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Valuation rings of dimension one as limits of smooth algebras
by
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Abstract

As in Zariski's Uniformization Theorem we show that a valuation ring V of characteristic $p>0$ of dimension one is a filtered direct limit of smooth $\mathbf{F}_p$-algebras under some conditions of transcendence degree. Under mild conditions, the algebraic immediate extensions of valuation rings are dense if they are filtered direct limit of smooth morphisms.

Key Words: Immediate extensions of valuations rings, pseudo convergent sequences, pseudo limits, smooth morphisms, Henselian rings.
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Introduction

Zariski predicted, and proved in characteristic 0 in [20], that any integral algebraic variety X equipped with a dominant morphism $v : Spec(V) \to X$ from a valuation ring V can be desingularized along V : there should exist a proper birational map $\tilde{X} \to X$ for which the lift $\tilde{v} : Spec(V) \to \tilde{X}$ of v supplied by the valuative criterion of properness would factor through the regular locus of $\tilde{X}$. This local form of resolution of singularities remains open in positive and mixed characteristic, and implies that every valuation ring V should be a filtered direct limit of regular rings. A filtered direct limit (in other words a filtered colimit) is a limit indexed by a small category that is filtered (see [4, 002V] or [4, 04AX]). A filtered union is a filtered direct limit in which all objects are subobjects of the final colimit, so that in particular all the transition arrows are monomorphisms. There exists several nice extensions of Zariski's Uniformization Theorem as for example recently the result of B. Antieau, R. Datta [1, Theorem 4.1.1], which says that every perfect valuation ring of characteristic $p>0$ is a filtered union of its smooth $\mathbf{F}_p$-subalgebras. This result is an application of [18, Theorem 1.2.5] which relies on some results from [3]. Also E. Elmanto and M. Hoyois proved that an absolute integrally closed valuation ring of residue field of characteristic $p>0$ is a filtered union of its regular finitely generated $\mathbf{Z}$-subalgebras (see [1, Corollary 4.2.4]).

The goal of this paper is to establish a kind of Zariski's Uniformization Theorem in characteristic $p>0$ and dimension one.

Theorem 1. *Let V be a one dimensional valuation ring containing a perfect field F of characteristic $p>0$, k its residue field, Γ its value group and K its fraction field. Then the following statements hold*

1. *if $k \subset V$ and $K = k(x)$ for some system of algebraically independent elements $x = (x_1, \ldots, x_r) \in V^r$ over k such that $\Gamma = \oplus_{i=1}^{r-1} \mathbf{Z} val(x_i)$ then V is a filtered union of its polynomial k-subalgebras, in particular of its smooth F-subalgebras,*

2. *if Γ is free of rank r, $k \subset V$ and $x = (x_1, \ldots, x_r) \in V^r$ is a system of elements such that $\Gamma = \oplus_{i=1}^{r} \mathbf{Z} val(x_i)$ and $K/k(x)$ is algebraic then V is a filtered union of its smooth F-subalgebras, in particular of its smooth $\mathbf{F}_p$-subalgebras,*

3. *if Γ is free of rank r, k/F is a field extension of finite type and $trdeg_F K = trdeg_F k + r$, then V is a filtered union of its smooth F-subalgebras, in particular of its smooth $\mathbf{F}_p$-subalgebras.*

The proof is given in Corollaries 2, 4, 5. The idea of the proof of Theorem 1 (1) is to see that $W = V \cap k(x)$ is a filtered union of its localizations of polynomial k-subalgebras and then to reduce to show that the immediate extension $W \subset V$ is a filtered union of its smooth W-subalgebras. This is done somehow in [15, Proposition 18] when $k \supset \mathbf{Q}$. If char $k > 0$ this does not work (see e.g. [13, Remark 6.10]), the reason being that in general a valuation ring could have a so called defect as was noted by Ostrowski. The Generalized Stability Theorem of F. V. Kuhlmann [7, Theorem 1.1] says in particular that W is defectless (see Corollary 1) and so we may show (1) using Lemma 3. For Theorem 1 (2) we see that $W \subset V$ is dense and we use a Néron-Schappacher Theorem (see e.g. [13, Theorem 4.1]) or a kind of General Néron Desingularization (see Lemma 4) which allow us to handle dense extensions.

When V contains $\mathbf{Q}$ then an immediate algebraic extension of valuation rings $V \subset V'$ is dense by Ostrowski's Defektsatz [11, Sect. 9, No 57]. It follows that $V \subset V'$ is a filtered direct limit of smooth V-algebras (see e.g. [15, Proposition 9]). The converse is also mainly true as shows the following result (see Theorem 7).

Theorem 2. *Let V' be an immediate algebraic extension of a valuation ring V, $\hat{V}$ the completion of V and K, $\hat{K}$ the fraction fields of V, $\hat{V}$. Assume that $trdeg_K \hat{K} \geq 1$, $\hat{K}/K$ is separable, V' is a filtered direct limit of smooth V-algebras and either V is Henselian, or* $\dim V = 1$. *Then $V \subset V'$ is dense.*

We owe thanks to F. V. Kuhlmann who hinted us some mistakes in some preliminary forms of the paper. Also we owe thanks to Kestutis Česnavičius who hinted us some mistakes in the preliminary forms of Theorem 6.

1 The Defect

Let V be a valuation ring with value group Γ, K its fraction field and its valuation val: $K^* = K \setminus \{0\} \to \Gamma$. Let F be a finite field extension of K, v_i, $1 \leq i \leq r$ the valuations of F extending val and V_i be the valuation rings of F defined by v_i. Let e_i, f_i be the ramification index respectively the degree of the residue field extension of V_i over V. It is well known that

$$[F : K] \geq \sum_{i=1}^{r} e_i f_i.$$

If this inequality is equality for all finite (resp. finite separable) field extensions F of V we say that V is *defectless* (resp. *separably defectless*, see [8] for details and examples).

We remind the following important result for the existence of defectless valuation rings.

Theorem 3. *([2, Theorem 2, VI, (8.5)]) Let F be a finite field extension of K and B be the integral closure of V in F. Then the above inequality is equality if and only if B is a free V-module.*

Lemma 1. *Let V be a valuation ring and S a multiplicative closed system from V. If V is defectless (resp. separably defectless) then the valuation ring $S^{-1}V$ is defectless (resp. separably defectless) too.*

Proof. Assume V is defectless (the proof in the separably defectless case is similar). Let F be a finite field extension of the fraction field K of V (and of $S^{-1}V$ too) and B, B' be the integral closures of V, respectively $S^{-1}V$ in F. By Theorem 3 we have to show that B' is a free $S^{-1}V$-module. As V is defectless by the quoted theorem we see that B is a free V-module and so $S^{-1}B$ is a free $S^{-1}V$-module.

We claim that $B' \cong S^{-1}B$ which is enough. Indeed, let $z \in F$ be integral over $S^{-1}V$. We have

$$z^e + \sum_{i=0}^{e-1} (a_i/s) z^i = 0$$

for some $e \in \mathbf{N}$, $a_i \in V$ and $s \in S$. Then

$$(sz)^e + \sum_{i=0}^{e-1} s^{e-i-1} a_i (sz)^i = 0$$

and so $sz \in B$, which shows our claim.

□

Next it is very useful the following particular form of the Generalized Stability Theorem of F. V. Kuhlmann (see [7, Theorem 1.1], [8, Theorem 5.1]).

Theorem 4. *Let $V \subset V'$ be an extension of valuation rings with the same residue field, $\Gamma \subset \Gamma'$ its value group extension and $K \subset K'$ its fraction field extension. Assume K'/K is a finite type field extension and Γ'/Γ is a finitely generated free $\mathbf{Z}$-module of rank trdegK'/K. If V is defectless (resp. separably defectless) then V' is defectless (resp. separably defectless) too.*

Corollary 1. *Let V be a valuation ring of characteristic $p > 0$ with finitely generated value group Γ and fraction field K. Assume V contains its residue field k and $K = k(y)$ for some elements $y = (y_1, \dots, y_r)$ of V such that val(y_i), $1 \leq i \leq r$ is a basis of the free $\mathbf{Z}$-module Γ. Then V is defectless.*

For the proof note that k is defectless for the trivial valuation, y is algebraically independent over k by [2, Theorem 1, (10.3)] and apply the above theorem to $k \subset V$.

Proposition 1. *(Kuhlmann, [7, Theorem 1.1]) Let V be a valuation ring, $\mathfrak{m}$ its maximal ideal and $X = (X_1, \dots, X_n)$ some variables. If V is defectless then the valuation ring $V_1 = V[X]_{\mathfrak{m}V[X]}$ is defectless too.*

Proof. Applying induction on n we reduce to the case $n = 1$. Let Γ be the value group of V and set $\Gamma' = \Gamma \oplus \mathbf{Z}\gamma$ where γ is taken to be positive but $<$ than all positive elements of Γ and let w be the valuation on $K(X)$ extending val by $X \to \gamma$. By Theorem 4 we see that W, the valuation ring of w, is defectless. Note that $\mathfrak{m}W$ is a prime ideal of W and $W_{\mathfrak{m}W} \cong V_1$ is defectless by Lemma 1. □

An inclusion $V \subset V'$ of valuation rings is an *immediate extension* if it is local as a map of local rings and induces equalities between the value groups and the residue fields of V and V'. It is dense if for any $x' \in V'$ and $\gamma \in \Gamma$ there exists $x \in V$ such that $\mathrm{val}(x - x') > \gamma$.

Let λ be a fixed limit ordinal and $v = \{v_i\}_{i<\lambda}$ a sequence of elements in V indexed by the ordinals i less than λ. Then v is *pseudo convergent* if

$\mathrm{val}(v_i - v_{i''}) < \mathrm{val}(v_{i'} - v_{i''})$ for $i < i' < i'' < \lambda$ (see [5], [16]). A *pseudo limit* of v is an element $x \in V$ if

$\mathrm{val}(x - v_i) = \mathrm{val}(v_i - v_{i'}))$ for $i < i' < \lambda$. We say that v is

1. *algebraic* if some $f \in V[T]$ satisfies $\mathrm{val}(f(v_i)) < \mathrm{val}(f(v_{i'}))$ for large enough $i < i' < \lambda$;
2. *transcendental* if each $f \in V[T]$ satisfies $\mathrm{val}(f(v_i)) = \mathrm{val}(f(v_{i'}))$ for large enough $i < i' < \lambda$.

When for any $\gamma \in \Gamma$ it holds $\mathrm{val}(v_i - v_{i'}) > \gamma$ for $i < i' < \lambda$ large we call v *fundamental*.

Lemma 2. *Let $V \subset V'$ be an immediate extension of one dimensional valuation rings and $K \subset K'$ their fraction field extension. If V is separably defectless and K'/K is separable and algebraic then K'/K is dense. Moreover, if K'/K is not separable but V is defectless then K'/K is still dense.*

Proof. We may reduce to the case when K'/K is finite separable. Let K^h, K'^h be the Henselizations of K respectively K' and $L = K^h(K') \subset K'^h$. By [8, Theorem 2.3] we have K^h separably defectless and so we have $[L : K^h] = e_{L/K^h} f_{L/K^h} = 1$ because there exists an unique extension of the valuation of V to K^h. Thus $L \subset K^h$ and $K \subset K^h$ is dense because $\dim K = 1$. The second statement goes similarly. □

Example 1. *We consider [13, Example 3.1.3] inspirated by [11]. Let k be a field of characteristic $p > 0$, X a variable, $\Gamma = \mathbf{Q}$ and K the fraction field of the group algebra $k[\Gamma]$, that is the rational function in $\{X^q\}_{q\in\mathbf{Q}}$. Let P be the field of all formal sums $z = \sum_{n\in\mathbf{N}} a_n X^{\gamma_n}$ where $(\gamma_n)_{n\in\mathbf{N}}$ is a monotonically increasing sequence from Γ and $a_n \in k$. Set val$(z) = \gamma_s$, where $s = \min\{n \in \mathbf{N} : a_n \neq 0\}$ if $z \neq 0$ and let V' be the valuation ring defined by val: $P^* \to \Gamma$.*

Let $\rho_n = (p^{n+1} - 1)/(p - 1)p^{n+1}$,

$$y = -1 + \sum_{n\geq 0} (-1)^n X^{\rho_n}$$

and $a_i = -1 + \sum_{0\leq n\leq i} (-1)^n X^{\rho_n}$. We have $1 + \rho_n = p\rho_{n+1}$ for $n \geq 0$ and $p\rho_0 = 1$ and y is a pseudo limit of the pseudo convergent sequence $a = (a_i)_{i\in\mathbf{N}}$, which has no pseudo limit in K. Then x is a root of the separable polynomial $g = Y^p + XY + 1 \in K[Y]$ and the algebraic separable extension $V = V' \cap K \subset V' \cap K(y)$ is not dense. The above lemma cannot be applied because V is not separably defectless.

Lemma 3. *Let $V \subset V'$ be an immediate extension of one dimensional valuation rings, $K \subset K'$ their fraction field extension and $x \in V'$ a transcendental element over K. Assume V is separably defectless and x is a pseudo limit of a pseudo convergent sequence a over V which has no pseudo limit in V. Then a is transcendental.*

Proof. Let $a = (a_j)_{j<\lambda}$ and assume a is algebraic. Let $h \in V[X]$ be a primitive polynomial of minimal degree among the polynomials $f \in V[X]$ such that $\mathrm{val}(f(a_j)) < \mathrm{val}(f(a_{j+1}))$ for all $j < \lambda$. We may take h separable. Indeed, assume that h is not separable. We have $\mathrm{val}(h(x)) > \mathrm{val}(h(a_j))$ for $j < \lambda$ large. Choose $b \in V$ such that $\mathrm{val}(bx) > \mathrm{val}(h(x))$ and set $h' = h + bX$. Note that $\mathrm{val}(x) = \mathrm{val}(a_j)$ for j large, which implies $\mathrm{val}(ba_j) = \mathrm{val}(bx) >$ $\mathrm{val}(h(x)) > \mathrm{val}(h(a_j))$ for j large. It follows that $\mathrm{val}(h'(a_j)) = \mathrm{val}(h(a_j))$ and we may replace h by h', which is separable.

By [5, Theorem 3] there exists a finite separable immediate extension $L = K(z)$ of K such that $h(z) = 0$ and z is a pseudo limit of a. By Lemma 2 L is dense over K and so a has a pseudo limit in K (see e.g. [13, Lemma 2.5]), which is false. □

Proposition 2. *Let $V \subset V'$ be an immediate extension of one dimensional valuation rings and $K \subset K'$ their fraction field extension. Assume V is separably defectless and $K' = K(x)$ for some $x \in K'$ which is transcendental over V. Then V' is a filtered union of localizations of its polynomial V-subalgebras in one variable.*

Proof. Then x is a pseudo limit of a pseudo convergent sequence a over V which has no pseudo limit in V by [5, Theorem 1]. But a is transcendental by Lemma 3 and using [12, Lemma 3.2] or [15, Lemma 15] we are done. □

Corollary 2. *Let V be a one dimensional valuation ring, K its fraction field, k its residue field and Γ its value group. Assume $k \subset V$ and $K = k(x)$ for some system of algebraically independent elements $x = (x_1, \dots, x_r) \in V^r$ over k such that either $\Gamma = \oplus_{i=1}^{r} \mathbf{Z} val(x_i)$, or $\Gamma = \oplus_{i=1}^{r-1} \mathbf{Z} val(x_i)$. Then V' is a filtered union of localizations of its polynomial V-subalgebras in r variables.*

Proof. The first case is a consequence of [2, Theorem 1, VI, (10.3)] and [12, Lemma 4.6] (see also [15, Lemma 26 (1)]). In the second case we see that V is an immediate extension of $W = V \cap k(x_1, \dots, x_{r-1})$. As above W is a filtered union of localizations of its polynomial V-subalgebras in $(r-1)$ variables. By Corollary 1 we have W defectless and using Proposition 2 we are done. □

2 Valuation rings as limits of smooth algebras

Proposition 3. *Let $V \subset V'$ be an immediate extension of one dimensional valuation rings and $K \subset K'$ their fraction field extension. Assume V is separably defectless and K'/K is algebraic separable. Then V' is a filtered union of its smooth V-subalgebras.*

Proof. By Lemma 2 the extension $V \subset V'$ is dense. Then by a Néron-Scappacher Theorem (see e.g. [13, Theorem 4.1]) we see that V' is a filtered direct limit of smooth V-algebras. $\square$

Corollary 3. *Let V be a one dimensional valuation ring of characteristic $p > 0$ with a free (over $\mathbf{Z}$) value group Γ of rank r and fraction field K. Assume V contains its residue field k and $x = (x_1, \ldots, x_r) \in V^r$ is a system of elements such that $\Gamma = \oplus_{i=1}^r \mathbf{Z} val(x_i)$ and $K/k(x)$ is algebraic separable. Then V is a filtered union of its smooth k-subalgebras.*

Proof. By Corollary 1 we see that $W = V \cap k(x)$ is defectless and so V is a filtered union of its smooth W-subalgebras using the above proposition. But W is a filtered union of its smooth k-algebras (see [2, Theorem 1, VI, (10.3)], [12, Lemma 4.6], [15, Lemma 26 (1)]), which is enough. $\square$

We need the following lemma ([15, Lemma 7] which is an extension of [6, Proposition 3], and [14, Proposition 5]).

Lemma 4. *For a commutative diagram of ring morphisms*

$$\begin{tikzcd}[column sep=small] & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & V \\ & A' \arrow[ur] & \end{tikzcd} \quad \textit{that factors as follows} \quad \begin{tikzcd}[column sep=small] & B \arrow[d, "b \mapsto a"] \arrow[dr] & \\ A \arrow[ur] \arrow[r] & A'/a^3A' \arrow[r] & V/a^3V \end{tikzcd}$$

with B finitely presented over A, a $b \in B$ that is standard over A (this means a special element from the ideal $H_{B/A}$ defining the non smooth locus of B over A, for details see for example [15, Lemma 4]), and a nonzerodivisor $a \in A'$ that maps to a nonzerodivisor in V that lies in every maximal ideal of V, there is a smooth A'-algebra S such that the original diagram factors as follows:

$$\begin{tikzcd}[column sep=small] & B \arrow[rr] \arrow[dr] & & V. \\ A \arrow[ur] \arrow[r] & A' \arrow[r] & S \arrow[ur] & \end{tikzcd}$$

In fact the separability condition is not necessary in Corollary 3.

Corollary 4. *Let V be a one dimensional valuation ring of characteristic $p > 0$ with a free (over $\mathbf{Z}$) value group Γ of rank r and fraction field K. Assume V contains its residue field k and $x = (x_1, \ldots, x_r) \in V^r$ is a system of elements such that $\Gamma = \oplus_{i=1}^r \mathbf{Z} val(x_i)$ and $K/k(x)$ is algebraic. Then V is a filtered union of its smooth k-subalgebras.*

Proof. As in Corollary 3 we see that $W = V \cap k(x)$ is defectless and so the algebraic extension $W \subset V$ is dense by Lemma 2. Let $E \subset V$ be a finitely generated F-subalgebra and $w : E \to V$ its inclusion. Assume $E = F[Y]/I$, for $Y = (Y_1, \ldots, Y_n)$. Using [17, Lemma 1.5] it is enough to show that w factors through a smooth F-algebra. Note that K/F is separable because F is perfect. Thus E/F is separable and $w(H_{E/F}) \neq 0$, let us assume that $w(H_{E/F})V \supset zV$ for some $z \in W$, $z \neq 0$. Replacing z by a power of it we may assume that $z = \sum_i^s b_i b_i'$ for some $b_i = \det(\partial f_{ij}/\partial Y_{j_i})$ for some systems of polynomials f_i from I and $b_i'' \in F[Y]$ which kills $I/(f_i)$. Similarly as in [6, Lemma 4] we may assume that we can take $s = 1$, that is for some polynomials $f = (f_1, \ldots, f_r)$ from I, we have $z \in NME$ for some $N \in ((f) : I)$ and a $r \times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$ (since V is a valuation ring this reduction is much easier). Thus we may assume z is standard over F, which is necessary later to apply Lemma 4.

Set $E' = E \otimes_F W$ and let $w' : E' \to V$ be the map induced by w. We have $w'(H_{E'/W}) \supset zV$ and note that w' factors modulo z^3 through the smooth W/z^3W-algebra $W/z^3W \cong V/z^3V$ since $W \subset V$ is dense. Then using Lemma 4 we see that w' factors through a smooth W-algebra. Since W is a filtered direct limit of smooth k-algebra as in Corollary 3 we see that w factors through a smooth k-algebra even F-algebra. Actually, we may apply as in Proposition 3 a variant of Néron-Schappacher Theorem to get V as a filtered union of its smooth W-subalgebras and we are done. ∎

Corollary 5. *Let V be a one dimensional valuation ring containing a perfect field F of characteristic $p > 0$, Γ its value group, K its fraction field and k its residue field. Assume Γ is free of rank r, k/F is a field extension of finite type and trdeg$_F K$ =trdeg$_F k + r$. Then V is a filtered union of its smooth F-subalgebras, in particular of its smooth $\mathbf{F}_p$-algebras.*

Proof. Let $E \subset V$ be a finitely generated F-subalgebra and $w : E \to V$ its inclusion. Using [17, Lemma 1.5] it is enough to show that w factors through a smooth F-algebra. Using Lemma 4 we may replace V by its completion as in [15, Proposition 9]. So we may assume V is Henselian. A lifting of k to V could be done when V is Henselian and char $k = p = 0$ (see [19, Theorem 2.9]) but the proof goes in the same way when $p > 0$ and k is separably generated over F. In particular the lifting could be done when k/F is of finite type because F is perfect. Thus we may assume $k \subset V$ and $\text{trdeg}_k K = r$. Choose some elements $x = (x_1, \ldots, x_r) \in V^r$ such that $\Gamma = \oplus_{i=1}^r \mathbf{Z}\text{val}(x_i)$. Then $W = V \cap k(x) \subset V$ is algebraic and we may proceed as in Corollary 4. ∎

3 Algebraic pseudo convergent sequences

Let V' be an immediate extension of a valuation ring V, $K \subset K'$ their fraction field extension, $v = (v_j)_{j<\lambda}$ a pseudo-convergent sequence in V which is not fundamental and has a pseudo limit x in V', but having no pseudo limit in K. Suppose that $K' = K(x)$. We need the following result [9, Theorem 1.8].

Theorem 5. *(Kuhlmann-Ćmiel) Assume that x is not a root of an irreducible polynomial $f \in K[X]$ and either $\dim V = 1$, or V is Henselian. Then there is a $\nu < \lambda$ such that val$(f(z))$ <val$(f(v_j))$ for all $\nu < j < \lambda$ and $z \in V$.*

Assume that x is transcendental over K.

Theorem 6. *The following statements are equivalent:*

1. *for every polynomial $f \in V[X]$ with $f(x) \neq 0$ there exists a $y \in V$ such that val$(f(y))$ =val$(f(x))$,*

2. *for every polynomial $f \in V[X]$ with $f(x) \neq 0$ there exists a $y \in V$ such that val$(f(y)) \geq$val$(f(x))$*

3. v is transcendental,

4. V' is a filtered union of its localizations of polynomial V-subalgebras in one variable.

Proof. First assume that (4) holds and let f be as in (1). Let $d \in V$, $z \in V'$ be such that $f(x) = dz$ and $zz' = 1$ for some $z' \in V'$. Then the solution x, z, z' in V' of the polynomials $F_1 = f - dZ, F_2 = ZZ' - 1 \in V[X, Z, Z']$ must be contained in a localization of a polynomial V-subalgebra $C = V[u']_{P(u')}$ of V', where $P \in V[U]$ and $u' \in V'$ is transcendental over V. Choose a $u \in V$ such that $u \equiv u'$ modulo $\mathfrak{m}'$ the maximal ideal of V'. Then $P(u)$ is a unit in V and the map $\rho : C \to V$ given by $u' \to u$ is a retraction of $V \subset C$ and $y = \rho(x), \rho(z), \rho(z')$ is a solution of F_1, F_2 in V, and so $\mathrm{val}(f(y)) = \mathrm{val}(d) = \mathrm{val}(f(x))$.

Clearly (2) follows from (1) and assume that (2) holds but v is algebraic. This is not possible as F. V. Kuhlmann said, the proof being given mainly in Theorem 5 (see also [10, Lemma 5.4]). Let $h \in V[T]$ be a polynomial with minimal degree among the polynomials $f \in V[T]$ such that $\mathrm{val}(f(v_i)) < \mathrm{val}(f(v_j))$ for large $i < j < \lambda$. Then h is irreducible (see the proof of [16, (II,4), Lemma 12]). Let $q \in \mathrm{Spec}\, V'$ be the minimal prime over the ideal of $h(x)V'$ and $q' \in \mathrm{Spec}\, V'$ the maximal prime ideal of V' which does not contain $h(x)$. Thus $\mathrm{height}(q/q') = 1$.

Assume that $x \in q'$. We claim that $h(0) \in q'$. Indeed, otherwise $x \not| h(0)$ and so $\mathrm{val}(h(0)) < \mathrm{val}(x)$. But $\mathrm{val}(x) = \mathrm{val}(v_j)$ for j large, since $\deg h > 1$ (v has no pseudo limits in K). As $h(0) \equiv h(x)$ modulo x and $h(0) \equiv h(v_j)$ modulo v_j for j large we get $\mathrm{val}(h(0)) = \mathrm{val}(h(x)) = \mathrm{val}(h(v_j))$ for j large in contradiction with the choice of h.

Thus $h(0) \in q'$ and so $h(x) \in (x, h(0)) \subset q'$, which is false. Therefore, we get $x \notin q'$ too and our problem can be reduced somehow to the case of the one dimensional valuation ring extension $W = (V/q' \cap V)_{q \cap V} \subset W' = (V'/q')_q$. More precisely, by Theorem 5 applied to $W \subset W'$ we get for j large that $\mathrm{val}(h(x)) > \mathrm{val}(f(v_j)) > \mathrm{val}(h(z))$ for all $z \in K$ with $h(z) \in q$. In particular, $h(z) \notin q'$ for all $z \in K$ because $h(x) \notin q'$. If $h(z) \notin q$ then $h(x) \not| h(z)$ and so $\mathrm{val}(h(z)) < \mathrm{val}(h(x))$. Thus h fails the condition (2).

If v is transcendental then V' is a filtered union of its localizations of polynomial V-subalgebras in one variable by [12, Lemma 3.2] (see also [15, Lemma 15]). ∎

Corollary 6. *In the assumptions of the above theorem, if v is algebraic then V' is not a filtered union of its localizations of polynomial V-subalgebras in one variable.*

Lemma 5. *In the assumptions of the above theorem, assume that V is Henselian and there exists an immediate extension of valuation rings $V' \subset V_1$ such that V_1 is a filtered direct limit of smooth V-algebras. Then for every polynomial $f \in V[X]$ such that $f(x) \neq 0$ there exists a $y \in V$ such that val$(f(y)) =$val$(f(x))$.*

Proof. Fix $f \in V[X]$. Let $d \in V$, $z \in V'$ be such that $f(x) = dz$ and $zz' = 1$ for some $z' \in V'$. Since V_1 is a filtered direct limit of smooth V-algebras, the solution x, z', z' in V' of the polynomials $F_1 = f - dZ, F_2 = ZZ' - 1 \in V[X, Z, Z']$ must come from a solution $\bar{x}$, $\bar{z}$, $\bar{z}'$ in a smooth V-algebra C. But V is Henselian and so there exists a retraction $\rho : C \to V$ of $V \subset C$. Thus $y = \rho(\bar{x}), \rho(\bar{z}), \rho(\bar{z}')$ is a solution of F_1, F_2 in V, and so $\mathrm{val}(f(y)) = \mathrm{val}(d) = \mathrm{val}(f(x))$. ∎

Lemma 6. *In the assumptions of the above theorem, assume that* $\dim V = 1$ *and there exists an immediate extension of valuation rings* $V' \subset V_1$ *such that* V_1 *is a filtered direct limit of smooth* V*-algebras. Let* $\hat{V}$ *be the completion of* V *and* K, $\hat{K}$ *the fraction fields of* V, $\hat{V}$. *Suppose that* $\hat{K}/K$ *is separable. Then for every polynomial* $f \in V[X]$ *with* $f(x) \neq 0$ *there exists a* $y \in V$ *such that* $val(f(y)) = val(f(x))$,

Proof. Let $\hat{V}_1$ be the completion of V_1 and K_1 the fraction field of V_1. Then $\hat{V}_1 \cap \hat{K} = \hat{V}$. Note that $K_2 = \hat{K}(V_1)$ is a localization of $\hat{K} \otimes_K K_1$ since $\hat{K}/K$ is separable. Then $V_2 = \hat{V}_1 \cap K_2$ is a localization of $\hat{V} \otimes_V V_1$ and so V_2 is a filtered direct limit of smooth $\hat{V}$-algebras. As $\hat{V}$ is Henselian we get $\mathrm{val}(f(z)) = \mathrm{val}(f(x))$ for some $z \in \hat{V}$ using the above lemma. Choose $y \in V$ such that $\mathrm{val}(y - z) > \mathrm{val}(f(x))$. Then $\mathrm{val}(f(y)) = \mathrm{val}(f(x))$. □

Theorem 7. *Let* V' *be an immediate algebraic extension of a valuation ring* V, $\hat{V}$ *the completion of* V *and* K, $\hat{K}$ *the fraction fields of* V, $\hat{V}$. *Assume that* $trdeg_K \hat{K} \geq 1$, $\hat{K}/K$ *is separable, there exists an immediate extension of valuation rings* $V' \subset V_1$ *such that* V_1 *is a filtered direct limit of smooth* V*-algebras and either* V *is Henselian, or* $\dim V = 1$. *Then* $V \subset V'$ *is dense.*

Proof. Assume that $V \subset V'$ is not dense and let $x \in V'$ which is not in $\hat{V}$. Then x is a pseudo limit of a pseudo convergent sequence v which is not fundamental and has no pseudo limits in V by [5, Theorem 1]. Note that v is algebraic because x is algebraic over K. Choose a transcendental $t \in \hat{V}$ over K. Multiplying t with a constant of V of high enough value we may assume that $x + t$ is still a pseudo limit of v.

Let $\hat{V}_1$ be the completion of V_1 and set $V_2 = \hat{V}_1 \cap \hat{K}(x + z)$. As in the proof of the above lemma we see that V_2 is a filtered direct limit of smooth $\hat{V}$-algebras. Thus for every polynomial $f \in V[X]$ such that $f(x) \neq 0$ there exists a $y \in V$ such that $\mathrm{val}(f(y)) = \mathrm{val}(f(x + z))$ using Lemmas 5, 6. By Theorem 6 applied to $x + z$ we get v transcendental, which is false. □

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