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An algebraic approach to residues
by
I-Chiau Huang

Abstract

Using basic tools from commutative algebra, a new and elementary framework is constructed to host algebraic operations, which are also provided by analytic or cohomological residues. Lagrange inversion formulas are built into the framework. Local duality is proved.

Key Words: Differential, generating function, Lagrange inversion, power series ring, residue.
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1 Introduction

The purpose of this article is to provide an elementary approach to residues, which first appeared in Cauchy's work in complex integration of one variable. G. P. Egorychev uses these integrations systematically for problems in combinatorial analysis [1]. Residues are also important ingredients of Grothendieck duality, which works not only for complex numbers but for arbitrary field and even in more general context [6]. It is observed that local cohomology residues in the duality theory provide another framework for computations in combinatorial analysis [8]. In this article, we construct a new framework for residues without homological algebra or the theory of integrations.

1.1 Analytic Origin

We begin with a quick look on analytic residues for the case of several variables. Reference is [5, Chapter 5]. Let ϵ be a positive real number and U be the ball $\{z \in \mathbb{C}^n\colon \|z\| < \epsilon\}$. We consider holomorphic functions $f_1, \ldots, f_n$ in a neighborhood of the closure $\overline{U}$ of U with the assumption that $f_1, \ldots, f_n$ have the origin as isolated common zero. Let Γ be the real n-cycle defined by

$$\Gamma = \{z\colon \|f_i(z)\| = \epsilon\}$$

and oriented by

$$d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0.$$

For any holomorphic function g in a neighborhood of $\overline{U}$, the analytic residue is given by

$$\operatorname{res} \frac{g dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \frac{g dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}. \tag{1.1}$$

In particular,

$$\operatorname{res} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{i_1} \cdots z_n^{i_n}} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Analytic residues satisfy the linearity and transformation laws and has local duality. These algebraic operations are sufficient for many applications and thus call for a purely algebraic interpretation.

1.2 Homological Approach

Generating functions are one of the most useful machinery for combinatorial problems. Starting from a sequences of numbers with combinatorial interests, a function is generated in the guise of a power series. In most situations, the generating function is analytic in a certain region so integrations are available. Coefficients in the power series can be extracted using analytic residues. If g above is represented as $\sum c_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}$, then

$$\operatorname{res} \frac{g dz_1 \wedge \cdots \wedge dz_n}{z_1^{i_1+1} \cdots z_n^{i_n+1}} = c_{i_1 \cdots i_n}.$$

We remark that the above notation although popular does not embody perfectly its usage. For instance, if we switch the role of two variables from $z_1, \ldots, z_n$, the generating function maybe remains the same. However, we obtain a negative sign due to the wedge product from the analytic residue.

With algebraic operations on functions, analytic residues gives rise to identities of numbers. See the method of coefficients for more information and other aspects [1]. Such an analytic approach is intuitively clear. However, convergence questions and the integrability of functions have to be treated, even thought they are usually not essential to actual applications. For a specific example of the issue, one may look at the proof of Lagrange inversion [19, Theorem 5.1.1]. To remove the unnecessary analytic restrictions in a systematic way, we mention an early work [18] for a historical perspective.

Detached from analytic conditions, the formal aspect of analytic residues is popular in applications as well. Instead of working on functions defined by power series, one can perform algebraic operations directly on power series regardless convergence. The analytic procedure of taking residues can be replaced by the easy operation of extracting coefficients in a power series. Indeed, the operation $[z^n]f := a_n$ is widely used for a power series $f = a_0 + a_1 z + a_2 z^2 + \cdots$. The operator $[z^{-1}]$ is called the formal residue operator in [4]. See also [19]. As in the analytic theory, the operation of taking a derivative is also available in the formal framework. However, differentials are not singled out as objects for algebraic operations in the formal side. In fact, Jacobian matrices occur in wedge products of differentials from changes of variables. With Jacobian determinants as an ingredient, Lagrange inversion is a phenomenon of changes of variables. For a sequence $a_0, a_1, a_2, \ldots$ of numbers, besides the generating function represented by the power series $a_0 + a_1 z + a_2 z^2 + \cdots$, there also generates a differential $(a_0 + a_1 z + a_2 z^2 + \cdots) dz$ retaining the information of changes of variables. The method of generating differentials has emerged from such a viewpoint [11].

Local cohomology for certain modules of differentials provides a satisfactory framework to host residues [8]. On the one hand, the cohomological framework does not have the restriction of the analytic framework; on the other hand, the cohomology framework keeps the elegance of the formal framework. For the wide applicability of cohomology residues, see [10].

1.3 Main Results

Throughout this article, κ is a field. A power series ring A of n variables with coefficients in κ can be characterized as a complete regular local ring of Krull dimension n and with coefficient field κ. The power series ring A is naturally endowed with a metric. As the definition of regular local rings, the maximal ideal $\mathfrak{m}$ of A can be generated by n elements $X_1, \ldots, X_n$. Every element in A can be written uniquely as an infinite sum $\sum a_{i_1\cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$, where $a_{i_1\cdots i_n} \in \kappa$. We call the chosen elements $X_1, \ldots, X_n$ *variables* of A and write $A = \kappa[\![X_1, \ldots, X_n]\!]$. For the intrinsic meaning of power series rings, in this paper, we only need to know a special case of Nakayama's lemma: Elements are variables of A if and only if their images form a basis of the $A/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ [14, Theorem 2.3].

A regular local ring is a Cohen-Macaulay ring, in which any system of parameters forms a regular sequence [14, Theorem 17.4]. Recall that n elements $f_1, \ldots, f_n$ in $\mathfrak{m}$ form a *system of parameters* of A, if $\mathfrak{m}^s$ is contained in the ideal $\langle f_1, \ldots, f_n \rangle$ generated by $f_1, \ldots, f_n$ for some integer s. Recall also that a sequence of elements $f_1, \ldots, f_m$ in $\mathfrak{m}$ is a *regular sequence*, if f_i is not a zero divisor in $A/\langle f_1, \ldots, f_{i-1} \rangle$ for $1 \leq i \leq m$. Let $f_1, \ldots, f_n$ be a system of parameters of A and $g dX_1 \wedge \cdots \wedge dX_n$ be a differential, where $g \in A$ and $X_1, \ldots, X_n$ are variables of A. Our main construction gives an element

$$\operatorname{res} \begin{bmatrix} g dX_1 \wedge \cdots \wedge dX_n \\ f_1, \ldots, f_n \end{bmatrix} \in \kappa,$$

which resembles the integration of differentials (1.1).

A module of differentials is a universal object in certain category. In Section 2, we work in a power series ring with fixed variables so that the language of category theory can be avoided. Besides basic properties of residues, local duality is proved. In Section 3, we study residues for different choices of variables. The invariance law asserts that residues are independent of the choice of variables. This paper concludes with a short discussion on Lagrange inversion in Section 4. Among vast amount of applications, we will perform residue calculus on Catalan numbers and Motzkin numbers to show the idea. These simple examples exhibit specific changes of variables.

2 Variables fixed

In this section, we work in a power series ring A over κ with fixed variables $X_1, \ldots, X_n$. We start with partial derivatives defined on A. For each $1 \leq i \leq n$, any element f of A can be written uniquely as $\sum_{j \geq 0} f_j X_i^j$, where f_j does not involve the variable X_i. The *partial derivative* $\partial/\partial X_i$ given by

$$\frac{\partial f}{\partial X_i} = \sum_{j \geq 1} j f_j X_i^{j-1}$$

is κ-linear and satisfies the Leibniz rule, that is,

$$\frac{\partial(fg)}{\partial X_i} = f\frac{\partial g}{\partial X_i} + g\frac{\partial f}{\partial X_i}.$$

Let Ω be the free A-module of rank n with basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$. The map $d\colon A \to \Omega$ given by

$$df = \frac{\partial f}{\partial X_1}\mathbf{u}_1 + \cdots + \frac{\partial f}{\partial X_n}\mathbf{u}_n$$

is also κ-linear and satisfies the Leibniz rule. In particular, $dX_i = \mathbf{u}_i$ for all i. So we can write

$$df = \frac{\partial f}{\partial X_1}dX_1 + \cdots + \frac{\partial f}{\partial X_n}dX_n.$$

The A-module $\Omega^n := \wedge^n\Omega$ is free with a basis $dX_1 \wedge \cdots \wedge dX_n$. Elements of Ω^n are simply called *differentials*. Let g be a power series written as $\sum c_{i_1\cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$, where $c_{i_1\cdots i_n} \in \kappa$. Given positive integers $i_1, \ldots, i_n$, we denote

$$\operatorname{res}\begin{bmatrix} g dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1+1}, \ldots, X_n^{i_n+1} \end{bmatrix} := c_{i_1\cdots i_n}. \tag{2.1}$$

In particular,

$$\operatorname{res}\begin{bmatrix} dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1}, \ldots, X_n^{i_n} \end{bmatrix} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

Note that

$$\operatorname{res}\begin{bmatrix} g dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1+1}, \ldots, X_n^{i_n+1} \end{bmatrix} = \operatorname{res}\begin{bmatrix} g X_1^{j_1} \cdots X_n^{j_n} dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1+j_1+1}, \ldots, X_n^{i_n+j_n+1} \end{bmatrix}.$$

Note also that (2.1) vanishes, if $g \in \langle X_1^{i_1+1}, \ldots, X_n^{i_n+1}\rangle$.

The following lemma is taken from the proof of [17, Proposition 5.1.14].

Lemma 1. *Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be elements in a commutative ring R satisfying*

$$\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

for $a_{ij}, b_{ij} \in R$. Then $g_1 \cdots g_n \left(\det(a_{ij}) - \det(b_{ij})\right) \in \langle g_1^2, \ldots, g_n^2\rangle$.

Proof. By the transition of the matrix (a_{ij}) to the matrix (b_{ij}) by a single row at a time, we may assume that the matrices (a_{ij}) and (b_{ij}) differ only on the last row. Let (c_{ij}) be the matrix whose rows are the same as those of (a_{ij}) except the last row, while $c_{nj} = a_{nj} - b_{nj}$ for all j. Then $\det(c_{ij}) = \det(a_{ij}) - \det(b_{ij})$ and

$$\sum_{j=1}^{n} c_{ij} f_j = \begin{cases} g_i, & \text{if } 0 \le i < n; \\ 0, & \text{if } i = n. \end{cases}$$

Working on the adjoint of the matrix (c_{ij}), we have

$$\det(c_{ij})\langle f_1,\ldots,f_n\rangle \subset \langle g_1,\ldots,g_{n-1}\rangle.$$

Since $g_n \in \langle f_1,\ldots,f_n\rangle$, the above inclusion implies

$$g_n \det(c_{ij}) \in \langle g_1,\ldots,g_{n-1}\rangle.$$

Multiplying the above containment by the product $g_1\cdots g_{n-1}$, we obtain

$$(g_1\cdots g_n)\left(\det(a_{ij})-\det(b_{ij})\right) \in \langle g_1^2,\ldots,g_{n-1}^2\rangle \subset \langle g_1^2,\ldots,g_n^2\rangle.$$

□

Let $f_1,\ldots,f_n$ be a system of parameters of A. Assume that

$$\begin{pmatrix} X_1^m \\ \vdots \\ X_n^m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

and

$$\begin{pmatrix} X_1^{m'} \\ \vdots \\ X_n^{m'} \end{pmatrix} = \begin{pmatrix} a'_{11} & \cdots & a'_{1n} \\ \vdots & \ddots & \vdots \\ a'_{n1} & \cdots & a'_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

for some $m, m' \in \mathbb{N}$ and $a_{ij}, a'_{ij} \in A$. Applying Lemma 1 to the situation

$$\begin{aligned} \begin{pmatrix} X_1^{m+m'} \\ \vdots \\ X_n^{m+m'} \end{pmatrix} &= \begin{pmatrix} a_{11}X_1^{m'} & \cdots & a_{1n}X_1^{m'} \\ \vdots & \ddots & \vdots \\ a_{n1}X_n^{m'} & \cdots & a_{nn}X_n^{m'} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \\ &= \begin{pmatrix} a'_{11}X_1^{m} & \cdots & a'_{1n}X_1^{m} \\ \vdots & \ddots & \vdots \\ a'_{n1}X_n^{m} & \cdots & a'_{nn}X_n^{m} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned} & (X_1\cdots X_n)^{m+2m'}\det(a_{ij}) - (X_1\cdots X_n)^{2m+m'}\det(a'_{ij}) \\ \in\ & \langle X_1^{2m+2m'},\ldots,X_n^{2m+2m'}\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} & \operatorname{res}\begin{bmatrix} g\det(a_{ij})dX_1\wedge\cdots\wedge dX_n \\ X_1^m,\ldots,X_n^m \end{bmatrix} \\ =\ & \operatorname{res}\begin{bmatrix} g\det(a_{ij})(X_1\cdots X_n)^{m+2m'}dX_1\wedge\cdots\wedge dX_n \\ X_1^{2m+2m'},\ldots,X_n^{2m+2m'} \end{bmatrix} \\ =\ & \operatorname{res}\begin{bmatrix} g\det(a'_{ij})(X_1\cdots X_n)^{2m+m'}dX_1\wedge\cdots\wedge dX_n \\ X_1^{2m+2m'},\ldots,X_n^{2m+2m'} \end{bmatrix} \\ =\ & \operatorname{res}\begin{bmatrix} g\det(a'_{ij})dX_1\wedge\cdots\wedge dX_n \\ X_1^{m'},\ldots,X_n^{m'} \end{bmatrix} \end{aligned}$$

for any $g \in A$. So we can extend the notation (2.1).

Definition 1 (residue)**.** *Given a differential $\omega = g dX_1 \wedge \cdots \wedge dX_n$ and a system of parameters $f_1, \ldots, f_n$ of A, we choose $m \in \mathbb{N}$ and $a_{ij} \in A$ such that*

$$\begin{pmatrix} X_1^m \\ \vdots \\ X_n^m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{2.3}$$

Write $g \det(a_{ij}) = \sum c_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$ for $c_{i_1 \cdots i_n} \in \kappa$. The residue of ω with respect to the sequence $f_1, \ldots, f_n$ is defined to be the element

$$\operatorname{res}\begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} := c_{(m-1)\cdots(m-1)}$$

in κ.

The following linearity and transformation laws for residues are easy to verify.

Linearity Law. *For differentials $\omega_1, \omega_2 \in \Omega^n$, constants $c_1, c_2 \in \kappa$ and a system of parameters $f_1, \ldots, f_n$ of A,*

$$\operatorname{res}\begin{bmatrix} c_1\omega_1 + c_2\omega_2 \\ f_1, \ldots, f_n \end{bmatrix} = c_1 \operatorname{res}\begin{bmatrix} \omega_1 \\ f_1, \ldots, f_n \end{bmatrix} + c_2 \operatorname{res}\begin{bmatrix} \omega_2 \\ f_1, \ldots, f_n \end{bmatrix}.$$

Transformation Law. *For a differential $\omega \in \Omega^n$ and two systems of parameters $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_n$ of A,*

$$\operatorname{res}\begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \operatorname{res}\begin{bmatrix} \det(r_{ij})\omega \\ f'_1, \ldots, f'_n \end{bmatrix}$$

if $f'_i = \sum_{j=1}^n r_{ij} f_j$ for $1 \le i \le n$.

As special cases of the transformation law, we have

$$\operatorname{res}\begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix} = \operatorname{res}\begin{bmatrix} f_1^{i_1} \cdots f_n^{i_n}\omega \\ f_1^{i_1+1}, \ldots, f_n^{i_n+1} \end{bmatrix}$$

and

$$\operatorname{res}\begin{bmatrix} \omega \\ f_{\sigma(1)}, \ldots, f_{\sigma(n)} \end{bmatrix} = \operatorname{res}\begin{bmatrix} (-1)^{\operatorname{sgn}(\sigma)}\omega \\ f_1, \ldots, f_n \end{bmatrix}$$

for any permutation σ of the sequence $1, \ldots, n$.

Vanishing Law. *Let $f_1, \ldots, f_n$ be a system of parameters of A. If $h \in \langle f_1, \ldots, f_n \rangle$, then*

$$\operatorname{res}\begin{bmatrix} h dX_1 \wedge \cdots \wedge dX_n \\ f_1, \ldots, f_n \end{bmatrix} = 0.$$

Proof. Choose $a_{ij} \in A$ and $m \in \mathbb{N}$ satisfying (2.3). Multiplying (2.3) by the adjoint of the matrix (a_{ij}), we see that $\det(a_{ij})\langle f_1, \ldots, f_n \rangle \subset \langle X_1^m, \ldots, X_n^m \rangle$. Hence $h \det(a_{ij}) \in \langle X_1^m, \ldots, X_n^m \rangle$ and

$$\operatorname{res}\begin{bmatrix} h dX_1 \wedge \cdots \wedge dX_n \\ f_1, \ldots, f_n \end{bmatrix} = \operatorname{res}\begin{bmatrix} h \det(a_{ij}) dX_1 \wedge \cdots \wedge dX_n \\ X_1^m, \ldots, X_n^m \end{bmatrix} = 0.$$

∎

We remark that the linearity, vanishing and transformation laws together with the formalism (2.2) determine the residue map. For a system of parameters $f_1, \ldots, f_n$ of A, the vanishing law gives rise to a well-defined symmetric pairing

$$A/\langle f_1, \ldots, f_n\rangle \times A/\langle f_1, \ldots, f_n\rangle \to \kappa$$

by setting

$$(\bar{g}, \bar{h}) \mapsto \operatorname{res}\left[\begin{array}{c} ghdX_1 \wedge \cdots \wedge dX_n \\ f_1, \ldots, f_n \end{array}\right],$$

where $\bar{g}$ and $\bar{h}$ are the images of g and h under the canonical surjective map

$$A \to A/\langle f_1, \ldots, f_n\rangle,$$

respectively. Local duality is interpreted as the non-degeneracy of the above pairing. To prove it, we need a fact available in [10, Lemma 2.2].

Lemma 2. *Let $f_1, \ldots, f_m$ be a regular sequence of a Noetherian local ring R. If an element $g \in R$ satisfies $gf_1^s \ldots f_m^s \in \langle f_1^{s+1}, \ldots, f_m^{s+1}\rangle$ for some $s \geq 0$, then $g \in \langle f_1, \ldots, f_m\rangle$.*

Local Duality. *Let $g \in A$ and $f_1, \ldots, f_n$ be a system of parameters of A. If*

$$\operatorname{res}\left[\begin{array}{c} ghdX_1 \wedge \cdots \wedge dX_n \\ f_1, \ldots, f_n \end{array}\right] = 0$$

for all $h \in A$, then $g \in \langle f_1, \ldots, f_n\rangle$.

Proof. By Lemma 2, it suffices to prove that $g(f_1 \cdots f_n)^s \in \langle f_1^{s+1}, \ldots, f_n^{s+1}\rangle$ for some $s \geq 0$. Besides a_{ij} and m chosen in (2.3), we choose furthermore $b_{ij} \in A$ and $\ell \in \mathbb{N}$ such that

$$\begin{pmatrix} f_1^{\ell+1} \\ \vdots \\ f_n^{\ell+1} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} X_1^m \\ \vdots \\ X_n^m \end{pmatrix}.$$

Then

$$\begin{pmatrix} f_1^{\ell+1} \\ \vdots \\ f_n^{\ell+1} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

Since

$$\begin{pmatrix} f_1^{\ell+1} \\ \vdots \\ f_n^{\ell+1} \end{pmatrix} = \begin{pmatrix} f_1^{\ell} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_n^{\ell} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

we may apply Lemma 1 to obtain

$$(f_1 \cdots f_n)^{2\ell+1} - (f_1 \cdots f_n)^{\ell+1} (\det a_{ij})(\det b_{ij}) \in \langle f_1^{2\ell+2}, \ldots, f_n^{2\ell+2}\rangle. \tag{2.4}$$

Write $g\det(a_{ij}) = \sum c_{i_1\cdots i_n} X_1^{i_1}\cdots X_n^{i_n}$ for $c_{i_1\cdots i_n}\in\kappa$. For non-negative integers $j_1,\ldots,j_n$ less than m,

$$\begin{aligned}
c_{j_1\cdots j_n} &= \operatorname{res}\begin{bmatrix} g\det(a_{ij})dX_1\wedge\cdots\wedge dX_n \\ X_1^{j_1+1},\ldots,X_n^{j_n+1}\end{bmatrix} \\
&= \operatorname{res}\begin{bmatrix} X_1^{m-j_1-1}\cdots X_n^{m-j_n-1} g\det(a_{ij})dX_1\wedge\cdots\wedge dX_n \\ X_1^{m},\ldots,X_n^{m}\end{bmatrix} \\
&= \operatorname{res}\begin{bmatrix} X_1^{m-j_1-1}\cdots X_n^{m-j_n-1} g dX_1\wedge\cdots\wedge dX_n \\ f_1,\ldots,f_n\end{bmatrix} = 0.
\end{aligned}$$

Hence $g\det a_{ij}\in\langle X_1^m,\ldots,X_n^m\rangle$. Since

$$(\det b_{ij})\langle X_1^m,\ldots,X_n^m\rangle \subset \langle f_1^{\ell+1},\ldots,f_n^{\ell+1}\rangle,$$

it follows that

$$g(f_1\cdots f_n)^{\ell+1}(\det a_{ij})(\det b_{ij}) \in \langle f_1^{2\ell+2},\ldots,f_n^{2\ell+2}\rangle. \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$g(f_1\cdots f_n)^{2\ell+1} \in \langle f_1^{2\ell+2},\ldots,f_n^{2\ell+2}\rangle.$$

□

3 Change of Variables

In this section, we study residues for different choices of variables. Working in the $A/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$, it is easy to see that the following operations on a set of variables $Y_1,\ldots,Y_n$ result in another set of variables.

($\mathcal{P}$) Permute $Y_1,\ldots,Y_n$.

($\mathcal{A}$) Add an element of $\langle Y_2,\ldots,Y_n\rangle\kappa[\![Y_2,\ldots,Y_n]\!]$ to Y_1.

($\mathcal{M}$) Multiply Y_1 by an invertible power series.

As a special case of [7, (5.1)], we have the following lemma. We include a proof for reader's convenience.

Lemma 3. *Any set of variables $Y_1,\ldots,Y_n$ can be obtained from $X_1,\ldots,X_n$ by a sequence of the above operations ($\mathcal{P}$), ($\mathcal{A}$) and ($\mathcal{M}$).*

Proof. Write $Y_i = f_i + \sum a_{ij}X_j$, where $a_{ij}\in\kappa$ and $f_i\in\mathfrak{m}^2$. Then $\det(a_{ij})\neq 0$. As a standard procedure in linear algebra, by a sequence of the operations

($\mathcal{P}$) permuting $Y_1,\ldots,Y_n$,

($\mathcal{A}_1$) adding an element of $\kappa Y_2+\cdots+\kappa Y_n$ to Y_1,

($\mathcal{M}_0$) multiplying Y_1 by a non-zero element in κ, we may assume $a_{ij} = \delta_{ij}$, the Kronecker delta. Since $X_1, Y_2, \ldots, Y_n$ are variables, we may write $f_1 = X_1 f_1' + f_1''$, where $f_1' \in \mathfrak{m}$ and $f_1'' \in \langle Y_2, \ldots, Y_n \rangle \kappa[\![Y_2, \ldots, Y_n]\!]$. By using the operation $(\mathcal{A})$, we may assume $f_1'' = 0$. Now $Y_1 = (1 + f_1')X_1$, where $1 + f_1'$ is invertible. By using the operation $(\mathcal{M})$, we can change Y_1 to X_1. Repeating these operations, we can change Y_i to X_i one by one to finish the proof. ∎

With respect to another set of variables $Y_1, \ldots, Y_n$ of A, we may also define partial derivatives $\partial/\partial Y_j$. For each i, the chain rule

$$\frac{\partial f}{\partial X_i} = \frac{\partial Y_1}{\partial X_i}\frac{\partial f}{\partial Y_1} + \cdots + \frac{\partial Y_n}{\partial X_i}\frac{\partial f}{\partial Y_n} \tag{3.1}$$

clearly holds if f is a variable Y_j. Since partial derivatives $\partial/\partial X_i$ and $\partial/\partial Y_j$ all satisfy the Leibniz rule, the chain rule (3.1) holds if f is a monomial $Y_1^{i_1} \cdots Y_n^{i_n}$. Since partial derivatives are κ-linear, the chain rule (3.1) holds if f is a polynomial in the variables $Y_1, \ldots, Y_n$. Given $f \in A$ and $s \in \mathbb{N}$, we choose a polynomial f_s such that $f - f_s \in \mathfrak{m}^{2s}$. Then

$$\begin{aligned} & \frac{\partial f}{\partial X_i} - \left(\frac{\partial Y_1}{\partial X_i}\frac{\partial f}{\partial Y_1} + \cdots + \frac{\partial Y_n}{\partial X_i}\frac{\partial f}{\partial Y_n}\right) \\ = \quad & \frac{\partial (f - f_s)}{\partial X_i} - \left(\frac{\partial Y_1}{\partial X_i}\frac{\partial (f - f_s)}{\partial Y_1} + \cdots + \frac{\partial Y_n}{\partial X_i}\frac{\partial (f - f_s)}{\partial Y_n}\right) \in \mathfrak{m}^s. \end{aligned}$$

Since the containment holds for arbitrary s, the chain rule (3.1) holds in general.

For $f \in A$,

$$\begin{aligned} \frac{\partial f}{\partial Y_1} dY_1 + \cdots + \frac{\partial f}{\partial Y_n} dY_n &= \frac{\partial f}{\partial Y_1}\left(\sum \frac{\partial Y_1}{\partial X_i} dX_i\right) + \cdots + \frac{\partial f}{\partial Y_n}\left(\sum \frac{\partial Y_n}{\partial X_i} dX_i\right) \\ &= \left(\sum \frac{\partial f}{\partial Y_i}\frac{\partial Y_i}{\partial X_1}\right) dX_1 + \cdots + \left(\sum \frac{\partial f}{\partial Y_i}\frac{\partial Y_i}{\partial X_n}\right) dX_n \\ &= \frac{\partial f}{\partial X_1} dX_1 + \cdots + \frac{\partial f}{\partial X_n} dX_n. \end{aligned}$$

Hence

$$df = \frac{\partial f}{\partial Y_1} dY_1 + \cdots + \frac{\partial f}{\partial Y_n} dY_n$$

for any set of variables $Y_1, \ldots, Y_n$ and $f \in A$. The Jacobian matrix

$$\frac{\partial (Y_1, \ldots, Y_n)}{\partial (X_1, \ldots, X_n)} := \begin{pmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y_n}{\partial X_1} & \cdots & \frac{\partial Y_n}{\partial X_n} \end{pmatrix}$$

and its determinant appear in the relation

$$dY_1 \wedge \cdots \wedge dY_n = \left| \frac{\partial (Y_1, \ldots, Y_n)}{\partial (X_1, \ldots, X_n)} \right| dX_1 \wedge \cdots \wedge dX_n.$$

The A-module Ω^n can be also generated freely by $dY_1 \wedge \cdots \wedge dY_n$.

We say that the invariance law holds for variables $Y_1, \ldots, Y_n$ of A, if

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

We need the following result from [7, (5.6)] for invariance law.

Lemma 4. *Let R be a commutative ring. For $f \in R[\![X]\!]$ and $n \in \mathbb{N}$, the coefficients of X^n in f^n and in $f^{n-1}X(df/dX)$ are the same.*

Proposition 1. *Invariance law holds for any set of variables of A.*

Proof. By Lemma 3, it suffices to prove that if the invariance law holds for variables $Z_1, \ldots, Z_n$, then the invariance law also holds for the variables $Y_1, \ldots, Y_n$ obtained from $Z_1, \ldots, Z_n$ by one of the operations $(\mathcal{P})$, $(\mathcal{A})$ and $(\mathcal{M})$. For the operation $(\mathcal{P})$, this is easy to verify.

For the operation $(\mathcal{A})$, we have $Y_i = Z_i$ for $i > 1$ and $Y_1 = Z_1 - g$ for some $g \in \langle Z_2, \ldots, Z_n \rangle \kappa[\![Z_2, \ldots, Z_n]\!]$. In such a case,

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = \operatorname{res}\begin{bmatrix} dZ_1 \wedge \cdots \wedge dZ_n \\ (Z_1 - g)^{i_1}, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix}.$$

We choose $s \in \mathbb{N}$ such that $g^s \in \langle Z_2^{i_2}, \ldots, Z_n^{i_n} \rangle \kappa[\![Z_2, \ldots, Z_n]\!]$. Then

$$\begin{aligned} \operatorname{res}\begin{bmatrix} dZ_1 \wedge \cdots \wedge dZ_n \\ (Z_1 - g)^{i_1}, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix} &= \operatorname{res}\begin{bmatrix} (Z_1^s - g^s)^{i_1} dZ_1 \wedge \cdots \wedge dZ_n \\ (Z_1 - g)^{i_1} Z_1^{si_1}, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix} \\ &= \operatorname{res}\begin{bmatrix} (\sum_{j=0}^{s-1} Z_1^{s-j-1} g^j)^{i_1} dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1^{si_1}, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix}. \end{aligned}$$

The degree of $(\sum_{j=0}^{s-1} Z_1^{s-j-1} g^j)^{i_1}$ as a polynomial in the variable Z_1 with coefficients in $\kappa[\![Z_2, \ldots, Z_n]\!]$ is $si_1 - i_1$. If $i_1 > 1$, then $si_1 - i_1 < si_1 - 1$ and

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = 0.$$

If $i_1 = 1$, then

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = \sum_{j=0}^{s-1} \operatorname{res}\begin{bmatrix} g^j dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1^{j+1}, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix} = \operatorname{res}\begin{bmatrix} dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1, Z_2^{i_2}, \ldots, Z_n^{i_n} \end{bmatrix}.$$

For the operation $(\mathcal{M})$, we have $Y_i = Z_i$ for $i > 1$ and $Y_1 = Z_1/f$ for some invertible $f \in A$. In such a case,

$$\begin{aligned} \operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} &= \operatorname{res}\begin{bmatrix} f^{i_1}(f^{-1}dZ_1 - Z_1 f^{-2} df) \wedge dZ_2 \wedge \cdots \wedge dZ_n \\ Z_1^{i_1}, \ldots, Z_n^{i_n} \end{bmatrix} \\ &= \operatorname{res}\begin{bmatrix} (f^{i_1-1} - Z_1 f^{i_1-2} \frac{\partial f}{\partial Z_1}) dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1^{i_1}, \ldots, Z_n^{i_n} \end{bmatrix}. \end{aligned}$$

If $i_1 > 1$, we apply Lemma 4 for the situation $R = \kappa[\![Z_2, \ldots, Z_n]\!]$, $X = Z_1$ and $n = i_1 - 1$. Working in $R[\![Z_1]\!]$, the coefficient of $Z_1^{i_1-1}$ in $f^{i_1-1} - Z_1 f^{i_1-2}(\partial f/\partial Z_1)$ vanishes. In other words,

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = 0.$$

If $i_1 = 1$, then

$$\operatorname{res}\begin{bmatrix} Z_1 f^{i_1-2} \frac{\partial f}{\partial Z_1} dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1^{i_1}, \ldots, Z_n^{i_n} \end{bmatrix} = 0.$$

Hence

$$\operatorname{res}\begin{bmatrix} dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1}, \ldots, Y_n^{i_n} \end{bmatrix} = \operatorname{res}\begin{bmatrix} dZ_1 \wedge \cdots \wedge dZ_n \\ Z_1^{i_1}, \ldots, Z_n^{i_n} \end{bmatrix}.$$

□

By the linearity and vanishing laws, the proposition gives the following corollary.

Invariance Law. *Let $Y_1, \ldots, Y_n$ be variables of A. Given $g = \sum c_{i_1 \cdots i_n} Y_1^{i_1} \cdots Y_n^{i_n}$,*

$$\operatorname{res}\begin{bmatrix} g dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1+1}, \ldots, Y_n^{i_n+1} \end{bmatrix} = c_{i_1 \cdots i_n}.$$

4 Lagrange Inversion

Lagrange inversion is an important tool for combinatorial problems. Analytic approaches can be found in [16, Proposition 12.3.1] and [19, Theorem 5.1.1]. See [2, 15] for surveys. There are other kinds of inversions in combinatorics. We remark that Möbius inversion can be also regarded as Lagrange inversion [12]. In this section, we explain how Lagrange inversion itself and its applications come from the invariance law.

Let w be a power series in $\kappa[\![X]\!]$ defined by $w = X\phi$ for an invertible power series $\phi \in \kappa[\![X]\!]$. For the case that the coefficient field κ has characteristic zero, the Lagrange inversion formula is sometimes stated as

$$[X^n]w(X)^k = \frac{k}{n}[X^{n-k}]\phi(X)^n.$$

However, the notation above is redundant. Power series w and ϕ are already specified. It is not necessary to write them as $w(X)$ and $\phi(X)$. Note that w is also a variable. It would shed more light to Lagrange inversion by writing the formula as

$$[X^n]w(X)^k = \frac{k}{n}[w^{n-k}]\phi(w)^n$$

or simply

$$[X^n]w^k = \frac{k}{n}[w^{n-k}]\phi^n.$$

By the invariance law, the latter can be stated as

$$\operatorname{res}\begin{bmatrix} w^k dX \\ X^{n+1} \end{bmatrix} = \frac{k}{n}\operatorname{res}\begin{bmatrix} \phi^n dw \\ w^{n-k+1} \end{bmatrix}.$$

Our proof of Lagrange inversion keeps track the relation $w = X\phi$ between the variable X and the variable w. Write $w^k = \sum_{i\geq 0} c_i X^i$ for $c_i \in \kappa$. Then $dw^k = \sum_{i\geq 1} ic_i X^{i-1} dX$ and

$$\operatorname{res}\left[\begin{array}{c} w^k dX \\ X^{n+1} \end{array}\right] = c_n = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} dw^k \\ X^n \end{array}\right].$$

By the transformation law,

$$\frac{1}{n}\operatorname{res}\left[\begin{array}{c} dw^k \\ X^n \end{array}\right] = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} \phi^n dw^k \\ w^n \end{array}\right].$$

By the Leibniz rule, $dw^k = kw^{k-1}dw$ and

$$\frac{1}{n}\operatorname{res}\left[\begin{array}{c} \phi^n dw^k \\ w^n \end{array}\right] = \frac{k}{n}\operatorname{res}\left[\begin{array}{c} \phi^n w^{k-1} dw \\ w^n \end{array}\right].$$

By the transformation law again, we obtain the required formula

$$\operatorname{res}\left[\begin{array}{c} w^k dX \\ X^{n+1} \end{array}\right] = \frac{k}{n}\operatorname{res}\left[\begin{array}{c} \phi^n w^{k-1} dw \\ w^n \end{array}\right] = \frac{k}{n}\operatorname{res}\left[\begin{array}{c} \phi^n dw \\ w^{n-k+1} \end{array}\right].$$

Some applications of Lagrange inversion are routine residue calculus involving changes of variables.

Example 1. *Catalan numbers C_n are defined by the power series $C = \sum C_i X^i$ satisfying $C = 1 + XC^2$. Let $Y := C - 1$. Then $\kappa[\![X]\!] = \kappa[\![Y]\!]$. Indeed, $X = Y/(1+Y)^2$. For $n > 0$,*

$$C_n = \operatorname{res}\left[\begin{array}{c} YdX \\ X^{n+1} \end{array}\right] = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} dY \\ X^n \end{array}\right] = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} (1+Y)^{2n} dY \\ Y^n \end{array}\right] = \frac{1}{n}\binom{2n}{n-1}.$$

Example 2. *Motzkin numbers M_n are defined by the power series $M = \sum M_i X^i$ satisfying $M = 1 + XM + X^2M^2$. Let $Y := XM$. Then $\kappa[\![X]\!] = \kappa[\![Y]\!]$. Indeed, $X = Y/(1+Y+Y^2)$. For $n \geq 0$,*

$$M_n = \frac{1}{n+1}\operatorname{res}\left[\begin{array}{c} dY \\ X^{n+1} \end{array}\right] = \frac{1}{n+1}\operatorname{res}\left[\begin{array}{c} (1+Y+Y^2)^{n+1} dY \\ Y^{n+1} \end{array}\right].$$

Applying the binomial theorem to $1 + Y$ and Y^2, we obtain

$$M_n = \frac{1}{n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{i}\binom{n+1-i}{n-2i}.$$

Lagrange inversion in several variables is also a phenomenon of changes of variables. Consider a power series ring $\kappa[\![X_1, \ldots, X_n]\!] = \kappa[\![Y_1, \ldots, Y_n]\!]$, where $Y_i = X_i\varphi_i$ for an invertible φ_i. Then

$$dX_i = d\frac{Y_i}{\varphi_i} = \sum_{j=1}^{n}\frac{\partial(Y_i/\varphi_i)}{\partial Y_j}dY_j = \frac{1}{\varphi_i}\sum_{j=1}^{n}(\delta_{ij} - \frac{Y_i}{\varphi_i}\frac{\partial\varphi_i}{\partial Y_j})dY_j$$

and

$$dX_1 \wedge \cdots \wedge dX_n = (\varphi_1 \cdots \varphi_n)^{-1} \det(\delta_{ij} - \frac{Y_i}{\varphi_i}\frac{\partial \varphi_i}{\partial Y_j}) dY_1 \wedge \cdots \wedge dY_n.$$

For any power series g, we have the Lagrange-Good formula [3]

$$\begin{aligned} \operatorname{res}\begin{bmatrix} g dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1+1}, \ldots, X_n^{i_n+1} \end{bmatrix} &= \operatorname{res}\begin{bmatrix} g\varphi_1^{i_1+1} \cdots \varphi_n^{i_n+1} dX_1 \wedge \cdots \wedge dX_n \\ Y_1^{i_1+1}, \ldots, Y_n^{i_n+1} \end{bmatrix} \\ &= \operatorname{res}\begin{bmatrix} g\varphi_1^{i_1} \cdots \varphi_n^{i_n} \det(\delta_{ij} - \frac{Y_i}{\varphi_i}\frac{\partial \varphi_i}{\partial Y_j}) dY_1 \wedge \cdots \wedge dY_n \\ Y_1^{i_1+1}, \ldots, Y_n^{i_n+1} \end{bmatrix} \end{aligned}$$

over arbitrary coefficient field. There is a vast literature on Lagrange inversion. We mention a recent study [13]. For more Lagrange inversion formulas computed using residues, see [9].

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Institute of Mathematics
Academia Sinica, Taiwan, R.O.C.
E-mail: `ichuang@math.sinica.edu.tw`