly, there are as many separating chains (empty ones, inclusive) as odd sides. Since the parity of a separating chain and the parity of the number of odd vertices it contains are the same, the number of separating odd chains is even, so the number of separating even chains and $n$ share parity.

The proposition below completes Proposition 5.1.

**2. Proposition.** *No suitably triangulated $n$-gon has exactly one separating even chain. Consequently, if $n$ is odd, then any suitably triangulated $n$-gon has at least three separating even chains*; *in particular, if it has exactly three odd sides, then the three separating chains are all even.*

**Proof.** Clearly, only the first statement is to be dealt with. Suppose, if possible, that some suitably triangulated $n$-gon has exactly one separating even chain. Then $n$ is odd, and, by Proposition 5.1, the total number $k$ of odd sides is at least 3. Obviously, $k$ is odd, and there are exactly $k-1$ separating odd chains. For every possible integer $m$ and every separating chain of length $2m-1$, glueing $m$ cones with pairwise distinct apices, one along every other side in the chain, then yields a suitably triangulated $n'$-gon, for some (odd) $n' < n$, with exactly one odd side (since $k$ is odd). This contradicts Proposition 5.1 and concludes the proof.

**Remark.** Example 2.5 shows again that the lower bound in the second statement above is achieved for every odd $n$: There are exactly three odd sides, two separating chains are empty, and the third has length $n-3$, whence non-empty if $n \geq 5$. Let us just mention that, for every odd $n \geq 7$, there exist suitably triangulated $n$-gons with exactly three odd sides and only one separating empty chain; and for every odd $n \geq 11$, there exist suitably triangulated $n$-gons with exactly three odd sides whose separating chains are all non-empty. Such examples are obtained by suitable 'widening' surgery and triangulated disc insertion to replace a separating empty chain by a non-empty one.

The next proposition completes Corollary 4.2 and, along with this latter, settles the case for suitably triangulated quadrangles whose sides are not all even: Side parities alternate along the boundary — letting $n = 4$ in Example 2.5 illustrates this configuration.

**3. Proposition.** *If the separating chains of a suitably triangulated $n$-gon are all even, then the total number $k$ of odd sides is divisible by* 3*. Consequently, if $k$ is not divisible by* 3*, then at least two separating chains are odd*; *in particular, if $k = 2$, in which case $n$ is necessarily even, the separating chains are both odd.*

**Proof.** As before, only the first statement requires proof. Consider a suitably triangulated $n$-gon whose separating chains are all even. For every possible integer $m$ and every separating chain of length $2m$, glue $m$ cones with pairwise distinct apices, one along every other side in the chain, to end up with a suitably triangulated $k$-gon whose sides are all odd. Applied to this latter, Corollary 4.2 then shows $k$ divisible by 3.

Propositions 5.2 and 5.3 settle the case for suitably triangulated pentagons: Three sides are odd, two separating chains are empty, and the third has length 2 — letting $n = 5$ in Example 2.5 illustrates this configuration.

The lower bound in the second statement in 5.3 is achieved for every even $n \geq 4$, as Example 2.5 shows; in this case, $k = 2$. In fact, this lower bound is achieved for every $n \geq 6$, regardless of the class of $k$ modulo 3. More precisely, for every $n \geq 6$, there exists a suitably triangulated $n$-gon with exactly two even sides that are not adjacent — hence, exactly two separating odd chains, each of unit length. This is just the special case $\ell = 2$ of part **(a)** in the last proposition below.

Before stating and proving this proposition, notice the following simple trick: The length of a side can be increased by 2, and indeed by any even number, without changing any parity whatsoever: Choose any edge $ab$ along that side, consider the vertex $c$ opposite $ab$ in the unique triangle $ab$ faces, and subdivide this triangle by joining $c$ to two extra vertices on $ab$. We may and will therefore assume that the length of some even side, if any, is at least 4.

**4. Proposition.** **(a)** *If there exists a suitably triangulated $n$-gon with exactly $\ell$ even sides, $\ell \geq 2$, then there exists a suitably triangulated $(n+2)$-gon with exactly $\ell$ even sides, at least one of which is flanked by odd sides.*

**(b)** *For every even $n \geq 2$ and every positive even $\ell \leq n$, there exists a suitably triangulated $n$-gon with exactly $\ell$ even sides.*

**(c)** *For every odd $n \geq 5$ and every positive even $\ell \leq n-3$, there exists a suitably triangulated $n$-gon with exactly $\ell$ even sides.*

**Proof.** **(a)** To obtain the desired configuration from one whose existence is assumed, consider an even side of length at least 4, and glue a cone along the path formed by all inner edges of that side. This latter is thus replaced in the new configuration by a side of length 2 flanked by two sides of unit length. There are no other changes whatsoever along the remaining part of the boundary, and the interior vertices of the new configuration are all even. The conclusion follows.

**(b)** Induct on $n$. Corollary 2.6 settles the base case $n = 2$. For an even $n \geq 4$, the above argument and the induction hypothesis imply the conclusion for every positive even $\ell \leq n - 2$. The $n$ spoke ruffled umbrella settles the case $\ell = n$.

**(c)** Induct on $n$ again. The paragraph following the proof of 5.3 settles the base case $n = 5$. As in the proof of part **(b)**, for an odd $n \geq 7$, the conclusion holds for every positive even $\ell \leq n - 5$. Finally, Example 2.5 settles the case $\ell = n - 3$.

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