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A note on UN-rings

by

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Abstract

A ring is called a UN-ring if each nonunit is the product of a unit and a nilpotent. In this short note, answers are provided in response to two questions on UN-rings raised by Călugăreanu in [1].

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1 Introduction

Throughout, rings are associative with identity. Recall from [2] that a nonzero element in a ring is called fine if it is the sum of a unit and a nilpotent and a ring is a fine ring if every nonzero element is fine. Fine rings form a proper class of simple rings and were the topic of the paper [2]. As the multiplicative duals of these notions, an element in a ring is called UN if it is the product of a unit and a nilpotent and a ring is a UN-ring if every nonunit is UN. UN-rings were introduced by Călugăreanu in [1] and have been discussed in depth in [1] and [6]. This short note is written in response to two questions on UN-rings raised by Călugăreanu in [1]. An element in a ring is 2-good if it is the sum of two units, and a ring is 2-good if each of its elements is 2-good. In [1], among others, it was observed that every UN-ring whose identity is 2-good is a 2-good ring, and it was asked to refine the inclusion $\{UN$-rings with 2-good identity$\} \subset \{$2-good rings$\}$, that is, to find classes $\mathcal{C}$ of rings such that $\{UN$-rings with 2-good identity$\} \subset \mathcal{C} \subset \{$2-good rings$\}$. As observed here, this can be fufilled by letting $\mathcal{C} = \{$rings R: $R/J(R)$ is UN with 2-good identity$\}$. It is a known result in [4] that matrix rings over elementary divisor rings are 2-good. The author in [1] was motivated to ask whether matrix rings over elementary divisor UN-rings are UN. Here we prove that matrix rings over right Hermite UN-rings are UN. Consequently, matrix rings over elementary divisor UN-rings are UN.

For a ring R, we denote by $C(R), J(R)$, $U(R)$ and $\mathrm{nil}(R)$ the center, the Jacobson radical, the unit group and the set of nilpotents of R, respectively. We write $\mathbb{M}_n(R)$ for the ring of $n \times n$ matrices over R whose identity is denoted by I_n.

2 The results

Lemma 1. *Let I be a nil ideal of a ring R and $a \in R$. Then $a \in R$ is UN iff $a + I \in R/I$ is UN.*

Proof. The necessity is clear. For the sufficiency, write $\bar{x} = x + I$ for $x \in R$. Write $\bar{a} = \bar{u}\bar{b}$ where $\bar{u}$ is a unit in R/I and $\bar{b}$ is nilpotent in R/I. As I is nil, $u \in U(R)$ and $b \in \mathrm{nil}(R)$. Moreover, $a = ub + c$ for some $c \in I$. Thus, $a = u(b + u^{-1}c)$ with $b + u^{-1}c \in \mathrm{nil}(R)$. □

Lemma 2. *Let $a \in R$ be a UN-element. If a power of a is central, then a is nilpotent.*

Proof. Write $a = ub$ where $u \in U(R)$ and $b \in \mathrm{nil}(R)$. Suppose a^n is central for some $n \geq 1$. Then $a^n a = a^n ub = ua^n b = (ua^{n-1})ab = (ua^{n-1})ub^2$ and, similarly, $(a^n)^k a = (ua^{n-1})^k ub^{k+1}$ for any $k \geq 1$. Hence a is nilpotent. □

Corollary 1. *[6, Proposition 0(b)] Let R be a UN-ring. Then $C(R)$ is a local ring with nil Jacobson radical.*

Corollary 2. *Let R be a commutative ring and $n \geq 1$. Then $\mathbb{M}_n(R)$ is a UN-ring iff R is a local ring with $J(R)$ nil.*

Proof. Let $T = \mathbb{M}_n(R)$. As $R \cong C(T)$, the necessity follows from Corollary 1. For the sufficiency, $J(R)$ being nil implies that $\mathbb{M}_n(J(R)) = J(T)$ is nil. Since $R/J(R)$ is a field, $T/J(T) \cong \mathbb{M}_n(R/J(R))$ is UN by [1, Corollary 8]. So T is UN by Lemma 1. □

Lemma 3. *If $a \in R$ is a UN-element, then uav is UN for any $u, v \in U(R)$.*

Proof. Write $a = wb$ where $w \in U(R)$ and $b \in \mathrm{nil}(R)$. Then $uav = (uwv)(v^{-1}bv)$ is UN. □

An $m \times n$ matrix A admits diagonal reduction if A is equivalent to a diagonal matrix $\mathrm{diag}(d_1, d_2, \cdots)$ with the property that every Rd_iR is contained in $Rd_{i+1} \cap d_{i+1}R$. A ring R is called right Hermite if every 1×2 matrix over R admits diagonal reduction. If every 2×1 matrice over R admits diagonal reduction, R is a left Hermite ring, and if both, R is an Hermite ring.

Theorem 1. *Matrix rings over a right Hermite UN-ring are UN.*

Proof. Let R be a right Hermite UN-ring and $n \geq 1$. We show that $\mathbb{M}_n(R)$ is UN by induction on n. This is certainly true for $n = 1$. Assume that $n > 1$ and that $\mathbb{M}_m(R)$ is UN for every $0 < m < n$. We next show that every non-invertible matrix $A \in \mathbb{M}_n(R)$ is UN. By [5, Theorem 3.5], there exists an invertible matrix $U \in \mathbb{M}_n(R)$ such that AU is lower triangular. Hence we can assume that $A = (a_{ij})$ is lower triangular. If every entry a_{ii} is a nonunit, then $a_{ii} = u_{ii}b_{ii}$ where $u_{ii} \in U(R)$ and $b_{ii} \in \mathrm{nil}(R)$. Thus, $U := \begin{pmatrix} u_{11}^{-1} & & 0 \\ & \ddots & \\ 0 & & u_{nn}^{-1} \end{pmatrix}$ is invertible and $UA = \begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ * & & b_{nn} \end{pmatrix}$ is nilpotent; so A is UN. Hence, we can assume that some a_{ii} is a unit. As A is not invertible, some a_{jj} is a nonunit. Thus, we can write $\{1, 2, \ldots, n\}$ as a disjoint union of two nonempty subsets I and J such that a_{ii} is a unit for all $i \in I$ and a_{jj} is a nonunit for all $j \in J$. For each $j \in J$, write

$a_{jj} = v_{jj}d_j$ where $v_{jj} \in U(R)$ and $d_j \in \mathrm{nil}(R)$. Applying the following elementary row operations to A: multiplying row i by a_{ii}^{-1} (for all $i \in I$) and multiplying row j by v_{jj}^{-1} (for all $j \in J$). These operations bring A to a lower triangular matrix A_1 whose (i,i)-entry is 1 (for all $i \in I$) and whose (j,j)-entry is d_j (for all $j \in J$). By adding a certain multiple of row i of A_1 to row l $(l = i+1, \ldots, n)$ and adding a certain multiple of column i of A_1 to column l $(l = 1, \ldots, i-1)$ can bring A_1 to a lower triangular matrix A_2 whose (i,i)-entry, which is 1, is the only nonzero entry in row i and in column i (for all $i \in I$) and whose (j,j)-entry is d_j, which is nilpotent, (for all $j \in J$). Finally, A_2 can be carried to the block matrix $A_3 := \begin{pmatrix} I_k & 0 \\ 0 & T \end{pmatrix}$, where $k = |I|$ and $T = \begin{pmatrix} t_{n-k} & & 0 \\ & \ddots & \\ * & & t_1 \end{pmatrix}$ is a lower triangular nilpotent matrix. If $n-k = 1$, then $A_3 = \begin{pmatrix} I_k & 0 \\ 0 & t_1 \end{pmatrix}$. Let P be the $n \times n$ permutation matrix corresponding to the n-cycle $(1, 2, \ldots, n)$. Then PA_3 is nilpotent by [1, Lemma 6], so A_3 is UN. If $n-k > 1$, write $A_3 = \begin{pmatrix} B & 0 \\ X & t_1 \end{pmatrix}$ where B is non-invertible. By induction hypothesis, B is UN, so $B = U_1T_1$ where U_1 is an invertible mtraix in $\mathbb{M}_{n-1}(R)$ and T_1 is nilpotent in $\mathbb{M}_{n-1}(R)$. Then $V := \begin{pmatrix} U_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ is invertible and $VA_3 = \begin{pmatrix} T_1 & 0 \\ X & t_1 \end{pmatrix}$ is nilpotent, so A_3 is UN. Therefore, by Lemma 3, A is UN. □

Following [4], a ring R is called an elementary divisor ring if, for every $n \geq 1$, every matrix in $\mathbb{M}_n(R)$ is equivalent to a diagonal matrix. Every elementary divisor ring is both a left and a right Hermite ring, and an example is given in [3, Example 4.11] of a commutative Hermite ring that is not an elementary divisor ring. In [1], Călugăreanu asks whether matrix rings over elementary divisor UN-rings are also UN-rings. The answer is yes by the next result which is an immediate consequence of Theorem 1.

Corollary 3. *Matrix rings over an elementary divisor UN-ring are UN.*

By [1, Corollary 5], UN-rings whose identity is the sum of two units are 2-good. A question there asks to refine the inclusion {UN-rings with 2-good identity} $\subset$ {2-good rings}, that is, to find classes $\mathcal{C}$ of rings such that {UN-rings with 2-good identity} $\subset \mathcal{C} \subset$ {2-good rings}. Here we have a natural pick of $\mathcal{C}$ to fufil this.

Example 1. *Let $\mathcal{C}$ be the class of rings R such that $R/J(R)$ is a UN-ring with 2-good identity. It is clear that {UN-rings with 2-good identity} $\subseteq \mathcal{C} \subseteq$ {2-good rings}. To see the inclusions are proper, note that $S := \mathbb{M}_n(\mathbb{Z})$ $(n \geq 2)$ is 2-good by [4, Theorem 11], but S is not contained in $\mathcal{C}$ by Corollary 2. Moreover, $\mathbb{Q}[[t]] \in \mathcal{C}$, but $\mathbb{Q}[[t]]$ is not UN by Lemma 2 (as the central element t is not nilpotent).*

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References

[1] G. Călugăreanu, UN-rings, *J. Alg. Appl.*, **15**, 1650182 (2016).

[2] G. Călugăreanu and T.Y. Lam, Fine rings: A new class of simple rings, *J. Alg. Appl.*, **15**, 1650173 (2016).

[3] L. Gillman and M. Henriksen, Rings of continuous functions in which finitely generated ideals are principal, *Trans. AMS.*, **82**, 366-391 (1956).

[4] M. Henriksen, Two classes of rings generated by their units, *J. Algebra*, **31**, 182-193 (1974).

[5] I. Kaplansky, Elementary divisors and modules, *Trans. AMS.*, **66**, 464-491 (1949).

[6] P. Vámos, On rings whose nonunits are a unit multiple of a nilpotent, *J. Algebra Appl.*, **16**, 1750140 (2017).

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