Bull. Math. Soc. Sci. Math. Roumanie
Tome 64 (112), No. 2, 2021, 105–131

**Symbolic strong persistence property under monomial operations and strong persistence property of cover ideals**

by

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**Abstract**

Let $R$ be a commutative Noetherian ring and $I$ be an ideal of $R$. Then, $I$ has the strong persistence property if $(I^{k+1} :_R I) = I^k$ for all $k$. Also, we say that $I$ has the symbolic strong persistence property if $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k$, where $I^{(k)}$ denotes the $k$-th symbolic power of $I$. In this paper, by using some monomial operations, such as expansion, weighting, monomial multiple, monomial localization, and contraction, we introduce several methods for constructing new monomial ideals which have the symbolic strong persistence property based on the monomial ideals which have the symbolic strong persistence property. We also probe the strong persistence property of the cover ideal of the union of two finite simple graphs.

**Key Words**: Strong persistence property, associated primes, cover ideals, symbolic strong persistence property.

**2010 Mathematics Subject Classification**: Primary 13C13, 13B25, 13A30, 05E40; Secondary 13P25, 05C25.

# 1 Introduction

Let $I$ be an ideal in a commutative Noetherian ring $R$. A prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of $I$ if there exists an element such as $c$ in $R$ such that $\mathfrak{p} = (I :_R c)$. The set of associated primes of $I$, denoted by $\mathrm{Ass}_R(R/I)$, is the set of all prime ideals associated to $I$. A well-known result of Brodmann [5] showed that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k\geq 1}$ of associated prime ideals is stationary for large $k$, that is, there exists a positive integer $k_0$ such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all integers $k \geq k_0$. The minimal such $k_0$ is called the *index of stability* of $I$ and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set* of associated prime ideals of $I$, which is denoted by $\mathrm{Ass}^\infty(I)$. In general, studying the stable set and the index of stability for ideals is complicated, refer to [14, 20, 23] for more details. One can ask many questions in the context of Brodmann's theorem. An ideal $I$ of $R$ satisfies the *persistence property* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I^{k+1})$ for all positive integers $k$. Furthermore, an ideal $I$ of $R$ has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all positive integers $k$. One can easily prove that the strong persistence property implies the persistence property, see [18, Proposition 2.9]. Ratliff [25] showed that $(I^{k+1} :_R I) = I^k$ for all large $k$. Now, let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ and $x_1, \ldots, x_n$ are indeterminates. Generally speaking, finding classes of monomial ideals which have the persistence property is intricate. Especially, it has been verified in [12] that there exists a square-free monomial ideals which does not satisfy the persistence property. However, it has been proved in [16] that all edge ideals of finite simple graphs have the strong persistence property; even, this result is true for every finite graph with loops, see [26]. Also, it is known by [10] that every polymatroidal ideal has the strong persistence property. In addition, based on [6], the cover ideals of perfect graphs satisfy the persistence property. It also has been established in [21] that the cover ideals of some imperfect graphs have the strong persistence property, that is, cycle graphs of odd orders, wheel graphs of even orders, and helm graphs of odd orders with greater than or equal to 5. Furthermore, according to [19], it has been introduced two classes of monomial ideals which have the strong persistence property, that is, unisplit and separable monomial ideals. Also, an ideal $I$ is called *normally torsion-free* if $\mathrm{Ass}(R/I^k) \subseteq \mathrm{Ass}(R/I)$ for all $k$, see [13, 27] for more information. Along this argument, the concept of symbolic strong persistence property was presented in [26]. An ideal $I$ in a commutative Noetherian ring $R$ has the *symbolic strong persistence property* if $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k$, where $I^{(k)} = \bigcap_{\mathfrak{p}\in \mathrm{Min}(I)}(I^k R_{\mathfrak{p}} \cap R)$ denotes the $k$-th symbolic power of $I$. It has been shown in [26] that the strong persistence property implies the symbolic strong persistence property, but little is known for the classes of monomial ideals which satisfy the symbolic strong persistence property. One of the main purposes in this paper is to introduce several methods for constructing new monomial ideals which have the symbolic strong persistence property based on the monomial ideals which have the symbolic strong persistence property.

This paper is organized as follows. In Section 2, by using the weighting operation, we first show that a monomial ideal has the symbolic strong persistence property if and only if its weighted ideal has the symbolic strong persistence property (Theorem 1). Next, by considering the contraction operation, our goal is to establish that if a monomial ideal has the symbolic strong persistence property, then its contracted ideal has the symbolic strong persistence property as well (Theorem 2). Especially, by using the monomial localization of a monomial ideal with respect to a monomial prime ideal, we prove that if a monomial ideal $I$ has the symbolic strong persistence property, then $I(\mathfrak{p})$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$, and hence has the symbolic strong persistence property (Theorem 3). After that, by means of the expansion operation, we verify that a monomial ideal has the symbolic strong persistence property if and only if its expansion has the symbolic strong persistence property (Theorem 4).

In Section 3, we give several new results on the strong persistence property and symbolic strong persistence property. For this purpose, we first demonstrate that if $I$ is an ideal in a commutative Noetherian ring $R$, then $I$ has the symbolic strong persistence property if and only if $I_{\mathfrak{p}}$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$, where $I_{\mathfrak{p}}$ denotes the localization of $I$ at $\mathfrak{p}$ (Theorem 5). Next, we present some calsses of square-free monomial ideals which have the strong persistence property (Theorems 6), and as an application, we re-prove that the cover ideal of any odd cycle has the strong persistence property (Theorem 7). After that, we focus on a useful theorem which tells us that if $I$ is a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ and $\mathcal{G}(I) = G_1 \cup \cdots \cup G_r$ such that $\{x_s : x_s | m \text{ for some } m \in G_i\} \cap \{x_t : x_t | m \text{ for some } m \in G_j\} = \emptyset$ for all $1 \leq i \neq j \leq r$, then $(G_i)R$ has the symbolic strong persistence property for some $1 \leq i \leq r$ if and only if $I$ has the symbolic strong persistence property (Theorem 9). We finish this section with a lemma which say that, under certain condition, a monomial ideal has the symbolic strong persistence property if and only if its monomial multiple has the symbolic strong persistence property.

Section 4 is concerned with the strong persistence property of the cover ideal of the union of two finite simple graphs. To do this, we start with a theorem which examines the relation between assoicated primes of powers of the cover ideal of the union of two finite simple graphs with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex (Theorem 11). We next concentrate on a theorem which tells us the relation between assoicated primes of powers of the cover ideal of the union of two finite simple connected graphs with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only one edge in common (Theorem 12). In particular, under the condition of Theorem 11 (respectively, Theorem 12) for two simple finite graphs $G$ and $H$, if $J(G)$ and $J(H)$ have the strong persistence property, then $J(G \cup H)$ has the strong persistence property. We close this section by expressing two counterexamples which explore the relation between assoicated primes of powers of the cover ideal of the union of two finite simple connected graphs with the assoicated primes of powers of the cover ideals of each of them, in a general case (Questions 2 and 3).

Throughout this paper, we denote the unique minimal set of monomial generators of a monomial ideal $I$ by $\mathcal{G}(I)$. Also, $R = K[x_1, \ldots, x_n]$ is a polynomial ring over a field $K$ and $x_1, \ldots, x_n$ are indeterminates. The symbol $\mathbb{N}$ will always denote the set of positive integers. A simple graph $G$ means that $G$ has no loop and no multiple edge. All graphs in this paper are undirected. Moreover, if $G$ is a finite simple graph, then $J(G)$ stands for the cover ideal of $G$.

## 2 Symbolic strong persistence property under some monomial operations

The aim of this section is to state some methods for constructing new monomial ideals which have the symbolic strong persistence property based on the monomial ideals which have the symbolic strong persistence property. To do this, we first show that a monomial ideal has the symbolic strong persistence property if and only if its weighted ideal has the symbolic strong persistence property. To see this, one requires to recall the definition of weighted ideals.

**Definition 1.** *A weight over a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ is a function $w\colon \{x_1, \ldots, x_n\} \to \mathbb{N}$, $w_i = w(x_i)$. Then $w_i$ is called the weight of the variable $x_i$. Given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we denote $m_w = x_1^{w_1\alpha_1} \cdots x_n^{w_n\alpha_n}$. If $I$ is a monomial ideal and $w$ a weight, the weighted ideal of $I$ is $I_w = \big(m_w \mid m \in \mathcal{G}(I)\big)$.*

For example, consider the monomial ideal $I = (x_1^2x_2x_3^6, x_2^3x_4x_5^4)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Also, let $w : \{x_1, x_2, x_3, x_4, x_5\} \to \mathbb{N}$ be a weight over $R$ with $w(x_1) = 2$, $w(x_2) = 4$, $w(x_3) = 2$, $w(x_4) = 3$, and $w(x_5) = 1$. Hence, the weighted ideal $I_w$ is given by $I_w = (x_1^4x_2^4x_3^{12}, x_2^{12}x_4^3x_5^4)$.

The following proposition and lemma are necessary for us to prove the subsequent theorem.

**Proposition 1.** *Let $I$, $J$, and $L$ be monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $w$ a weight over $R$. Then $(I :_R J) = L$ if and only if $(I_w :_R J_w) = L_w$.*

*Proof.* This result is a straightforward consequence of the fact that gcd and lcm of two monomials behave well with respect to taking weights. □

**Lemma 1.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $w$ a weight over $R$. Then $(I^{(k)})_w = (I_w)^{(k)}$ for all $k \in \mathbb{N}$.*

*Proof.* It is straightforward, and left to the reader. □

We are ready to state one of the main result of this section in the following theorem.

**Theorem 1.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $w$ a weight over $R$. Then $I$ has the symbolic strong persistence property if and only if $I_w$ has the symbolic strong persistence property.*

*Proof.* We can combine together Proposition 1 and Lemma 1 to obtain the claim. □

In what follows, our goal is to establish that if a monomial ideal has the symbolic strong persistence property, then its contracted ideal has the symbolic strong persistence property as well. To accomplish this, we first need to prove several auxiliary results as follows.

**Notation 1.** *Given $1 \leq i \leq n$ and a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in a polynomial ring $R = K[x_1, \ldots, x_n]$, we set $m_{x_i=1} := x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}$.*

**Definition 2.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ and $1 \leq i \leq n$. We define the contracted ideal of $I$, denoted by $I_{\setminus x_i}$, as the ideal of $R$ generated by $\{m_{x_i=1} \mid m \in \mathcal{G}(I)\}$.*

**Proposition 2.** *Let $A$ be a finite set of irreducible monomial ideals whose radicals are mutually incomparable with respect to inclusion. Then $\bigcap_{Q \in A} Q$ is a minimal primary decomposition.*

*Proof.* Assume that $A := \{Q_1, \ldots, Q_r\}$. Let $\mathfrak{p}_i = \sqrt{Q_i}$ for all $i$. If there exists $j$ such that $\cap_{i \neq j} Q_i \subseteq Q_j$, then $\cap_{i \neq j} \mathfrak{p}_i \subseteq \mathfrak{p}_j$, which implies that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for some $i \neq j$, a contradiction. □

**Lemma 2.** *Let $I$ and $J$ be monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $1 \leq i \leq n$. Then*

*(i)* $(I \cap J)_{\setminus x_i} = I_{\setminus x_i} \cap J_{\setminus x_i}$.

*(ii)* $(IJ)_{\setminus x_i} = I_{\setminus x_i} J_{\setminus x_i}$.

*(iii)* $(I^k)_{\setminus x_i} = (I_{\setminus x_i})^k$ *for all* $k \in \mathbb{N}$.

*(iv) If $I = Q_1 \cap \cdots \cap Q_r$ is a minimal primary decomposition of $I$, then*

$$I_{\setminus x_i} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j})} Q_j \text{ and } (I_{\setminus x_i})^{(1)} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j}), \sqrt{Q_j} \in \mathrm{Min}(I)} Q_j.$$

*(v)* $(I^{(k)})_{\setminus x_i} = (I_{\setminus x_i})^{(k)}$ *for all* $k \in \mathbb{N}$.

*Proof.* (i) It follows from the fact that $\mathrm{lcm}(u,v)_{x_i=1} = \mathrm{lcm}(u_{x_i=1}, v_{x_i=1})$.

(ii) This claim can be deduced from the fact that $(uv)_{x_i=1} = u_{x_i=1}v_{x_i=1}$.

(iii) The desired conclusion follows immediately from (ii).

(iv) If $I = Q_1 \cap \cdots \cap Q_r$ is a minimal primary decomposition of $I$, then part (i) implies that $I_{\setminus x_i} = (Q_1)_{\setminus x_i} \cap \cdots \cap (Q_r)_{\setminus x_i}$. In addition, for each $j = 1, \ldots, r$, we have

$$(Q_j)_{\setminus x_i} = \begin{cases} R & \text{if } x_i \in \mathcal{G}(\sqrt{Q_j}) \\ Q_j & \text{if } x_i \notin \mathcal{G}(\sqrt{Q_j}). \end{cases}$$

Accordingly, we get $I_{\setminus x_i} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j})} Q_j$. Based on Proposition 2, one can conclude the minimal primary decomposition of $I_{\setminus x_i}$. Finally, since $\mathfrak{p} \in \mathrm{Min}(I_{\setminus x_i})$ if and only if $\mathfrak{p} \in \mathrm{Min}(I)$ and $x_i \notin \mathcal{G}(\mathfrak{p})$, we gain the following equality

$$(I_{\setminus x_i})^{(1)} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j}),\, \sqrt{Q_j} \in \mathrm{Min}(I)} Q_j.$$

(v) Let $I^k = Q_1 \cap \cdots \cap Q_r$ be a minimal primary decomposition of $I^k$, where $\sqrt{Q_j}$ is a minimal prime of $I$ if and only if $j \leq r''$, and $x_i \notin \mathcal{G}(\sqrt{Q_j})$ if and only if $j \leq r' \leq r''$. We thus have $I^{(k)} = Q_1 \cap \cdots \cap Q_{r''}$. By parts (i) and (iii), one can conclude that $(I_{\setminus x_i})^k = (I^k)_{\setminus x_i} = (Q_1)_{\setminus x_i} \cap \cdots \cap (Q_r)_{\setminus x_i}$. Furthermore, part (iv) and the fact that $L^{(k)} = (L^k)^{(1)}$ for every monomial ideal $L$, yield that $(I_{\setminus x_i})^{(k)} = Q_1 \cap \cdots \cap Q_{r'} = (I^{(k)})_{\setminus x_i}$. □

**Proposition 3.** *Let* $I$*,* $J$*, and* $L$ *be monomial ideals in a polynomial ring* $R = K[x_1, \ldots, x_n]$ *over a field* $K$ *such that* $(I :_R J) = L$*, and* $1 \leq i \leq n$*. Then* $(I_{\setminus x_i} :_{R_{\setminus x_i}} J_{\setminus x_i}) = L_{\setminus x_i}$*.*

*Proof.* Let $m_{x_i=1} \in \mathcal{G}(L_{\setminus x_i})$ and $a_{x_i=1} \in \mathcal{G}(J_{\setminus x_i})$, where $m \in \mathcal{G}(L)$ an $a \in \mathcal{G}(J)$. Since $ma \in I$, this gives that $(ma)_{x_i=1} \in I_{\setminus x_i}$. It follows now from $(ma)_{x_i=1} = m_{x_i=1}a_{x_i=1} \in I_{\setminus x_i}$that $L_{\setminus x_i} \subseteq (I_{\setminus x_i} :_{R_{\setminus x_i}} J_{\setminus x_i})$.

To conclude our argument, one has to establish the reverse inclusion. To do this, consider $A$ as the set of exponents of $x_i$ in the monomials of $\mathcal{G}(I) \cup \mathcal{G}(J)$, and set $a := \max A$. Take a monomial $m \in (I_{\setminus x_i} :_{R_{\setminus x_i}} J_{\setminus x_i})$. This implies that $mx_i^a \in (I :_R J)$, and so $mx_i^a \in L$. That is, $mx_i^a = \ell f$, where $f \in \mathcal{G}(L)$ and $\ell$ is a monomial in $R$. We therefore have $m_{x_i=1} = (mx_i^a)_{x_i=1} = (\ell f)_{x_i=1} \in L_{\setminus x_i}$. Because $m_{x_i=1} \mid m$, this yields that $m \in L_{\setminus x_i}$, as required. □

As an immediate consequence of Proposition 3, we get the corollary below.

**Corollary 1.** *Let* $I$ *be a monomial ideal in a polynomial ring* $R = K[x_1, \ldots, x_n]$ *over a field* $K$*, and* $1 \leq i \leq n$*. If* $I$ *has the strong persistence property, then* $I_{\setminus x_i}$ *has the strong persistence property.*

*Proof.* Assume $I$ has the strong persistence property, and fix $k \geq 1$. On account of $(I^{k+1} :_R I) = I^k$, Proposition 3 implies that $((I^{k+1})_{\setminus x_i} :_{R_{\setminus x_i}} I_{\setminus x_i}) = (I^k)_{\setminus x_i}$. Here, Lemma 2 (iii) yields that $((I_{\setminus x_i})^{k+1} :_{R_{\setminus x_i}} I_{\setminus x_i}) = (I_{\setminus x_i})^k$. This means that $I_{\setminus x_i}$ has the strong persistence property, as desired. □

We are now in a position to express another main result of this section in Theorem 2.

**Theorem 2.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $1 \leq i \leq n$. If $I$ has the symbolic strong persistence property, then $I_{\backslash x_i}$ has the symbolic strong persistence property.*

*Proof.* Suppose that $I$ has the symbolic strong persistence property. Fix $k \geq 1$. In the light of $(I^{(k+1)} : I^{(1)}) = I^{(k)}$, it follows from Proposition 3 that $((I^{(k+1)})_{\backslash x_i} :_{R_{\backslash x_i}} (I^{(1)})_{\backslash x_i}) = (I^{(k)})_{\backslash x_i}$. By virtue of Lemma 2 (iv), one can deduce that $((I_{\backslash x_i})^{(k+1)} :_{R_{\backslash x_i}} (I_{\backslash x_i})^{(1)}) = (I_{\backslash x_i})^{(k)}$. That is, $I_{\backslash x_i}$ has the symbolic strong persistence property. This completes the proof. $\square$

To understand Theorem 3, one has to recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal as has been introduced in [10]. Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. We also denote by $V^*(I)$ the set of monomial prime ideals containing $I$. Let $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ be a monomial prime ideal with $\mathfrak{p} \in V^*(I)$. The *monomial localization* of $I$ with respect to $\mathfrak{p}$, denoted by $I(\mathfrak{p})$, is the ideal in the polynomial ring $R(\mathfrak{p}) = K[x_{i_1}, \ldots, x_{i_r}]$ which is obtained from $I$ by applying the $K$-algebra homomorphism $R \to R(\mathfrak{p})$ with $x_j \mapsto 1$ for all $x_j \notin \{x_{i_1}, \ldots, x_{i_r}\}$.

**Theorem 3.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ such that $I$ has the symbolic strong persistence property. Then $I(\mathfrak{p})$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$, and hence has the symbolic strong persistence property.*

*Proof.* Fix $k \geq 1$ and $\mathfrak{p} \in \mathrm{Min}(I)$. The assumption implies that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. This yields that $(I^{(k+1)} :_R I^{(1)})(\mathfrak{p}) = I^{(k)}(\mathfrak{p})$. It follows now from [24, Lemma 4.6 (iv)] that $(I^{(k+1)}(\mathfrak{p}) :_{R(\mathfrak{p})} I^{(1)}(\mathfrak{p})) = I^{(k)}(\mathfrak{p})$. Since $\mathfrak{p} \in \mathrm{Min}(I)$, by [24, Lemma 4.6 (vii)], we obtain $I^{(s)}(\mathfrak{p}) = I^s(\mathfrak{p})$ for all $s$. Thus, one can conclude that $(I^{k+1}(\mathfrak{p}) :_{R(\mathfrak{p})} I(\mathfrak{p})) = I^k(\mathfrak{p})$. On account of [24, Lemma 4.6 (ii)], we get $((I(\mathfrak{p}))^{k+1} :_{R(\mathfrak{p})} I(\mathfrak{p})) = (I(\mathfrak{p}))^k$. Therefore, $I(\mathfrak{p})$ has the strong persistence property, and hence has the symbolic strong persistence property, as required. $\square$

**Definition 3.** *([28, Definition 6.1.5]) Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. The support of $u$ is given by $\mathrm{supp}(u) := \{x_i | \ a_i > 0\}$. In addition, for a monomial ideal $I$ of $R$ with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, we define $\mathrm{supp}(I) := \bigcup_{i=1}^m \mathrm{supp}(u_i)$.*

To see an application of Theorem 3, one can consider Question 1. To do this, we begin with the definition of monomial ideals of clutter type in the following definition.

**Definition 4.** *[22] Let $I$ be a non-square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ with $\mathcal{G}(I) = \{u_1, \ldots, u_r\}$. We say that $I$ is of clutter type if $\sqrt{u_i} \nmid \sqrt{u_j}$ (or equivalently, $\mathrm{supp}(u_i) \nsubseteq \mathrm{supp}(u_j)$) for each $1 \leq i \neq j \leq r$.*

**Example 1.** *[22] Let $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$ be a monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field $K$. Then one can rapidly see that $I$ is of clutter type. Note that $I$ does not satisfy both the persistence property and strong persistence property since $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}_R(R/I) \setminus \mathrm{Ass}_R(R/I^2)$ and $(I^2 :_R I) \neq I$.*

**Question 1.** *Does every non-square-free monomial ideal of clutter type have the symbolic strong persistence property?*

The answer is negative. We provide a counterexample. To accomplish this, consider the monomial ideal $I = (x_1^4x_3, x_1^3x_2x_4, x_1x_2^3x_5, x_2^4x_6)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ over a field $K$. It is easy to check that $I$ is a non-square-free monomial ideal of clutter type. Furthermore, using Macaulay2 [8] implies that $(x_1, x_2) \in \mathrm{Min}(I)$. On account of Theorem 3, one can deduce that $I(\mathfrak{p}) = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4)$ has the symbolic strong persistence property, whereas the monomial ideal $I(\mathfrak{p})$ does not satisfy the symbolic strong persistence property since $(I(\mathfrak{p})^{(2)} : I(\mathfrak{p})^{(1)}) \neq I(\mathfrak{p})^{(1)}$.

Finally, we want to examine the symbolic strong persistence property under expansion operation. Indeed, we show that a monomial ideal has the symbolic strong persistence property if and only if its expansion has the symbolic strong persistence property. To do this, we recall the definition of the expansion of a monomial ideal, which has been introduced in [2]. Let $K$ be a field and $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$ in the variables $x_1, \ldots, x_n$. Fix an ordered $n$-tuple $(i_1, \ldots, i_n)$ of positive integers, and consider the polynomial ring $R^{(i_1,\ldots,i_n)}$ over $K$ in the variables

$$x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{n1}, \ldots, x_{ni_n}.$$

Let $\mathfrak{p}_j$ be the monomial prime ideal $(x_{j1}, x_{j2}, \ldots, x_{ji_j}) \subseteq R^{(i_1,\ldots,i_n)}$ for all $j = 1, \ldots, n$. Attached to each monomial ideal $I \subset R$ a set of monomial generators $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where $\mathbf{x}^{\mathbf{a}_i} = {x_1}^{a_i(1)} \cdots {x_n}^{a_i(n)}$ and $a_i(j)$ denotes the $j$th component of the vector $\mathbf{a}_i = (a_i(1), \ldots, a_i(n))$ for all $i = 1, \ldots, m$. We define the *expansion of $I$ with respect to the $n$-tuple* $(i_1, \ldots, i_n)$, denoted by $I^{(i_1,\ldots,i_n)}$, to be the monomial ideal

$$I^{(i_1,\ldots,i_n)} = \sum_{i=1}^{m} \mathfrak{p}_1^{a_i(1)} \cdots \mathfrak{p}_n^{a_i(n)} \subseteq R^{(i_1,\ldots,i_n)}.$$

We simply write $R^*$ and $I^*$, respectively, rather than $R^{(i_1,\ldots,i_n)}$ and $I^{(i_1,\ldots,i_n)}$. Note that the expansion operation is applied to the unique set of minimal monomial generators of the monomial ideal $I$.

For example, consider the polynomial ring $R = K[x_1, x_2, x_3, x_4]$ and the ordered 4-tuple $(1, 2, 2, 3)$. Then we have $\mathfrak{p}_1 = (x_{11})$, $\mathfrak{p}_2 = (x_{21}, x_{22})$, $\mathfrak{p}_3 = (x_{31}, x_{32})$, and $\mathfrak{p}_4 = (x_{41}, x_{42}, x_{43})$. Hence, for the monomial ideal $I = (x_1x_3, x_2x_4, x_3^2)$, the ideal

$$I^* \subseteq K[x_{11}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{43}]$$

is $\mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_4 + \mathfrak{p}_3^2$, namely

$$I^* = (x_{11}x_{31}, x_{11}x_{32}, x_{21}x_{41}, x_{21}x_{42}, x_{21}x_{43}, x_{22}x_{41}, x_{22}x_{42}, x_{22}x_{43}, x_{31}^2, x_{31}x_{32}, x_{32}^2).$$

Let us provide the other main result of this section in the subsequent theorem.

**Theorem 4.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. Then $I$ has the symbolic strong persistence property if and only if $I^*$ has the symbolic strong persistence property.*

*Proof.* We first note that $I = J$ if and only if $I^* = J^*$. To prove the forward implication, let $I$ have the symbolic strong persistence property. Fix $k \geq 1$. The assumption yields that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. In view of [2, Lemma 1.1 and Corollary 1.4], one has the following equalities

$$\begin{aligned}((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) &= ((I^{(k+1)})^* :_{R^*} (I^{(1)})^*)\\ &= (I^{(k+1)} :_R I^{(1)})^*\\ &= (I^{(k)})^*\\ &= (I^*)^{(k)}.\end{aligned}$$

This means that $I^*$ has the symbolic strong persistence property. Conversely, assume that $I^*$ has the symbolic strong persistence property. Accordingly, we have $((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) = (I^*)^{(k)}$. On account of [2, Lemma 1.1 and Corollary 1.5], we get $((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) = (I^{(k+1)} :_R I^{(1)})^*$ and $(I^*)^{(k)} = (I^{(k)})^*$. This implies that $(I^{(k+1)} :_R I^{(1)})^* = (I^{(k)})^*$, and so $(I^{(k+1)} :_R I^{(1)}) = (I^{(k)})$. That is, $I$ has the symbolic strong persistence property. $\square$

# 3 Some results on the (symbolic) strong persistence property

In this section, we give several results on the strong persistence property and symbolic strong persistence property. For this purpose, we begin with Theorem 5. To achieve this, one has to recall the following proposition.

**Proposition 4.** *([22]) Let $I$ be an ideal in a commutative Noetherian ring $R$. Also, let $I = Q_1 \cap \cdots \cap Q_t \cap Q_{t+1} \cap \cdots \cap Q_r$ be a minimal primary decomposition of $I$ with $\mathfrak{p}_i = \sqrt{Q_i}$ for each $i = 1, \ldots, r$, and $\mathrm{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then $I_{\mathfrak{p}_i} = (Q_i)_{\mathfrak{p}_i}$ for each $i = 1, \ldots, t$.*

**Theorem 5.** *Let $I$ be an ideal in a commutative Noetherian ring $R$. Then $I$ has the symbolic strong persistence property if and only if $I_{\mathfrak{p}}$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$, where $I_{\mathfrak{p}}$ denotes the localization of $I$ at $\mathfrak{p}$.*

*Proof.* ($\Rightarrow$) Assume that $I$ has the symbolic strong persistence property. Let $\mathrm{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Without loss of generality, it is enough to show that $(I^{k+1}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I^k_{\mathfrak{p}_1}$ for all $k \geq 1$. Fix $k \geq 1$. Also, for all $s$, suppose that

$$I^s = Q_{s,1} \cap \cdots \cap Q_{s,t} \cap Q_{s,t+1} \cap \cdots \cap Q_{s,r_s},$$

is a minimal primary decomposition of $I^s$ with $\sqrt{Q_{s,i}} = \mathfrak{p}_i$ for each $i = 1, \ldots, t$, and $\sqrt{Q_{s,i}}$ is not minimal for each $i = t+1, \ldots, r_s$. We therefore have

$$I^{(1)} = \bigcap_{i=1}^{t} Q_{1,i}, \ I^{(k)} = \bigcap_{i=1}^{t} Q_{k,i}, \text{ and } I^{(k+1)} = \bigcap_{i=1}^{t} Q_{k+1,i}.$$

Since $I$ has the symbolic strong persistence property, one has $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$, and hence $(I^{(k+1)}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I^{(1)}_{\mathfrak{p}_1}) = I^{(k)}_{\mathfrak{p}_1}$. It follows also from Proposition 4 that, for all $s$, $(I^s)_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ and $(I^{(s)})_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$. We thus gain $(I^{k+1}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I^k_{\mathfrak{p}_1}$, as required.

($\Leftarrow$) Assume that $I_{\mathfrak{p}}$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$. Want to show that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k \geq 1$. To accomplish this, fix $k \geq 1$. Our strategy is to use [17, Exercise 6.4]. For this purpose, one has to prove that $(I^{(k+1)}_{\mathfrak{q}} :_{R_{\mathfrak{q}}} I^{(1)}_{\mathfrak{q}}) = I^{(k)}_{\mathfrak{q}}$ for all $\mathfrak{q} \in \mathrm{Ass}_R(R/I^{(k)})$. With the notation which has been used in the proof of the forward implication, and by considering the fact that $\mathrm{Ass}_R(R/I^{(s)}) = \mathrm{Min}(I^s) = \mathrm{Min}(I)$ for all $s$, one can deduce that $\mathrm{Ass}_R(R/I^{(k)}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Without loss of generality, we need only demonstrate that $(I^{(k+1)}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I^{(1)}_{\mathfrak{p}_1}) = I^{(k)}_{\mathfrak{p}_1}$. Since $I_{\mathfrak{p}_1}$ has the strong persistence property, we obtain that $(I^{k+1}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I^k_{\mathfrak{p}_1}$. As $(I^s)_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ and $(I^{(s)})_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ for all $s$, one derives that $(I^{(k+1)}_{\mathfrak{p}_1} :_{R_{\mathfrak{p}_1}} I^{(1)}_{\mathfrak{p}_1}) = I^{(k)}_{\mathfrak{p}_1}$. This completes the proof. $\square$

To show the next corollary, one needs the following proposition.

**Proposition 5.** *Every irreducible primary monomial ideal has the strong persistence property. Especially, every prime monomial ideal has the strong persistence property.*

*Proof.* Fix $k \geq 1$, and assume that $Q = (x_{i_1}^{\alpha_1}, \ldots, x_{i_t}^{\alpha_t})$ is an irreducible primary monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ with $\alpha_1, \ldots, \alpha_t$ are positive integers and $\{x_{i_1}, \ldots, x_{i_t}\} \subseteq \{x_1, \ldots, x_n\}$. We prove that $(Q^{k+1} :_R Q) = Q^k$ for all positive integers $k$. Fix $k \geq 1$. Because $Q^k \subseteq (Q^{k+1} :_R Q)$, it remains to show that $(Q^{k+1} :_R Q) \subseteq Q^k$. To do this, one needs to verify that $(Q^{k+1} :_R (x_{i_1}^{\alpha_1})) \subseteq Q^k$. Here, inspired by [19, Remark 2.5], one can deduce the following equalities,

$$\begin{aligned}
Q^{k+1} :_R (x_{i_1}^{\alpha_1}) &= \Big( \sum_{\theta_1+\cdots+\theta_t=k+1} ((x_{i_1}^{\alpha_1})^{\theta_1} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}) \Big) :_R (x_{i_1}^{\alpha_1}) \\
&= \sum_{\theta_1+\cdots+\theta_t=k+1} ((x_{i_1}^{\alpha_1})^{\theta_1} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}) :_R (x_{i_1}^{\alpha_1}) \\
&= \sum_{\theta_1=0,\ \theta_2+\cdots+\theta_t=k+1} ((x_{i_2}^{\alpha_2})^{\theta_2} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}) \\
&+ \sum_{\theta_1\geq 1,\ \theta_1+\cdots+\theta_t=k+1} ((x_{i_1}^{\alpha_1})^{\theta_1-1} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}).
\end{aligned}$$

As $((x_{i_2}^{\alpha_2})^{\theta_2} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}) \subseteq Q^k$ with $\theta_2 + \cdots + \theta_t = k+1$ and $((x_{i_1}^{\alpha_1})^{\theta_1-1} \cdots (x_{i_t}^{\alpha_t})^{\theta_t}) \subseteq Q^k$ with $\theta_1 \geq 1,\ \theta_1 + \cdots + \theta_t = k+1$, we conclude that $(Q^{k+1} :_R (x_{i_1}^{\alpha_1})) \subseteq Q^k$, as required. $\square$

**Corollary 2.** *Let $I$ be an ideal in a commutative Noetherian ring $R$. Let $I = Q_1 \cap \cdots \cap Q_m \cap Q_{m+1} \cap \cdots \cap Q_r$ be a minimal primary decomposition of $I$ such that* $\mathrm{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$, *where $\mathfrak{p}_i = \sqrt{Q_i}$ for each $i = 1, \ldots, m$. If $Q_i$ has the strong persistence property for each $i = 1, \ldots, m$, then $I$ has the symbolic strong persistence property. In particular, every square-free monomial ideal has the symbolic strong persistence property.*

*Proof.* Fix $k \geq 1$. By virtue of Proposition 4, we get $I_{\mathfrak{p}_i} = (Q_i)_{\mathfrak{p}_i}$ for each $i = 1, \ldots, m$. Since $Q_i$ has the strong persistence property for each $i = 1, \ldots, m$, one has $(Q_i^{k+1} :_R Q_i) = Q_i^k$ for each $i = 1, \ldots, m$. Thus, $((Q_i)_{\mathfrak{p}_i}^{k+1} :_{R_{\mathfrak{p}_i}} (Q_i)_{\mathfrak{p}_i}) = (Q_i)_{\mathfrak{p}_i}^k$ for each $i = 1, \ldots, m$. This implies that $I_{\mathfrak{p}_i}$ has the strong persistence property for each $i = 1, \ldots, m$. Now, the claim follows from Theorem 5. The last assertion is a direct consequence from Proposition 5, which says that every prime monomial ideal has the strong persistence property. □

Along [21, Question 3.7], we present the following theorem.

**Theorem 6.** *Suppose that $I$ is a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. Also, for all $s \geq 2$, suppose that $\mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I) \cup \{\mathfrak{q}\}$ such that $\mathfrak{q}^{st}$ is the primary component of the embedded prime $\mathfrak{q}$, where*

$$t := \min\{|\{x_{j_1}, \ldots, x_{j_\ell}\}| \ : \ x_{j_1} \cdots x_{j_\ell} \in I\}.$$

*Then $I$ has the strong persistence property.*

*Proof.* To show the claim, it is sufficient to verify that $(I^{k+1} :_R I) \subseteq I^k$ for all $k$. Fix $k \geq 1$. Let $\mathrm{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Since $\mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I) \cup \{\mathfrak{q}\}$ and $\mathrm{Min}(I) = \mathrm{Ass}_R(R/I)$, we deduce from [28, Definition 4.3.22 and Proposition 4.3.25] and the assumption that $I^s = \mathfrak{p}_1^s \cap \cdots \cap \mathfrak{p}_r^s \cap \mathfrak{q}^{st}$ is a minimal primary decomposition of $I^s$ for all $s \geq 2$. Pick an arbitrary monomial $u$ in $(I^{k+1} :_R I)$. Fix $1 \leq j \leq r$. Since $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ for $1 \leq i \neq j \leq r$, this implies that one can choose an element such as $v_i \in \mathfrak{p}_i \setminus \cup_{j \neq i} \mathfrak{p}_j$. Let $\lambda$ be an element in $\mathfrak{p}_j$. Because $u \in (I^{k+1} :_R I)$, we have $uI \subseteq I^{k+1}$, and by $v_1 \cdots v_{j-1} \lambda v_{j+1} \cdots v_r \in \bigcap_{i=1}^r \mathfrak{p}_i$, one derives that $u\lambda v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \in \bigcap_{i=1}^r \mathfrak{p}_i^{k+1} \cap \mathfrak{q}^{(k+1)t}$. This yields that $u\lambda v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \in \mathfrak{p}_j^{k+1}$. Due to $v_i \notin \mathfrak{p}_j$ for $1 \leq i \neq j \leq r$, we obtain that $v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \notin \mathfrak{p}_j$. Thanks to $\mathfrak{p}_j^{k+1}$ is primary, one has $u\lambda \in \mathfrak{p}_j^{k+1}$, and hence $u \in (\mathfrak{p}_j^{k+1} :_R \mathfrak{p}_j)$. It follows from Proposition 5 that $\mathfrak{p}_j$ has the strong persistence property, and so $u \in \mathfrak{p}_j^k$. Accordingly, we have $u \in \bigcap_{i=1}^r \mathfrak{p}_i^k$. To finish the proof, one requires to establish $u \in \mathfrak{q}^{kt}$. Without loss of generality, let $x_1 \cdots x_t \in \bigcap_{i=1}^r \mathfrak{p}_i$. This implies that $ux_1 \cdots x_t \in I^{k+1}$. We thus have $ux_1 \cdots x_t \in \mathfrak{q}^{(k+1)t}$, and therefore there exists some monomial $h \in \mathcal{G}(\mathfrak{q}^{(k+1)t})$ such that $h | ux_1 \cdots x_t$. Assume that $h = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $u = x_1^{\beta_1} \cdots x_n^{\beta_n}$ with $\alpha_i \geq 0$ and $\beta_i \geq 0$ for each $i$. Since $h \in \mathcal{G}(\mathfrak{q}^{(k+1)t})$, this means that $h$ is a minimal generator of $\mathfrak{q}^{(k+1)t}$, and by considering the assumption that $\mathfrak{q}$ is a prime monomial ideal, one can conclude that $\alpha_1 + \cdots + \alpha_n = (k+1)t$. On account of $h | ux_1 \cdots x_t$, this yields that $(k+1)t \leq \beta_1 + \cdots + \beta_n + t$, and hence $kt \leq \beta_1 + \cdots + \beta_n$. Consequently, one can conclude that $u \in \mathfrak{q}^{kt}$. This implies that $u \in I^k$, and the proof is complete. □

It has already been proved that the cover ideal of any odd cycle graph satsfies the strong persistence property, see [21, Theorem 3.3]. As an application of Theorem 6, we re-prove this fact in Theorem 7. To do this, we need the following auxiliary lemma. Indeed, in Lemma 3, we give a minimal primary decomposition of the powers of the cover ideal of any odd cycle graph.

**Lemma 3.** *Let $C_{2n+1}$ be an odd cycle graph with $V(C_{2n+1}) = \{1, \dots, 2n+1\}$ and $E(C) = \{\{i, i+1\} \ : \ i = 1, \dots, 2n+1\}$, where $2n+2 = 1$. Then, for all $s \geq 2$,*

$$J(C_{2n+1})^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)},$$

*where $\mathfrak{m} = (x_1, \dots, x_{2n+1})$ is the unique homogeneous maximal ideal in the polynomial ring $R = K[x_1, \dots, x_{2n+1}]$ over a field $K$.*

*Proof.* Since $J(C_{2n+1}) = \bigcap_{i=1}^{2n+1}(x_i, x_{i+1})$, it is sufficient for us to prove that

$$(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}))^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)}. \tag{3.1}$$

Let $A$ (respectively, $B$) denote the ideal on the left-hand (respectively, right-hand) side of (3.1). Fix $s \geq 2$. We first show that $A \subseteq B$. As $\bigcap_{i=1}^{2n+1}(x_i, x_{i+1}) \subseteq (x_i, x_{i+1})$ for each $i = 1, \dots, 2n+1$, this implies that $(\bigcap_{i=1}^{2n+1}(x_i, x_{i+1}))^s \subseteq \bigcap_{i=1}^{2n+1}(x_i, x_{i+1})^s$. To complete the argument, it suffices to prove that $(\bigcap_{i=1}^{2n+1}(x_i, x_{i+1}))^s \subseteq \mathfrak{m}^{s(n+1)}$. To see this, consider a minimal generator $u$ in $(\bigcap_{i=1}^{2n+1}(x_i, x_{i+1}))^s$. We thus have $u = \prod_{i=1}^{s} g_i$, where each $g_i$ is a minimal generator of $\bigcap_{i=1}^{2n+1}(x_i, x_{i+1})$. By virtue of $\bigcap_{i=1}^{2n+1}(x_i, x_{i+1})$ is exactly the cover ideal of the odd cycle $C_{2n+1}$ and because of any minimal geneator of $J(C_{2n+1})$ corresponds to a minimal vertex cover of $C_{2n+1}$, and also by considering the fact that any minimal vertex cover of $C_{2n+1}$ has at least $n+1$ elements, we can deduce that $\deg u = \sum_{i=1}^{s} \deg g_i \geq s(n+1)$. This implies that $u \in \mathfrak{m}^{s(n+1)}$. Accordingly, one has $A \subseteq B$.

We now verify that $B \subseteq A$. For this purpose, select a minimal generator $u$ in $B$. Let $u := x_1^{\ell_1} \cdots x_{2n+1}^{\ell_{2n+1}}$ with $\ell_i \geq 0$ for each $i = 1, \dots, 2n+1$. Our strategy is to use [21, Lemma 3.2]. To accomplish this, one has to demonstrate that $\ell_i + \ell_{i+1} \geq s$ for eech $i = 1, \dots, 2n+1$ and $\sum_{i=1}^{2n+1}[(\ell_i + \ell_{i+1}) - s] \geq s$. Fix $1 \leq i \leq 2n+1$. It follows from $u \in B$ that $u \in (x_i, x_{i+1})^s$, and so $x_i^{\alpha_i} x_{i+1}^{s-\alpha_i} | u$ for some $0 \leq \alpha_i \leq s$. This implies that $\ell_i + \ell_{i+1} \geq s$. To finish the proof, we establish $\sum_{i=1}^{2n+1}[(\ell_i + \ell_{i+1}) - s] \geq s$. Since $u \in B$, one has $u \in \mathfrak{m}^{s(n+1)}$, and so $x_1^{\theta_1} \cdots x_{2n+1}^{\theta_{2n+1}} | u$ with $\sum_{i=1}^{2n+1} \theta_i = s(n+1)$. This gives that

$$\begin{aligned}
\sum_{i=1}^{2n+1} [(\ell_i + \ell_{i+1}) - s] &= \sum_{i=1}^{2n+1} (\ell_i + \ell_{i+1}) - s(2n+1) \\
&= 2 \sum_{i=1}^{2n+1} \ell_i - s(2n+1) \\
&\geq 2 \sum_{i=1}^{2n+1} \theta_i - s(2n+1) \\
&= s.
\end{aligned}$$

We therefore have $u \in A$, and thus $B \subseteq A$, as required. $\square$

**Theorem 7.** *The cover ideal of any odd cycle graph satsfies the strong persistence property.*

*Proof.* Let $C_{2n+1}$ be an odd cycle graph with $V(C_{2n+1}) = \{1, \dots, 2n+1\}$ and $E(C) = \{\{i, i+1\} : i = 1, \dots, 2n+1\}$, where $2n+2 = 1$. Fis $s \geq 2$. It follows from Lemma 3 that

$$J(C_{2n+1})^s = (\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}))^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)}, \tag{3.2}$$

where $\mathfrak{m} = (x_1, \dots, x_{2n+1})$ is the unique homogeneous maximal ideal in the polynomial ring $R = K[x_1, \dots, x_{2n+1}]$ over a field $K$. In other words, (3.2) is a minimal primary decomposition of $J(C_{2n+1})^s$. This gives rise to the following equality

$$\mathrm{Ass}_R(R/J(C_{2n+1})^s) = \mathrm{Ass}_R(R/J(C_{2n+1})) \cup \{\mathfrak{m}\}.$$

In the light of any minimal geneator of $J(C_{2n+1})$ corresponds to a minimal vertex cover of $C_{2n+1}$, and by remembering the fact that any minimal vertex cover of $C_{2n+1}$ has at least $n+1$ elements, and also $J(C_{2n+1}) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R(R/J(C_{2n+1}))} \mathfrak{p}$, one can derive

$$t = \min\{|\{x_{j_1}, \dots, x_{j_\ell}\}| : x_{j_1} \cdots x_{j_\ell} \in J(C_{2n+1})\} = n+1.$$

Now, Theorem 6 gives that $J(C_{2n+1})$ satisfes the strong persistence property, as claimed. □

To establish Lemma 4, we require to know the following proposition.

**Proposition 6.** *Suppose that $I_1$ and $I_2$ are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field $K$ such that $\mathcal{G}(I_1) \subseteq R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subseteq R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $m, 1 \leq m < n$. Then $(I_1 I_2)^{(k)} = I_1^{(k)} I_2^{(k)}$ for all $k \geq 1$.*

*Proof.* Fix $k \geq 1$. Since $I_1$ and $I_2$ are generated by disjoint sets of variables, [11, Lemma 1.1] implies that $I_1 I_2 = I_1 \cap I_2$. In particular, we have $\mathrm{Min}(I_1 \cap I_2) = \mathrm{Min}(I_1) \cup \mathrm{Min}(I_2)$. Hence, one can conclude the following equalities

$$\begin{aligned}
(I_1 \cap I_2)^{(k)} &= \bigcap_{\mathfrak{p} \in \mathrm{Min}(I_1 \cap I_2)} ((I_1 \cap I_2)^k R_\mathfrak{p} \cap R) \\
&= \bigcap_{\mathfrak{p} \in \mathrm{Min}(I_1)} ((I_1)^k R_\mathfrak{p} \cap R) \cap \bigcap_{\mathfrak{p} \in \mathrm{Min}(I_2)} ((I_2)^k R_\mathfrak{p} \cap R) \\
&= I_1^{(k)} \cap I_2^{(k)}.
\end{aligned}$$

We therefore obtain the following equalities

$$(I_1 I_2)^{(k)} = (I_1 \cap I_2)^{(k)} = I_1^{(k)} \cap I_2^{(k)} = I_1^{(k)} I_2^{(k)}.$$

This completes our argument. □

The theorem below is crucial for us to show Lemma 4.

**Theorem 8.** *([9, Theorem 3.4])* $(I+J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}$.

In order to establish Theorem 9, one needs to prove the following lemma.

**Lemma 4.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field $K$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subseteq R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subseteq R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. Then $I^{(s)} = \bigcap_{i=1}^{s}(I_1^{(i)} + I_2^{(s+1-i)})$ for all $s \in \mathbb{N}$.*

*Proof.* Fix $s \in \mathbb{N}$, and set $L_t := I_1^{(t)} + \sum_{i=0}^{t-1} I_1^{(i)} \cap I_2^{(s-i)}$ with $1 \leq t \leq s$. In what follows, want to prove that $L_t = \bigcap_{i=1}^{t}(I_1^{(i)} + I_2^{(s+1-i)})$. To achieve this, we proceed by induction on $t$. One can easily see that the assertion is true for the case in which $t = 1$. Now, suppose, inductively, that $t > 1$ and that the result has been proved for all $r$ less than $t$ with $t \leq s$. It follows also from the inductive hypothesis that $L_{t-1} = \bigcap_{i=1}^{t-1}(I_1^{(i)} + I_2^{(s+1-i)})$. It is well-known that if $\alpha \leq \beta$, then $I^{(\beta)} \subseteq I^{(\alpha)}$. Hence, $I_1^{(t)} \subseteq I_1^{(t-1)}$, and so $I_1^{(t)} \cap I_1^{(t-1)} = I_1^{(t)}$. In addition, if $0 \leq i \leq t-2 \leq s$, then $s+1-t \leq s-i$, and thus $I_2^{(s-i)} \subseteq I_2^{(s+1-t)}$. This implies $\sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \subseteq I_2^{(s+1-t)}$, and so $I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} = \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}$. Now, by considering the fact that $I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \subseteq \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}$, we get

$$
\begin{aligned}
&I_1^{(t)} \cap I_1^{(t-1)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} + I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\
&= I_1^{(t)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}.
\end{aligned}
$$

Hence, one can derive the following equalities

$$
\begin{aligned}
L_t &= I_1^{(t)} + \sum_{i=0}^{t-1} I_1^{(i)} \cap I_2^{(s-i)} \\
&= I_1^{(t)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\
&= I_1^{(t)} \cap I_1^{(t-1)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} \\
&+ I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} + I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap (I_1^{(t-1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}) \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap L_{t-1} \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap \bigcap_{i=1}^{t-1}(I_1^{(i)} + I_2^{(s+1-i)}) \\
&= \bigcap_{i=1}^{t}(I_1^{(i)} + I_2^{(s+1-i)}).
\end{aligned}
$$

This completes the inductive step, and so the claim has been proved by induction. Especially, one can conclude that $L_s = \bigcap_{i=1}^{s}(I_1^{(i)} + I_2^{(s+1-i)})$, and hence

$$\bigcap_{i=1}^{s}(I_1^{(i)} + I_2^{(s+1-i)}) = I_1^{(s)} + \sum_{i=0}^{s-1} I_1^{(i)} \cap I_2^{(s-i)} = \sum_{i=0}^{s} I_1^{(i)} \cap I_2^{(s-i)}.$$

On account of $\mathcal{G}(I_1) \subseteq R_1 = K[x_1, \ldots, x_m]$ and $\mathcal{G}(I_2) \subseteq R_2 = K[x_{m+1}, \ldots, x_n]$, Proposition 6 yields that $I_1^{(i)} \cap I_2^{(s-i)} = I_1^{(i)} I_2^{(s-i)}$ for each $i = 0, \ldots, s$. Therefore, we get $\bigcap_{i=1}^{s}(I_1^{(i)} + I_2^{(s+1-i)}) = \sum_{i=0}^{s} I_1^{(i)} I_2^{(s-i)}$. Finally, it follows from Theorem 8 that $\bigcap_{i=1}^{s}(I_1^{(i)} + I_2^{(s+1-i)}) = (I_1 + I_2)^{(s)} = I^{(s)}$, as required. $\square$

We are now ready to express and prove Theorem 9.

**Theorem 9.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ and $\mathcal{G}(I) = G_1 \cup \cdots \cup G_r$ such that*

$$\{x_s : x_s | m \text{ for some } m \in G_i\} \cap \{x_t : x_t | m \text{ for some } m \in G_j\} = \emptyset,$$

*for all $1 \leq i \neq j \leq r$. Then $(G_i)R$ has the symbolic strong persistence property for some $1 \leq i \leq r$ if and only if $I$ has the symbolic strong persistence property.*

*Proof.* It is enough to show our claim only for $r = 2$. To accomplish this, let $I$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subseteq R_1 = K[x_1, \ldots, x_m]$ and $\mathcal{G}(I_2) \subseteq R_2 = K[x_{m+1}, \ldots, x_n]$ for some positive integer $1 \leq m < n$. We first show the forward implication. Without loss of generality, assume that $I_1$ has the symbolic strong persistence property. Also, let $\mathcal{G}(I_1^{(1)}) = \{v_1, \ldots, v_t\}$. Our main aim is to prove that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k$. Now, fix $k \in \mathbb{N}$. By virtue of the assumption, one has

$(I_1^{(i)} :_R I_1^{(1)}) = I_1^{(i-1)}$ for all $i$. By applying Lemma 4, we have the following equalities

$$\begin{aligned}
(I^{(k+1)} :_R I_1^{(1)}) &= (\bigcap_{i=1}^{k+1}(I_1^{(i)} + I_2^{(k+2-i)}) :_R I_1^{(1)}) \\
&= \bigcap_{i=1}^{k+1}\bigcap_{j=1}^{t}((I_1^{(i)} :_R v_j) + (I_2^{(k+2-i)} :_R v_j)) \\
&= \bigcap_{i=1}^{k+1}\bigcap_{j=1}^{t}((I_1^{(i)} :_R v_j) + I_2^{(k+2-i)}) \\
&= \bigcap_{i=1}^{k+1}(I_2^{(k+2-i)} + \cap_{j=1}^{t}(I_1^{(i)} :_R v_j)) \\
&= \bigcap_{i=1}^{k+1}(I_2^{(k+2-i)} + (I_1^{(i)} :_R I_1^{(1)})) \\
&= \bigcap_{i=2}^{k+1}(I_2^{(k+2-i)} + I_1^{(i-1)}) \\
&= \bigcap_{\ell=1}^{k}(I_2^{(k+1-\ell)} + I_1^{(\ell)}) \\
&= I^{(k)}.
\end{aligned}$$

Thanks to $I^{(k)} \subseteq (I^{(k+1)} :_R I^{(1)})$ and $(I^{(k+1)} :_R I^{(1)}) \subseteq (I^{(k+1)} :_R I_1^{(1)})$, one can conclude that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. This completes the proof.

To establish the converse implication, suppose, on the contrary, that $I_1$ and $I_2$ do not satisfy the symbolic strong persistence property. This implies that there exist a positive integer $k_1$ (respectively, $k_2$) and a monomial $m_1$ (respectively, $m_2$) such that $m_1 \in \mathcal{G}(I_1^{(k_1+1)} :_R I_1^{(1)}) \setminus \mathcal{G}(I_1^{(k_1)})$ (respectively, $m_2 \in \mathcal{G}(I_2^{(k_2+1)} :_R I_2^{(1)}) \setminus \mathcal{G}(I_2^{(k_2)})$). Note that, in general, if $\alpha \leq \beta$, then $I^{(\beta)} \subseteq I^{(\alpha)}$. Take the nonnegative integer $a_1$ (respectively, $a_2$) such that $m_1 \in I_1^{(a_1)} \setminus I_1^{(a_1+1)}$ (respectively, $m_2 \in I_2^{(a_2)} \setminus I_2^{(a_2+1)}$). This gives that $a_1 \leq k_1 - 1$ (respectively, $a_2 \leq k_2 - 1$). Put $m := m_1m_2$ and $b := a_1 + a_2$. Thus, one has $m \in I^{(b)}$. We claim that $m \notin I^{(b+1)}$. Suppose, on the contrary, that $m \in I^{(b+1)}$. Since, by Theorem 8, $I^{(b+1)} = \sum_{\ell=0}^{b+1} I_1^{(\ell)} I_2^{(b+1-\ell)}$, one obtains that $m \in I_1^{(\ell)} I_2^{(b+1-\ell)}$ for some $0 \leq \ell \leq b+1$. We thus have there exist $u_1 \in \mathcal{G}(I_1^{(\ell)})$ and $u_2 \in \mathcal{G}(I_2^{(b+1-\ell)})$ such that $u_1u_2|m_1m_2$. Consequently, $u_1|m_1m_2$ and $u_2|m_1m_2$. Because $\gcd(u_1, m_2) = 1$ and $\gcd(u_2, m_1) = 1$, one has $u_1|m_1$ and $u_2|m_2$. This implies that $m_1 \in I_1^{(\ell)}$ and $m_2 \in I_2^{(b+1-\ell)}$. It follows from $m_1 \in I_1^{(a_1)} \setminus I_1^{(a_1+1)}$ and $m_2 \in I_2^{(a_2)} \setminus I_2^{(a_2+1)}$ that $a_1 \geq \ell$ and $a_2 \geq b + 1 - \ell$. This gives that $a_1 + a_2 \geq b + 1$, which contradicts the fact that $b = a_1 + a_2$. Accordingly, $m \notin I^{(b+1)}$. By setting $s_1 := k_1 + 1 + a_2$ and $s_2 := a_1 + k_2 + 1$, one can easily see that $s_i \geq a_1 + a_2 + 2$ for each $i = 1, 2$. Hence, $s := \min\{s_1, s_2\} \geq a_1 + a_2 + 2$. In addition, it is routine to check that if $u \in \mathcal{G}(I_1^{(1)})$ (respectively, $u \in \mathcal{G}(I_2^{(1)})$), then $mu \in I^{(s_1)}$ (respectively, $mu \in I^{(s_2)}$). Therefore, we get $m \in (I^{(s)} :_R I^{(1)}) \setminus I^{(s-1)}$, which contradicts the assumption that $I$ has the symbolic strong persistence property. This finishes the proof. $\square$

The following lemma says that, under certain condition, a monomial ideal has the symbolic strong persistence property if and only if its monomial multiple has the symbolic strong persistence property.

**Lemma 5.** *Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$, and $h$ be a monomial in $R$. Also, let $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$. Then $I$ has the symbolic strong persistence property if and only if $hI$ has the symbolic strong persistence property.*

*Proof.* To simplify notation, set $L := hI$. We first assume that $I$ has the symbolic strong persistence property. This means that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k$. In order to complete the argument, one has to show that $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$ for all $k$. Fix $k \geq 1$. It follows readily from Proposition 6 that $(hI)^{(k)} = h^k I^{(k)}$. In view of $(L^{(k+1)} :_R h) = (h^{k+1} I^{(k+1)} :_R h) = h^k I^{(k+1)}$, one can conclude that $(L^{(k+1)} :_R L^{(1)}) = (h^k I^{(k+1)} :_R I^{(1)})$. By virtue of $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$, we get $\gcd(h, v) = 1$ for all $v \in \mathcal{G}(I^{(1)})$. The assumption yields that $(L^{(k+1)} :_R L^{(1)}) = h^k I^{(k)} = (hI)^{(k)}$. Therefore, $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$, and the proof is over.

To establish the converse implication, suppose that $L$ has the symbolic strong persistence property. Want to prove that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k$. To do this, fix $k \geq 1$. Because of $L$ has the symbolic strong persistence property, we get $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$, and hence $(h^{k+1} I^{(k+1)} :_R hI^{(1)}) = h^k I^{(k)}$. This yields that $(h^k I^{(k+1)} :_R I^{(1)}) = h^k I^{(k)}$. Moreover, a similar argument gives that $(h^k I^{(k+1)} :_R I^{(1)}) = h^k (I^{(k+1)} :_R I^{(1)})$, and so $h^k (I^{(k+1)} :_R I^{(1)}) = h^k I^{(k)}$. This implies that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$, that is, $I$ has the symbolic strong persistence property, as claimed. □

## 4 Strong persistence property of the cover ideals

The main aim of this section is to explore the strong persistence property of the cover ideal of the union of two finite simple graphs. To accomplish this, we have to recall the following results.

The following lemma examines the relation between assoicated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex.

A repeated application of [21, Theorem 2.5] yields the following lemma:

**Lemma 6.** *([22]) Let $G = (V(G), E(G))$ be a finite simple connected graph and $T$ be a tree such that $|V(G) \cap V(T)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(T)$ and $E(L) := E(G) \cup E(T)$. Then*

$$\operatorname{Ass}_R(R/J(L)^s) = \operatorname{Ass}_{R_1}(R_1/J(G)^s) \cup \operatorname{Ass}_{R_2}(R_2/J(T)^s),$$

*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(T)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$.*

The next lemma explores the relation between assoicated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only a path in common. In fact, a repeated application of Lemma 6 gives the following lemma:

**Lemma 7.** *([22]) Let $G = (V(G), E(G))$ be a finite simple connected graph, $T_1, \dots, T_r$ be some trees with $V(G) \cap V(T_i) = \{v_i\}$ for each $i = 1, \dots, r$, and $P = (V(P), E(P))$ be a path with $V(P) = \{v_1, \dots, v_r, v_{r+1}, \dots, v_m\} \subseteq V(G)$, and*

$$E(P) = \{\{v_i, v_{i+1}\} \ : \ \text{for } i = 1, \dots, m-1\} \subseteq E(G).$$

*Let $T = (V(T), E(T))$ be the tree with*

$$V(T) = \Big(\bigcup_{i=1}^{r} V(T_i)\Big) \cup V(P) \text{ and } E(T) = \Big(\bigcup_{i=1}^{r} E(T_i)\Big) \cup E(P).$$

*Also, let $L = (V(L), E(L))$ be the finite simple graph such that*

$$V(L) := V(G) \cup V(T) \text{ and } E(L) := E(G) \cup E(T).$$

*Then*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R'}(R'/J(G)^s) \cup \mathrm{Ass}_{R''}(R''/J(T)^s),$$

*for all $s$, where $R' = K[x_\alpha : \alpha \in V(G)]$, $R'' = K[x_\alpha : \alpha \in V(T)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$.*

In the next theorem, we turn our attention to study the strong persistence property of the cover ideal of the union of two finite simple connected graphs.

**Theorem 10.** *([22]) Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $J(G)$ and $J(H)$ have the strong persistence property. Also, let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$, $E(L) := E(G) \cup E(H)$, and*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s),$$

*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$. Then, under each of the following cases, $J(L)$ has the strong persistence property.*

*(i)* $V(G) \cap V(H) = \{v\}$.

*(ii)* $V(G) \cap V(H) = \{v, w\}$ *and* $E(G) \cap E(H) = \{\{v, w\}\}$.

*(iii)* $V(G) \cap V(H) = \{v, w, z\}$ *and* $E(G) \cap E(H) = \{\{v, w\}, \{w, z\}\}$.

To verify Theorem 11, one needs to know the following auxiliary propositions as well.

**Proposition 7.** *([22]) Let $I$ be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field $K$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ and $\mathrm{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. Then, the following statements hold.*

*(i) If $x_i | u_t$ for some $i$, $1 \leq i \leq n$, and for some $t$, $1 \leq t \leq m$, then there exists some $j$, $1 \leq j \leq s$, such that $x_i \in \mathfrak{p}_j$.*

*(ii) If $x_i \in \mathfrak{p}_j$ for some $i$, $1 \leq i \leq n$, and for some $j$, $1 \leq j \leq s$, then there exists some $t$, $1 \leq t \leq m$, such that $x_i | u_t$.*

*Especially,* $\bigcup_{j=1}^{s} \mathrm{supp}(\mathfrak{p}_j) = \bigcup_{t=1}^{m} \mathrm{supp}(u_t)$.

**Proposition 8.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s) \subseteq \mathrm{Ass}_R(R/J(L)^s).$$

*Proof.* To simplify our notation, put $I_1 := J(G)$, $I_2 := J(H)$, and $I := J(L)$. Take an arbitrary element $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ in $\mathrm{Ass}_{R_1}(R_1/I_1^s) \cup \mathrm{Ass}_{R_2}(R_2/I_2^s)$ with $\{i_1, \ldots, i_r\} \subseteq V(L)$. Let $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/I_1^s)$. In view of Proposition 7, one has $\mathrm{supp}(\mathfrak{p}) \subseteq \bigcup_{u \in \mathcal{G}(I_1)} \mathrm{supp}(u)$. This leads to $\{i_1, \ldots, i_r\} \subseteq V(G)$. It follows from [6, Lemma 2.11] that $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(G_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \ldots, x_{i_r}]$ and $G_{\mathfrak{p}}$ is the induced subgraph of $G$ on the vertex set $\{i_1, \ldots, i_r\} \subseteq V(G)$. Thanks to $G_{\mathfrak{p}} = L_{\mathfrak{p}}$, we get $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$. Once again, using [6, Lemma 2.11] yields that $\mathfrak{p} \in \mathrm{Ass}_R(R/I^s)$. A similar discussion shows that if $\mathfrak{p} \in \mathrm{Ass}_{R_2}(R_2/I_2^s)$, then $\mathfrak{p} \in \mathrm{Ass}_R(R/I^s)$. Therefore, one can conclude that

$$\mathrm{Ass}_{R_1}(R_1/I_1^s) \cup \mathrm{Ass}_{R_2}(R_2/I_2^s) \subseteq \mathrm{Ass}_R(R/I^s),$$

as claimed. □

The following theorem examines the relation between assoicated primes of powers of the cover ideal of the union of two finite simple graphs with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex.

**Theorem 11.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $|V(G) \cap V(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s),$$

*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$. In particular, if $J(G)$ and $J(H)$ have the strong persistence property, then $J(L)$ has the strong persistence property.*

*Proof.* Fix $s \geq 1$, and let $V(G) \cap V(H) = \{v\}$. If $G$ or $H$ is a tree, then the claim follows rapidly from Lemma 6 and Theorem 10. Hence, we can assume that neither $G$ nor $H$ is a tree. On account of Proposition 8, we gain the following containment

$$\mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s) \subseteq \mathrm{Ass}_R(R/J(L)^s).$$

In order to complete the proof, it is sufficient to verify the reverse inclusion. To achieve this, pick an arbitrary element $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ in $\mathrm{Ass}_R(R/J(L)^s)$. Based on Proposition 7, we have $\{i_1, \ldots, i_r\} \subseteq V(L)$. Moreover, it follows from [6, Lemma 2.11] that $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \ldots, x_{i_r}]$ and $L_{\mathfrak{p}}$ is the induced subgraph of $L$

on the vertex set $\{i_1, \ldots, i_r\}$. Hereafter, we assume that $\mathfrak{p}$ is the maximal ideal in the polynomial ring $R = K[\mathfrak{p}]$. To simplify the notation, put $\Gamma := \{x_c : c \in V(G) \setminus \{v\}\}$ and $\Lambda := \{x_c : c \in V(H) \setminus \{v\}\}$. Take an arbitrary element $u$ in $\mathcal{G}(J(L))$. Because of neither $G$ nor $H$ is a tree, one can conclude that neither $G$ nor $H$ is a star graph or an edge, and so $\mathrm{supp}(u) \cap \Gamma \neq \emptyset$ and $\mathrm{supp}(u) \cap \Lambda \neq \emptyset$. Since $J(L) = J(G) \cap J(H)$, this implies that $u = ab/\gcd(a, b)$ for some $a \in \mathcal{G}(J(G))$ and $b \in \mathcal{G}(J(H))$. Hence, one may consider the following cases:

**Case 1.** $a = f$ and $b = g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Hence, one has $u = fg$.

**Case 2.** $a = x_v f$ and $b = g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). We thus have $u = x_v fg$.

**Case 3.** $a = f$ and $b = x_v g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Therefore, one obtains that $u = x_v fg$.

**Case 4.** $a = x_v f$ and $b = x_v g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Due to $\gcd(a, b) = x_v$, one can conclude that $u = x_v fg$.

The argument above gives that if $u \in \mathcal{G}(J(L))$, then $u = Afg$, where $A | x_v$ and $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$) such that $Af \in J(G)$ (respectively, $Ag \in J(H)$). Since $\mathfrak{p} \in \mathrm{Ass}_R(R/J(L)^s)$, we get there exists some monomial $h$ in $R$ such that $\mathfrak{p} = (J(L)^s :_R h)$. Assume that $h = h_1 h_2 x_v^\rho$ with $h_1$ (respectively, $h_2$) is a monomial in the variables $\Gamma$ (respectively, $\Lambda$), and $\rho$ is a nonnegative integer. It follows readily from [15, Lemma 2.1] that $(J(G)^s :_R h_1 h_2 x_v^\rho) = (J(G)^s :_R h_1 x_v^\rho)$ and $(J(H)^s :_R h_1 h_2 x_v^\rho) = (J(H)^s :_R h_2 x_v^\rho)$. Since $(J(L)^s :_R h) \subseteq (J(G)^s :_R h)$ (respectively, $(J(L)^s :_R h) \subseteq (J(H)^s :_R h)$), one can conclude that $\mathfrak{p} \subseteq (J(G)^s :_R h_1 x_v^\rho)$ (respectively, $\mathfrak{p} \subseteq (J(H)^s :_R h_2 x_v^\rho)$). Our aim is to demonstrate that $h_1 x_v^\rho \notin J(G)$ or $h_2 x_v^\rho \notin J(H)$. Suppose, on the contrary, that $h_1 x_v^\rho \in J(G)$ and $h_2 x_v^\rho \in J(H)$. It follows from $h_1 x_v^\rho \in J(G)^s$ (respectively, $h_2 x_v^\rho \in J(H)^s$) that there exist square-free monomials $M_1, f_1, \ldots, f_s$ (respectively, $M_2, g_1, \ldots, g_s$) in the variables $\Gamma$ (respectively, $\Lambda$), and $A_1, \ldots, A_s$ (respectively, $A'_1, \ldots, A'_s$) with $A_i | x_v$ (respectively, $A'_i | x_v$) and $A_i f_i \in J(G)$ (respectively, $A'_i g_i \in J(H)$) for each $i = 1, \ldots, s$, such that

$$h_1 x_v^\rho = (\prod_{i=1}^{s} A_i f_i) M_1 x_v^\theta \ (\text{respectively}, h_2 x_v^\rho = (\prod_{i=1}^{s} A'_i g_i) M_2 x_v^\lambda),$$

for some nonnegative integer $\theta$ (respectively, $\lambda$). We thus have $h_1 = (\prod_{i=1}^s f_i) M_1$, $h_2 = (\prod_{i=1}^s g_i) M_2$, and $(\prod_{i=1}^s A_i) x_v^\theta = x_v^\rho = (\prod_{i=1}^s A'_i) x_v^\lambda$. Accordingly, one has

$$h_1 h_2 x_v^\rho = (\prod_{i=1}^{s} f_i) M_1 (\prod_{i=1}^{s} g_i) M_2 x_v^\rho = (\prod_{i=1}^{s} f_i g_i) M_1 M_2 x_v^\rho.$$

Since $(\prod_{i=1}^s A_i) x_v^\theta = x_v^\rho = (\prod_{i=1}^s A'_i) x_v^\lambda$, we get $h_1 h_2 x_v^\rho \in J(L)^s$, and so $h \in J(L)^s$; this contradicts the fact that $\mathfrak{p} = (J(L)^s :_R h)$. We therefore gain $h_1 x_v^\rho \notin J(G)$ or $h_2 x_v^\rho \notin J(H)$. As $\mathfrak{p}$ is the maximal ideal, one has $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/J(G)^s)$ or $\mathfrak{p} \in \mathrm{Ass}_{R_2}(R_2/J(H)^s)$, and so we have the following equality

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s), \tag{4.1}$$

as required. The last claim is an immediate consequence of Theorem 10 and (4.1). □

To demonstrate Theorem 12, one needs to apply the following lemma.

**Lemma 8.** *Let $G = (V(G), E(G))$ be a finite simple connected graph and $H$ be a triangle graph such that $|V(G) \cap V(H)| = 2$ and $|E(G) \cap E(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s),$$

*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$. In particular, if $J(G)$ has the strong persistence property, then $J(L)$ has the strong persistence property.*

*Proof.* Let $V(H) = \{v, w, z\}$, $V(G) \cap V(H) = \{v, w\}$, and $E(G) \cap E(H) = \{\{v, w\}\}$. Since the claim is true for the case $s = 1$, we only argue for all $s \geq 2$. Fix $s \geq 2$. It follows at once from Proposition 8 the containment below

$$\mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s) \subseteq \mathrm{Ass}_R(R/J(L)^s).$$

For completing the proof, it is enough for us to prove the reverse inclusion. To accomplish this, select an arbitrary element $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ in $\mathrm{Ass}_R(R/J(L)^s)$. One can conclude from Proposition 7 that $\{i_1, \ldots, i_r\} \subseteq V(L)$. If $z \notin \{i_1, \ldots, i_r\}$, then $\{i_1, \ldots, i_r\} \subseteq V(G)$, and [6, Lemma 2.11] implies that $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/J(G)^s)$. We thus let $z \in \{i_1, \ldots, i_r\}$. In view of [3, Lemma 2.4], $L_\mathfrak{p}$ is a connected graph, where $L_\mathfrak{p}$ denotes the induced graph on $\{i_1, \ldots, i_r\}$. This leads to the following cases:

**Case 1.** $v \in \{i_1, \ldots, i_r\}$ but $w \notin \{i_1, \ldots, i_r\}$ (or $w \in \{i_1, \ldots, i_r\}$ but $v \notin \{i_1, \ldots, i_r\}$). We only consider the case $v \in \{i_1, \ldots, i_r\}$ but $w \notin \{i_1, \ldots, i_r\}$, while the other case is proved similarly. Because $\mathfrak{p} \in \mathrm{Ass}_R(R/J(L)^s)$, [6, Lemma 2.11] concludes that $\mathfrak{p} \in \mathrm{Ass}_R(R/J(L_\mathfrak{p})^s)$. Since $w \notin \{i_1, \ldots, i_r\}$, one derives that $V(L_\mathfrak{p}) = V(G_\mathfrak{p}) \cup \{v, z\}$ and $E(L_\mathfrak{p}) = E(G_\mathfrak{p}) \cup \{\{v, z\}\}$. By [21, Theorem 2.5], we have $\mathrm{Ass}(J(L_\mathfrak{p})^s) = \mathrm{Ass}(J(G_\mathfrak{p})^s) \cup \{(x_v, x_z)\}$. If $\mathfrak{p} = (x_v, x_z)$, then $\mathfrak{p} \in \mathrm{Min}(J(H))$, and hence $\mathfrak{p} \in \mathrm{Ass}_{R_2}(R_2/J(H)^s)$. If $\mathfrak{p} \in \mathrm{Ass}(J(G_\mathfrak{p})^s)$, then [6, Lemma 2.11] yields that $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/J(G)^s)$, and so $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s)$.

**Case 2.** $v, w \in \{i_1, \ldots, i_r\}$. Without loss of generality, one may assume that $i_1 = v$, $i_2 = w$, and $i_3 = z$. Based on [6, Corollary 4.5], the associated primes of $J(L)^s$ will correspond to critical chromatic subgraphs of size $s + 1$ in the $s$-th expansion of $L$. This means that one can take the induced subgraph on the vertex set $\{i_1, \ldots, i_r\}$, and then form the $s$-th expansion on this induced subgraph, and within this new graph find a critical $(s + 1)$-chromatic graph. Thanks to $z$ is only connected to $v$ and $w$ in the graph $L$, and since this induced subgraph is critical, if we remove the vertex $z$, we can color the resulting graph with $s$ colors. Also, by virtue of [4, Theorem 14.6], the vertex $z$ has to be adjacent to at least $s$ vertices. But the only things $z$ is adjacent to are the shadows of $z$ and the shadows of $v$ and $w$, and hence one has a clique among these vertices. We thus gain that $z$ and its neighbors will form a clique of size $s + 1$. On account of a clique is a critical graph, this gives that we do not need any element of $\{i_4, \ldots, i_r\}$ or their shadows when making the critical $(s+1)$-chromatic graph. This implies that $\mathfrak{p} = (x_v, x_w, x_z)$. Due to [21, Proposition 3.6], one can conclude that $(x_v, x_w, x_z) \in \mathrm{Ass}_{R_2}(R_2/J(H)^s)$.

Accordingly, we have the following equality

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s). \tag{4.2}$$

Since the cover ideal of any triangle has the strong persistence property, the last assertion is a straightforward consequence of Theorem 10 and (4.2). $\square$

The next theorem probes the relation between assoicated primes of powers of the cover ideal of the union of two finite simple connected graphs with the assoicated primes of powers of the cover ideals of each of them, under the condition that they have only one edge in common.

**Theorem 12.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $|V(G) \cap V(H)| = 2$ and $|E(G) \cap E(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s),$$

*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$. In particular, if $J(G)$ and $J(H)$ have the strong persistence property, then $J(L)$ has the strong persistence property.*

*Proof.* Fix $s \geq 1$. Suppose that $V(G) \cap V(H) = \{v, w\}$ and $E(G) \cap E(H) = \{\{v, w\}\}$. If $G$ or $H$ is a tree or triangle, then the assertion is true by virtue of Lemma 7 and Theorem 8. We therefore can assume that neither $G$ nor $H$ is a tree or triangle. As a direct consequence of Proposition 8, we have the following containment

$$\mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s) \subseteq \mathrm{Ass}_R(R/J(L)^s).$$

To finish the argument, one has to establish the reverse inclusion. To do this, choose an arbitrary element $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ in $\mathrm{Ass}_R(R/J(L)^s)$. By Proposition 7, one has $\{i_1, \ldots, i_r\} \subseteq V(L)$. On account of [6, Lemma 2.11], one can derive that $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \ldots, x_{i_r}]$ and $L_{\mathfrak{p}}$ is the induced subgraph of $L$ on the vertex set $\{i_1, \ldots, i_r\}$. We thus assume that $\mathfrak{p}$ is the maximal ideal in the polynomial ring $R = K[\mathfrak{p}]$. To simplify the notation, set

$$\Gamma := \{x_c : c \in V(G) \setminus \{v, w\}\} \text{ and } \Lambda := \{x_c : c \in V(H) \setminus \{v, w\}\}.$$

Pick an arbitrary element $u$ in $\mathcal{G}(J(L))$. Due to neither $G$ nor $H$ is a tree or triangle, we can deduce that $\mathrm{supp}(u) \cap \Gamma \neq \emptyset$ and $\mathrm{supp}(u) \cap \Lambda \neq \emptyset$. It follows also from $J(L) = J(G) \cap J(H)$ that $u = ab/\gcd(a, b)$ for some $a \in \mathcal{G}(J(G))$ and $b \in \mathcal{G}(J(H))$. Consequently, one of the following cases occurs:

**Case 1.** $a = x_v f$ and $b = x_v g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Because $\gcd(a, b) = x_v$, we have $u = x_v fg$.

**Case 2.** $a = x_v f$ and $b = x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Since $\gcd(f, g) = 1$, this implies that $u = x_v x_w fg$.

**Case 3.** $a = x_v f$ and $b = x_v x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). By $\gcd(a, b) = x_v$, this yields that $u = x_v x_w fg$.

**Case 4.** $a = x_w f$ and $b = x_v g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Similarly to Case 2, one has $u = x_v x_w fg$.

**Case 5.** $a = x_w f$ and $b = x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). It follows form $\gcd(a, b) = x_w$ that $u = x_w fg$.

**Case 6.** $a = x_w f$ and $b = x_v x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Since $\gcd(a, b) = x_w$, we thus have $u = x_v x_w fg$.

**Case 7.** $a = x_v x_w f$ and $b = x_v g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). Similarly to Case 6, one can conclude that $u = x_v x_w fg$.

**Case 8.** $a = x_v x_w f$ and $b = x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). On account of $\gcd(a, b) = x_w$, this gives that $u = x_v x_w fg$.

**Case 9.** $a = x_v x_w f$ and $b = x_v x_w g$, where $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$). According to $\gcd(a, b) = x_v x_w$, we get $u = x_v x_w fg$.

It follows from the discussion above that if $u \in \mathcal{G}(J(L))$, then $u = Afg$, where $A \neq 1$, $A | x_v x_w$, and $f$ (respectively, $g$) is a square-free monomial in the variables $\Gamma$ (respectively, $\Lambda$) such that $Af \in J(G)$ (respectively, $Ag \in J(H)$). Since $\mathfrak{p} \in \mathrm{Ass}_R(R/J(L)^s)$, we obtain that $\mathfrak{p} = (J(L)^s :_R h)$ for some monomial $h$ in $R$. Suppose that $h = h_1 h_2 x_v^{\rho} x_w^{\delta}$ with $h_1$ (respectively, $h_2$) is a monomial in the variables $\Gamma$ (respectively, $\Lambda$), and $\rho$ (respectively, $\delta$) is a nonnegative integer. One can concludes from [15, Lemma 2.1] that $(J(G)^s :_R h_1 h_2 x_v^{\rho} x_w^{\delta}) = (J(G)^s :_R h_1 x_v^{\rho} x_w^{\delta})$ and $(J(H)^s :_R h_1 h_2 x_v^{\rho} x_w^{\delta}) = (J(H)^s :_R h_2 x_v^{\rho} x_w^{\delta})$. Thanks to

$$(J(L)^s :_R h) \subseteq (J(G)^s :_R h) \text{ (respectively, } (J(L)^s :_R h) \subseteq (J(H)^s :_R h)),$$

this implies that $\mathfrak{p} \subseteq (J(G)^s :_R h_1 x_v^{\rho} x_w^{\delta})$ (respectively, $\mathfrak{p} \subseteq (J(H)^s :_R h_2 x_v^{\rho} x_w^{\delta})$). Our purpose is to verify $h_1 x_v^{\rho} x_w^{\delta} \notin J(G)$ or $h_2 x_v^{\rho} x_w^{\delta} \notin J(H)$. Assume to the contrary that $h_1 x_v^{\rho} x_w^{\delta} \in J(G)$ and $h_2 x_v^{\rho} x_w^{\delta} \in J(H)$. Because of $h_1 x_v^{\rho} x_w^{\delta} \in J(G)^s$ (respectively, $h_2 x_v^{\rho} x_w^{\delta} \in J(H)^s$), we have there exist square-free monomials $M_1, f_1, \ldots, f_s$ (respectively, $M_2, g_1, \ldots, g_s$) in the variables $\Gamma$ (respectively, $\Lambda$), and $A_1, \ldots, A_s$ (respectively, $A'_1, \ldots, A'_s$) with $A_i \neq 1$ (respectively, $A'_i \neq 1$), $A_i | x_v x_w$ (respectively, $A'_i | x_v x_w$), and $A_i f_i \in J(G)$ (respectively, $A'_i g_i \in J(H)$) for each $i = 1, \ldots, s$, such that

$$h_1 x_v^{\rho} x_w^{\delta} = (\prod_{i=1}^{s} A_i f_i) M_1 x_v^{\theta_1} x_w^{\theta_2} \text{ (respectively, } h_2 x_v^{\rho} x_w^{\delta} = (\prod_{i=1}^{s} A'_i g_i) M_2 x_v^{\lambda_1} x_w^{\lambda_2}),$$

for some nonnegative integer $\theta_1$ and $\theta_2$ (respectively, $\lambda_1$ and $\lambda_2$). Hence, one has

$$(\prod_{i=1}^{s} A_i) x_v^{\theta_1} x_w^{\theta_2} = x_v^{\rho} x_w^{\delta} = (\prod_{i=1}^{s} A'_i) x_v^{\lambda_1} x_w^{\lambda_2},$$

$h_1 = (\prod_{i=1}^{s} f_i) M_1$, and $h_2 = (\prod_{i=1}^{s} g_i) M_2$. This gives rise to the following equalities

$$h_1 h_2 x_v^{\rho} x_w^{\delta} = (\prod_{i=1}^{s} f_i) M_1 (\prod_{i=1}^{s} g_i) M_2 x_v^{\rho} x_w^{\delta} = (\prod_{i=1}^{s} f_i g_i) M_1 M_2 x_v^{\rho} x_w^{\delta}.$$

Due to $(\prod_{i=1}^{s} A_i)x_v^{\theta_1}x_w^{\theta_2} = x_v^{\rho}x_w^{\delta} = (\prod_{i=1}^{s} A'_i)x_v^{\lambda_1}x_w^{\lambda_2}$, we get $h_1h_2x_v^{\rho}x_w^{\delta} \in J(L)^s$. Hence, $h \in J(L)^s$, which contradicts the fact that $\mathfrak{p} = (J(L)^s :_R h)$. Accordingly, one can derive $h_1x_v^{\rho}x_w^{\delta} \notin J(G)$ or $h_2x_v^{\rho}x_w^{\delta} \notin J(H)$. On account of $\mathfrak{p}$ is the maximal ideal, this implies that $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/J(G)^s)$ or $\mathfrak{p} \in \mathrm{Ass}_{R_2}(R_2/J(H)^s)$, and so we get the following equality

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s). \tag{4.3}$$

We can now combine together Theorem 10 and (4.3) to obtain the last claim. □

As an application of Theorem 12, we present the following proposition.

**Proposition 9.** *(i) The cover ideal of every cycle graph has the strong persistence property.*

*(ii) The cover ideal of every cycle graph with one chord has the strong persistence property.*

*Proof.* (i) Let $C_n$ denote the cycle graph of order $n$. If $n$ is an even number, then $C_n$ is a bipartite graph, and by virtue of [7, Corollary 2.6], $J(C_n)$ is a normally torsion-free square-free monomial ideal, and [24, Theorem 6.10] implies that $J(C_n)$ has the strong persistence property. If $n$ is an odd number, then the claim follows immediately from [21, Theorem 3.3].

(ii) The assertion can be deduced from Theorem 12 and part (i). □

The next question examines the relation between assoicated primes of powers of the cover ideal of the union of two finite simple connected graphs with the assoicated primes of powers of the cover ideals of each of them, in a general case.

**Question 2.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then can we deduce that, for all $s$, one of the following statements holds?*

*(i)* $\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s)$.

*(ii)* $\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(H)^s) \cup \{\mathfrak{m}\}$,

*where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$, and $\mathfrak{m} = (x_\alpha : \alpha \in V(L))$ is the unique homogeneous maximal ideal of $R$.*

The answer is negative. We provide a counterexample. Let $G := \mathcal{K}_4$ be the complete graph with $V(G) = \{1, 2, 3, 4\}$ and

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

and $H := S_4$ be the star graph with $V(H) = \{1, 2, 3, 4, 5\}$ and

$$E(H) = \{\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}.$$

Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. By using Macaulay2 [8], we obtain that

$$(x_2, x_3, x_4, x_5) \in \mathrm{Ass}(J(L)^3) \setminus (\mathrm{Ass}(J(G)^3) \cup \mathrm{Ass}(J(H)^3)),$$

and

$$(x_1, x_2, x_4, x_5) \in \mathrm{Ass}(J(L)^4) \setminus (\mathrm{Ass}(J(G)^4) \cup \mathrm{Ass}(J(H)^4) \cup \mathfrak{m}),$$

where $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5)$. In view of [7, Corollary 2.6] and [24, Theorem 6.10], we get the cover ideal of every tree has the strong persistence property, and so $J(H)$ has the strong persistence property. On the other hand, note that the graph $L$ is exactly the complete graph $\mathcal{K}_5$. It follows also from [1, Corollary 1.7] that the cover ideal of every complete graph is normal, and by virtue of [24, Theorem 6.2], one can cnclude that $J(G)$ and $J(L)$ have the strong persistence property.

In the subsequent question, our aim is to investigate the relation between assoicated primes of powers of the cover ideal of the union of a finite simple connected graph and a complete graph with the assoicated primes of powers of the cover ideals of each of them, under the condition that they are common in a path with length 2.

**Question 3.** *Let $G = (V(G), E(G))$ be a finite simple connected graph and $\mathcal{K}_n$ the complete graph of order $n$ such that $|V(G) \cap V(\mathcal{K}_n)| = 3$ and $|E(G) \cap E(\mathcal{K}_n)| = 2$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(\mathcal{K}_n)$ and $E(L) := E(G) \cup E(\mathcal{K}_n)$. Then can we conclude that*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(\mathcal{K}_n)^s),$$

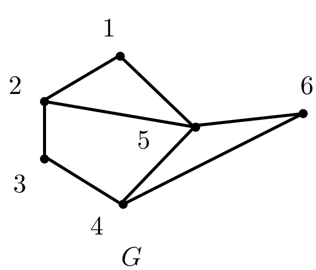
*for all $s$, where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(\mathcal{K}_n)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field $K$? In particular, if $J(G)$ has the strong persistence property, then does $J(L)$ have the strong persistence property?*

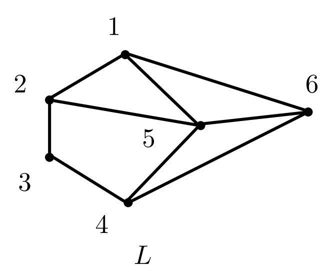
By giving a counterexample, we show that the answer is negative. To do this, consider the graph $G = (V(G), E(G))$, the left graph in the figure below, with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{4, 6\}, \{5, 6\}\},$$

and also the graph $L = G \cup \mathcal{K}_3$, the right graph in the figure below, with $V(\mathcal{K}_3) = \{1, 6, 5\}$, $E(\mathcal{K}_3) = \{\{5, 6\}, \{1, 6\}, \{1, 5\}\}$, $V(L) = \{1, 2, 3, 4, 5, 6\}$, and

$$V(L) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$



It is routine to check that

$$\begin{aligned}J(G) =&(x_1,x_2)\cap(x_2,x_3)\cap(x_3,x_4)\cap(x_4,x_5)\cap(x_5,x_1)\cap(x_5,x_2)\\ &\cap(x_4,x_6)\cap(x_5,x_6)\\ =&(x_2x_4x_5, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5),\end{aligned}$$

and

$$\begin{aligned}J(L) =&(x_1,x_2)\cap(x_2,x_3)\cap(x_3,x_4)\cap(x_4,x_5)\cap(x_5,x_1)\cap(x_5,x_2)\\ &\cap(x_1,x_6)\cap(x_5,x_6)\cap(x_4,x_6)\\ =&(x_2x_4x_5x_6, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5,\\ &x_1x_2x_4x_5).\end{aligned}$$

Using Macaulay2 [8] yields that

$$(x_1,x_2,x_3,x_4,x_5,x_6)\in \mathrm{Ass}(J(L)^3)\setminus(\mathrm{Ass}(J(G)^3)\cup\mathrm{Ass}(J(\mathcal{K}_3)^3)).$$

On the other hand, since the graph $G$ is the union of a cycle graph with one chord and a triangle, Proposition 9 (ii) and Lemma 8 imply that $J(G)$ has the strong persistence property. Furthermore, one can write

$$J(L) = x_5(x_2x_4x_6, x_2x_3x_6, x_1x_3x_6, x_1x_3x_4, x_1x_2x_4) + (x_1x_2x_4x_6).$$

Because $F := (x_2x_4x_6, x_2x_3x_6, x_1x_3x_6, x_1x_3x_4, x_1x_2x_4)$ is the cover ideal of the odd cycle $H$ with $V(H) = \{1,2,3,4,6\}$ and $E(H) = \{\{1,2\},\{2,3\},\{3,4\},\{4,6\},\{6,1\}\}$, Proposition 9 (i) gives that $F$ has the strong persistence property, and by virtue of $x_1x_2x_4x_6 \in F$, it follows from [21, Lemma 2.1] that $J(L) = x_5F + (x_1x_2x_4x_6)$ has the strong persistence property.

**Acknowledgement.** The authors are deeply grateful to the referee for careful reading of the manuscript and valuable suggestions which led to significant improvements in the quality of this paper. Jonathan Toledo was partly supported by CONACYT 740716.
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Received: 15.12.2019
Revised: 01.05.2021
Accepted: 01.05.2021

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