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A nonlinear elasticity system in Sobolev spaces with variable exponents

by

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Abstract

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in constant exponents Sobolev spaces. In this article we consider a Dirichlet problem for nonlinear elasticity system with laws of general behavior. The coefficients of elasticity depends on x and the density of the volumetric forces depends on the displacement. We consider this problem as a Leray-Lions operator and the main aim of this paper is to apply Galerkin techniques and monotone operator theory to prove a theorem of existence and uniqueness.

Key Words: Existence and uniqueness, spaces of Lebesgue and Sobolev with variable exponents, Dirichlet problem, nonlinear elasticity system, operator of Leray-Lions.
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1 Introduction

The study of PDE problems with variable exponents is a new and quite interesting topic. It comes from the theory of nonlinear elasticity, elastic mechanics, fluid dynamics, electrorheological fluids, and image processing, etc. (see [2], [16], [17]).
First, we introduce the notations needed in this article. Let Ω a connected open bounded domain of $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N} = 3$) with Lipschitz boundary Γ. To a given field of displacement u, we associate a nonlinear deformation tensor E defined by

$$E\left(\nabla u(x)\right) = \frac{1}{2}\left({}^{T}\nabla u + \nabla u + {}^{T}\nabla u \nabla u\right),$$

whose components are:

$$E_{ij}\left(\nabla u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right),\ 1 \leq i, j \leq 3. \tag{1.1}$$

The corresponding nonlinear constraints tensor $\sigma(u) = (\sigma_{ij}(u(x)))_{1\leq i,j\leq 3}$ is then given by:

$$\sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)),\ 1 \leq i, j \leq 3, \tag{1.2}$$

which describes a nonlinear relation between the stress tensor $(\sigma_{ij})_{i,j=1,2,3}$ and the deformation tensor $(E_{ij})_{i,j=1,2,3}$. The coefficients of elasticity a_{ijkh} satisfy the following symmetry properties:

$$a_{ijkh} = a_{jikh} = a_{ijhk}, \text{ for all } 1 \leq i, j, k, h \leq 3. \tag{1.3}$$

The aim of this paper is to prove the existence and uniqueness of weak solutions for the following nonlinear elliptic problem, encountered in the theory of nonlinear elasticity:

$$\begin{cases} -\sum\limits_{j=1}^{3} \dfrac{\partial}{\partial x_j} \sigma_{ij}(u(x)) = f_i(x, u(x)) \text{ in } \Omega, 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum\limits_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega, 1 \leq i, j \leq 3, \\ E_{ij}(\nabla u(x)) = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} + \sum\limits_{m=1}^{3} \dfrac{\partial u_m}{\partial x_i} \dfrac{\partial u_m}{\partial x_j} \right) \text{ in } \Omega, 1 \leq i, j \leq 3, \\ u_i = 0 \text{ on } \Gamma, 1 \leq i \leq 3. \end{cases}$$

This problem models the behavior of a heterogeneous material with Dirichlet's condition on the boundary. The consideration of this general material is in no way restrictive. Indeed, we can applied this study to the most particular elastic materials, but this particular case makes it easy, to describe the different stages of this work. The tensor of the constraints considered here is nonlinear and grouped, as special cases, some models used in Ciarlet [3], Lions [11] and Dautry-Lions [5]. Let us cite by way of example (see [3],[9]):

1. The problem of pure displacement for a homogeneous or heterogeneous material of St Vennan-Kirchhoff where:

   - the applied volumetric forces f are dead (does not depend on u),

   - the tensor of stress is in the form (material of St Vennan-Kirchhoff):

   $$\begin{cases} \sigma_{ij}(u(x)) = \lambda(trE(\nabla u(x))) + 2\mu E_{ij}(\nabla u(x)), \\ 1 \leq i, j \leq 3,\ \lambda > 0,\ \mu > 0, \end{cases}$$

2. The coefficients of elasticity have the form:

   $$a_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}), 1 \leq i,\ j,\ p,\ q \leq 3$$

   with, λ and μ depend on x or not,

3. The applied volumetric forces f have the form $f(\xi) = |\xi|^{p(x)-1}\xi$,

4. Some models called "LES"(Large Eddy Simulations) used in fluid mechanics. These problems are:

   $$-\operatorname{div}(\psi(x) a(\nabla u(x))) = f(x).$$

   For $\psi \equiv 1$ and $a(\xi) = |\xi|^{p(x)-2}\xi$, the above equation may be described by:

   $$-\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f.$$

   The operator $\Delta_{p(x)} : u \longrightarrow \Delta_{p(x)}(u) = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$ -Laplacian.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in constant exposants Sobolev spaces for example in [3] Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution, in [5] Dautry-Lions studied the linear problem in a regular boundary domain, in [12], [13], [14] Merouani studied the Lamé (elasticity) system in a polygonal boundary domain, in [18] Zoubai and Merouani studied the existence and uniqueness of the solutions of the nonlinear elasticity system by topological degree, and in [19] Zoubai and Merouani studied the existence and uniqueness of the solution of Neumann's problem, in Sobolev spaces with variable exponents.

The bibliography quoted here does not claim to be exhaustive and the deficiencies it certainly entails must be attributed to the author's ignorance and not to the author's ill will.

To solve our problem, we will consider an operator: $u \to A(u) = -\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(u(x))$ as operator of Leray-Lions [10], with Dirichlet's condition on Γ, and we prove a theorem of existence and uniqueness of solution using Galerkin techniques and monotone operator theory.

The appropriate Sobolev space to consider for this problem is the space $\left(W_0^{1,p(x)}(\Omega)\right)^3 \cap \left(W^{2,p(x)}(\Omega)\right)^3$, where $p(x)$ needs to satisfy the log-Hölder condition (see [6],[8]) to obtain suitable properties.

This paper is organized as follows:

- Notations and properties of variable exponent Lebesgue-Sobolev spaces,
- Some properties of the operator E_{ij},
- Hypotheses and main result,
- Proof of theorem,
- Conclusion and bibliography.

2 Properties of variable exponent Lebesgue-Sobolev spaces

In this section, we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, when Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}$ $(\mathbb{N} \geq 1)$ with a smooth boundary.

Let $p : \overline{\Omega} \to [1, +\infty)$ be a continuous, real-valued function. Denote by $p_- = \min_{x\in\overline{\Omega}} p(x)$ and $p_+ = \max_{x\in\overline{\Omega}} p(x)$.

We introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The following inequality will be used later

$$\min\left\{\|u\|_{L^{p(x)}(\Omega)}^{p_-},\|u\|_{L^{p(x)}(\Omega)}^{p_+}\right\} \leq \int_{\Omega} |u\left(x\right)|^{p(x)}\,dx \leq \max\left\{\|u\|_{L^{p(x)}(\Omega)}^{p_-},\|u\|_{L^{p(x)}(\Omega)}^{p_+}\right\}$$

for any $u \in L^{p(x)}\left(\Omega\right)$.

Lemma 1. *[4], [6], [7], [8]*

- *The space* $\left(L^{p(x)}\left(\Omega\right), \|.\|_{L^{p(x)}(\Omega)}\right)$ *is a Banach space.*
- *If* $p_- > 1$, *then* $L^{p(x)}\left(\Omega\right)$ *is reflexive and its conjugate space can be identified with* $L^{p'(x)}\left(\Omega\right)$ *where,* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. *Moreover, for any* $u \in L^{p(x)}\left(\Omega\right)$ *and* $v \in L^{p'(x)}\left(\Omega\right)$, *we have the Hölder inequality*

$$\int_{\Omega} |uv|\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

- *If* $p_+ < +\infty$, *then* $L^{p(x)}\left(\Omega\right)$ *is separable.*
- *Some embedding stay true, for example, if* $0 < |\Omega| < \infty$ *and* p_1, p_2 *are variable exponent such that* $p_1\left(x\right) \leq p_2\left(x\right)$ *almost everywhere in* Ω, *then we have the continuous injection.*

Now, we define also the variable Sobolev space by

$$W^{1,p(x)}\left(\Omega\right) = \left\{u \in L^{p(x)}\left(\Omega\right);\ \ |\nabla u| \in L^{p(x)}\left(\Omega\right)\right\},$$

endowed with the following norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Definition 1. *The variable exponent* $p : \overline{\Omega} \to [1,+\infty)$ *is said to satisfy the log-Hölder continuous condition if*

$$\forall x, y \in \overline{\Omega},\ \ |x-y| < 1,\ \ |p\left(x\right) - p\left(y\right)| < w\left(|x-y|\right),$$

where $w : (0,\infty) \to \mathbb{R}$ *is a nondecreasing function with* $\lim_{\alpha\to 0}\sup w\left(\alpha\right)\ln\left(\frac{1}{\alpha}\right) < \infty$.

Lemma 2. *[4], [6], [7], [8]*

- *If* $1 < p_- \leq p_+ < \infty$, *then the space* $\left(W^{1,p(x)}\left(\Omega\right), \|.\|_{W^{1,p(x)}(\Omega)}\right)$ *is a separable and reflexive Banach space.*
- *If* $p(x)$ *satisfies the log-Hölder continuous condition, then* $C^{\infty}\left(\Omega\right)$ *is dense in* $W^{1,p(x)}\left(\Omega\right)$. *Moreover, we can define the Sobolev space with zero boundary values,* $W_0^{1,p(x)}\left(\Omega\right)$ *as the completion of* $C_0^{\infty}\left(\Omega\right)$ *with respect to the norm* $\|.\|_{W^{1,p(x)}(\Omega)}$.

- *For all* $u \in W_0^{1,p(x)}(\Omega)$, *the Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)}$$

holds. Moreover, $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$ *is a norm in* $W_0^{1,p(x)}(\Omega)$.

Remark 1. *[1] Let* $a \geq 0, b \geq 0$ *and let* $1 \leq p_- \leq p_+ < +\infty$, *then*

$$(a+b)^{p(x)} \leq 2^{p_+-1}(a^{p(x)} + b^{p(x)}).$$

Throughout this paper, we shall assume that the variable exponent $p(x)$ satisfy the log-Hölder condition, and $\mathbb{N} < p_- \leq p_+ < \infty$ because if $p(x) > \mathbb{N}$ then $W^{1,p(x)}(\Omega) \subset C(\Omega)$ for every $x \in \Omega$.

3 Some properties of the operator E_{ij}

For the rest of this work, we will need some properties of the deformation tensor (1.1). For this, we have the following lemma:

Theorem 1. *(Some properties of the operator* E_{ij}*)*
For $u \in \mathbf{W}^{p(x)}(\Omega) = \left(W_0^{1,p(x)}(\Omega)\right)^3 \cap \left(W^{2,p(x)}(\Omega)\right)^3$, *with* $3 < p(x) < +\infty$, *the components* E_{kh} *of the deformation tensor of St. Venant* E *verify the following properties:*
1. *(Continuity)* E_{kh} *is a continuous function,* $k, h = 1$ *to* 3,
2. *(Coercivity)* $\exists \alpha > 0$; *such as* $E_{kh}(\xi)\xi_{ij} \geq \alpha |\xi|^{p(x)}, \forall i,j,k,h = 1$ *to* 3,
3. $E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega), \forall i,j,k,h = 1$ *to* 3,
4. *(Monotony) Let the functions* $\frac{\partial u_i}{\partial x_j} : \Omega \longrightarrow \left]-\infty, \frac{1}{3}\right], x \longrightarrow \frac{\partial u_i}{\partial x_j}(x)$ *and* $\frac{\partial u_j}{\partial x_i} : \Omega \longrightarrow \left]-\infty, \frac{1}{3}\right], x \longrightarrow \frac{\partial u_j}{\partial x_i}(x), i,j = 1$ *to* 3; *then the operators*

$$E_{ij}(.) \text{ of } \mathbf{W}^{p(x)}(\Omega) \text{ in } \left(\mathbf{W}^{p(x)}(\Omega)\right)', \ i,j = 1 \text{ to } 3,$$

are monotonous.

Proof of theorem 1:

1. The continuity of E_{kh} :
for $p(x) > 3$, the space $W^{1,p(x)}(\Omega)$ is an algebra, that is to say

$$u, v \in W^{1,p(x)}(\Omega) \Rightarrow uv \in W^{1,p(x)}(\Omega).$$

So we have for $u \in \mathbf{W}^{p(x)}(\Omega) = \left(W_0^{1,p(x)}(\Omega)\right)^3 \cap \left(W^{2,p(x)}(\Omega)\right)^3$:

$$\frac{\partial u_k}{\partial x_h}, \frac{\partial u_h}{\partial x_k} \text{ and } \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_h} \in W^{1,p(x)}(\Omega),$$

and therefore $E_{kh}(\nabla u) \in W^{1,p(x)}(\Omega)$. In addition, for $p(x) > 3$, we have the continuous injection $W^{1,p(x)}(\Omega) \hookrightarrow C(\Omega)$, so the continuity of $E_{kh}, k, h = 1$ to 3 are fulfilled.
2. The Coercivity:
for the coercivity of the components E_{kh} see [15].
3. $E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega), \forall i, j, k, h = 1$ à 3,
by exploiting the remark 1, we arrive at

$$\begin{aligned}
|E_{kh}(\nabla u)|^{p(x)} &= \left(\tfrac{1}{2}\right)^{p(x)} \left|\left(\tfrac{\partial u_k}{\partial x_h} + \tfrac{\partial u_h}{\partial x_k} + \sum_{m=1}^{3} \tfrac{\partial u_m}{\partial x_k}\tfrac{\partial u_m}{\partial x_h}\right)\right|^{p(x)}, \\
&\le \left(\tfrac{1}{2}\right)^{p(x)} \times 2^{p^+-1}\left[\left|\tfrac{\partial u_k}{\partial x_h} + \tfrac{\partial u_h}{\partial x_k}\right|^{p(x)} + \left(\left|\sum_{m=1}^{3} \tfrac{\partial u_m}{\partial x_k}\tfrac{\partial u_m}{\partial x_h}\right|\right)^{p(x)}\right], \\
&\le \left(\tfrac{1}{2}\right)^{p(x)} \times 2^{p^+-1}\left[2^{p^+-1}\left(\left|\tfrac{\partial u_k}{\partial x_h}\right|^{p(x)} + \left|\tfrac{\partial u_h}{\partial x_k}\right|^{p(x)}\right) + \left(\left|\sum_{m=1}^{3} \tfrac{\partial u_m}{\partial x_k}\tfrac{\partial u_m}{\partial x_h}\right|\right)^{p(x)}\right],
\end{aligned}$$

hence

$$E_{kh}(\nabla u) \in L^{p(x)}(\Omega),\ k, h = 1 \text{ to } 3,$$

and as $p(x) > p'(x)$ as soon as $p(x) > 3$ and Ω bounded, we have:

$$E_{kh}(\nabla u) \in L^{p'(x)}(\Omega),\ k, h = 1 \text{ to } 3.$$

Take then $v \in \mathbf{W}^{p(x)}(\Omega)$, we have $\frac{\partial v_i}{\partial x_j} \in L^{p(x)}(\Omega), \forall\, 1 \le i, j \le 3$. We therefore have by Hölder's inequality:

$$E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j} \in L^1(\Omega),\ i, j, k, h = 1 \text{ to } 3.$$

4. The monotony:
using the rule $\frac{1}{2}(a^2 + b^2) \ge -ab$, with $a = \frac{\partial u_m}{\partial x_i}$ and $b = \frac{\partial u_m}{\partial x_j}$, we have

$$\begin{aligned}
E_{ij}(u) &\ge \tfrac{1}{2}\left(\tfrac{\partial u_i}{\partial x_j} + \tfrac{\partial u_j}{\partial x_i}\right) - \tfrac{1}{4}\sum_{m=1}^{3}\left(\left(\tfrac{\partial u_m}{\partial x_i}\right)^2 + \left(\tfrac{\partial u_m}{\partial x_j}\right)^2\right) = \\
&\tfrac{1}{2}\left(\tfrac{\partial u_i}{\partial x_j} + \tfrac{\partial u_j}{\partial x_i}\right) - \tfrac{1}{4}\left(\sum_{m=1}^{3}\left(\tfrac{\partial u_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\tfrac{\partial u_m}{\partial x_j}\right)^2\right), i, j = 1 \text{ to } 3,
\end{aligned}$$

and consequently, $\forall i, j = 1$ to 3:

$$\begin{aligned}
\langle E_{ij}(u) - E_{ij}(v), u - v\rangle &\ge \tfrac{1}{2}\left\langle\left(\tfrac{\partial u_i}{\partial x_j} + \tfrac{\partial u_j}{\partial x_i}\right) - \left(\tfrac{\partial v_i}{\partial x_j} + \tfrac{\partial v_j}{\partial x_i}\right), u - v\right\rangle \\
-\tfrac{1}{4}\left\langle\left(\sum_{m=1}^{3}\left(\tfrac{\partial u_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\tfrac{\partial u_m}{\partial x_j}\right)^2\right) \right.&\left. - \left(\sum_{m=1}^{3}\left(\tfrac{\partial v_m}{\partial x_i}\right)^2 + \sum_{m=1}^{3}\left(\tfrac{\partial v_m}{\partial x_j}\right)^2\right), u - v\right\rangle.
\end{aligned} \tag{3.1}$$

To conclude, we must prove that the second member of (3.1) is ≥ 0. For that, we separate the second member of (3.1) in linear and nonlinear part.
Let the linear function $\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{A_{ij}} \mathbb{R}$, defined by

$$(A_{ij} \circ J_x)(x) = A_{ij}\left(\frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x)\right), i, j = 1 \text{ to } 3,$$

and the nonlinear function $\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R}$, defined by

$$(B_{ij}\circ J_x)(x) = B_{ij}\left(\frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right), i,j = 1 \text{ to } 3.$$

The functions A_{ij} and B_{ij} are continuous for $p(x) > 3$. It remains to show that, $\forall i,j = 1$ to 3, the A_{ij} are increasing on $\mathbb{R}$, the B_{ij} increasing on $\mathbb{R}^-$ and the $A_{ij} + B_{ij}$ increasing on $\left]-\infty, \frac{1}{3}\right]$.

1. Let us show that the A_{ij} are increasing: let the function

$$\Omega \xrightarrow{J_x} \mathbb{R} \xrightarrow{\frac{\partial u_i}{\partial x_j}} \mathbb{R}, \text{defined by } \left(\frac{\partial u_i}{\partial x_j} \circ J_x\right)(x) = \frac{\partial u_i}{\partial x_j}(x), i,j = 1 \text{ to } 3.$$

We note

$$\frac{\partial u}{\partial x_j}(x) = t_j \text{ and } \frac{\partial u}{\partial x_i}(x) = \tau_i,$$

and

$$\frac{\partial u_i}{\partial x_j}(x) = t_{ij} \text{ and } \frac{\partial u_j}{\partial x_i}(x) = \tau_{ji}.$$

The function $t \longmapsto \frac{1}{2}t$ of $\mathbb{R} \longrightarrow \mathbb{R}$, being increasing on $\mathbb{R}$, we have:

$$\frac{1}{2}\left\langle \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}, \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right\rangle = \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right) dx =$$
$$\frac{1}{2}\left\|\frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right\|^2_{L^2(\Omega)} \geq 0.$$

Therefore, the A_{ij} are increasing.

2. Let us show that the B_{ij} are increasing: let the function

$$\Omega \xrightarrow{J_x} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R},$$

defined by

$$(B_{ij} \circ J_x)(x) = B_{ij}\left(t_j, \tau_i\right) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right), i,j = 1 \text{ to } 3.$$

As in point 1., we note

$$t_{ij} = \frac{\partial u_i}{\partial x_j}(x), \tau_{ji} = \frac{\partial u_j}{\partial x_i}(x), \forall i,j = 1 \text{ to } 3,$$

$$B_{ij}\left(t_j, \tau_i\right) = -\frac{1}{4}\left(\sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}(x)\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(x)\right)^2\right)$$

so

$$\begin{aligned} B_{ij}\left(t_j, \tau_i\right) &= -\frac{1}{4}\left(\sum_{i=1}^{3} t_{ij}^2 + \sum_{j=1}^{3} \tau_{ji}^2\right) \\ &\geq -\frac{1}{4}\left(6 \times \underset{1 \leq i\ ,\ j \leq 3}{Max}\ \left(t_{ij}^2, \tau_{ji}^2\right)\right) = -\frac{3}{2}\varkappa^2. \end{aligned}$$

The function $f(\varkappa) = -\frac{3}{2}\varkappa^2$ being continuous and increasing on $\mathbb{R}^-$, we deduce that the B_{ij} are increasing on $\mathbb{R}^-$.

3. We show that the $A_{ij} + B_{ij}$, are increasing: the proofs of points 1. and 2. imply that the sum $A_{ij} + B_{ij}$, $\forall i, j = 1$ to 3, corresponds to the sum of the two functions $f(\varkappa) + g(\varkappa) = \varkappa - \frac{3}{2}\varkappa^2, \mathbb{R} \longrightarrow \mathbb{R}$, obviously continuous and increasing on $\left]-\infty, \frac{1}{3}\right]$, as the derivative of the convex function $h(x) = \frac{1}{2}x^2 - \frac{1}{2}x^3$ on $\left]-\infty, \frac{1}{3}\right]$. So, (3.1) is verified and consequently

$$\langle E_{ij}(u) - E_{ij}(v), u - v\rangle \geq \langle (A_{ij} + B_{ij})(u) - (A_{ij} + B_{ij})(v), u - v\rangle \geq 0, \forall i, j = 1 \text{ to } 3.$$

In other words, the $E_{ij}(u), i, j = 1$ to 3, are monotonous $\mathbf{W}^{p(x)}(\Omega)$ in $\left(\mathbf{W}^{p(x)}(\Omega)\right)'$, $i, j = 1$ to 3.

Corollary 1. *Under the same assumptions, of the above theorem, the operator*

$$-div(a_{ijkh}(x)\ E_{ij}(.)) \text{ is monotonous of } \mathbf{W}^{p(x)}(\Omega) \text{ in } \left(\mathbf{W}^{p(x)}(\Omega)\right)'.$$

under the assumption

$$\exists \alpha, \beta \in \mathbb{R}_+^*;\ \alpha \leq a_{ijkh}(x) \leq \beta,\ a.e., \forall i, j, k, h = 1 \text{ to } 3.$$

Proof of corollary 1:
We have

$$\forall (u, v) \in \mathbf{W}^{p(x)}(\Omega)^2, \langle -div(a_{ijkh}(x)\ E_{ij}(u)) - (-div(a_{ijkh}(x)\ E_{ij}(v)), u - v\rangle \geq \alpha \sum_{i,j=1}^{3} \int_\Omega (E_{ij}(u) - E_{ij}(v)) \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j}\right) dx.$$

The point **4**. of theorem 1 implies that

$$\sum_{i,j=1}^{3} \int_\Omega (E_{ij}(u) - E_{ij}(v)) \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j}\right) dx \geq 0,$$

hence the desired result.

4 Hypotheses and main result

We consider the problem (4.1), with the hypotheses (4.2),

$$\begin{cases} -\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(u(x)) = f_i(x, u(x)) \text{ in } \Omega, 1 \leq i \leq 3, \\ \sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\ E_{kh}(\nabla u(x)) \text{ in } \Omega, 1 \leq i, j \leq 3, \\ u_i = 0 \text{ on } \Gamma, 1 \leq i \leq 3. \end{cases} \tag{4.1}$$

$$\begin{cases} \forall i,j,k,h=1 \text{ to } 3: \\ (1)\ a_{ijkh}\in L^{\infty}(\Omega);\ \exists\alpha_0>0;\ a_{ijkh}\geq\alpha_0 \text{ a.e. in } \Omega, \\ (2)\ f=(f_1,f_2,f_3)\in\left(L^{\frac{p(x)}{p(x)-1}}(\Omega)\right)^3. \end{cases} \tag{4.2}$$

Let us look for an adequate weak form of (4.1).
Let $u\in\mathbf{W}^{p(x)}(\Omega)=\left(W_0^{1,p(x)}(\Omega)\right)^3\cap\left(W^{2,p(x)}(\Omega)\right)^3$ equipped with $\|.\|_{\mathbf{W}^{p(x)}(\Omega)}=\|.\|_{\left(W_0^{1,p(x)}(\Omega)\right)^3}$.
From the theorem 1, we have

$$E_{kh}(\nabla u)\frac{\partial v_i}{\partial x_j}\in L^1(\Omega),\ i,j,k,h=1 \text{ to } 3.$$

It is therefore natural to look $u\in\mathbf{W}^{p(x)}(\Omega)$ and take the test functions in $\mathbf{W}^{p(x)}(\Omega)$. We also recall that if $f(.,s)\in\left(L^{p'(x)}(\Omega)\right)^3$, the mapping $v\to\int_\Omega f(x,u(x))v(x)\,dx$ acting from $\mathbf{W}^{p(x)}(\Omega)$ to $\mathbb{R}$, is an element of $\left(\mathbf{W}^{p(x)}(\Omega)\right)'$. We denote by f this element, that is to say for $f\in\left(L^{p'(x)}(\Omega)\right)^3$, we have

$$\langle f,v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)',\mathbf{W}^{p(x)}(\Omega)}=\int_\Omega f(x,u(x))\,v(x)\,dx,\forall v\in\mathbf{W}^{p(x)}(\Omega).$$

The weak form of (4.1) is thus:

$$\begin{cases} u\in\mathbf{W}^{p(x)}(\Omega), \\ \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial vi}{\partial xj}dx \\ =\langle f,v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)',\mathbf{W}^{p(x)}(\Omega)},\forall v\in\mathbf{W}^{p(x)}(\Omega). \end{cases} \tag{4.3}$$

Theorem 2. *Under the hypotheses (4.2), there exist $u\in\mathbf{W}^{p(x)}(\Omega)$ solution of (4.3). If, moreover, $(E_{kh}(\xi)-E_{kh}(\eta))(\xi_{ij}-\eta_{ij})>0$, for all $\xi,\eta\in\mathbb{R}^{3\times3},\xi_{ij},\eta_{ij}\in\mathbb{R},\xi_{ij}\neq\eta_{ij}$, and f does not depend on u then there exist a unique solution u of (4.3).*

For the proof of this theorem, we will need the following lemmas:

Lemma 3. *Let $p:\Omega\to]1,+\infty[$. If $f_n\to f$ in $L^{p(x)}(\Omega)$ and $g_n\to g$ weakly in $L^{p'(x)}(\Omega)$. So*

$$\int_\Omega f_n\ g_n dx\to\int_\Omega f\ g dx \text{ when } n\to\infty.$$

Demonstration of Lemma 3
We have:

$$\begin{aligned}\left|\int_\Omega(f_n\ g_n-f\ g)\,dx\right| &=\left|\int_\Omega(f_n\ g_n-f\ g-f\ g_n+f\ g_n)\,dx\right| \\ &=\left|\int_\Omega[(f_n-f)\ g_n+f\ (g_n-g)]\,dx\right| \\ &\leq\int_\Omega|f_n-f|\ |g_n|\,dx+\left|\int_\Omega f\ (g_n-g)dx\right| \\ &\leq 2\|f_n-f\|_{L^{p(x)}(\Omega)}\|g_n\|_{L^{p'(x)}(\Omega)}+\left|\langle g_n-g,f\rangle_{L^{p'(x)}(\Omega),L^{p(x)}(\Omega)}\right|\to 0.\end{aligned}$$

Lemma 4. *(Finite-dimensional coercive operator) Let V be a finite-dimensional space, and $T : V \to V'$ continuous. We suppose that T is coercive, namely:*

$$\frac{\langle T(v).v\rangle_{V',V}}{\|v\|_V} \to +\infty \text{ when } \|v\|_V \to +\infty.$$

Then, for every $b \in V'$ there exists $v \in V$ such that $T(v) = b$.

5 Proof of theorem

Study of finite dimension problem

Since $\mathbf{W}^{p(x)}(\Omega)$ is separable, then, there exists a countable family $(f_n)_{n\in\mathbb{N}^*}$ dense in $\mathbf{W}^{p(x)}(\Omega)$. Let $V_n = Vect\{f_i, i = 1, ..., n\}$ be the vector space generated by the first n functions of this family. So we have $\dim V_n \le n$, $V_n \subset V_{n+1}$ for all $n \in \mathbb{N}^*$ and we have $\overline{\underset{n\in\mathbb{N}}{\cup} V_n} = \mathbf{W}^{p(x)}(\Omega)$.
We deduce that for all $v \in \mathbf{W}^{p(x)}(\Omega)$ there exists a sequence $v_n \in V_n$, such that $v_n \to v$ in $\mathbf{W}^{p(x)}(\Omega)$ when $n \to +\infty$.
In the first step, we fix $n \in \mathbb{N}^*$ and look for u_n solution of the following problem, posed in finite dimension:

$$\begin{cases} u_n \in V_n, \\ \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial v_i}{\partial x_j}dx \\ = \langle f, v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)',\mathbf{W}^{p(x)}(\Omega)}, \forall v \in V_n. \end{cases} \tag{5.1}$$

The application $v \to \langle f, v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)',\mathbf{W}^{p(x)}(\Omega)}$ is a linear mapping of V_n to $\mathbb{R}$ (it is also continuous because $\dim V_n < +\infty$). We denote by b_n this application. So $b_n \in V_n'$ and

$$\langle b_n, v\rangle_{V_n',V_n} = \langle f, v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)',\mathbf{W}^{p(x)}(\Omega)}.$$

Let $u \in V_n$. We denote by $T_n(u)$ the mapping of V_n into V_n' which has $v \in V_n$ associated

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx.$$

This application is linear and continuous, so it is also an element of V_n' and we have

$$\langle T_n(u), v\rangle_{V_n',V_n} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx.$$

We have thus defined an application T of V_n to V_n'. We shall show that T is continuous and coercive. We can thus deduce by the Lemma 4, that T is surjective, and therefore that there exists $u_n \in V_n$ satisfying $T(u_n) = b_n$, quecisely u_n is the solution of the problem (5.1).
(a) Continuity of T_n. To ease the writing, we note $V = V_n$ equipped with $\|.\|_V = \|.\|_{\mathbf{W}^{p(x)}(\Omega)}$ and note $T = T_n$. Let $u, \overline{u} \in V$, we have:

-Œ$_{kh}\left(\nabla\overline{u}\right))\dfrac{\partial v_i}{\partial x_j}dx$

Putting

$$a=\left\|a_{ijkh}\right\|_{L^{\infty}(\Omega)},$$

we obtain by Hölder inequality

$$\begin{aligned}\left\|T\left(u\right)-T\left(\overline{u}\right)\right\|_{V'}&=\sup_{v\in V,\ \|v\|_V=1}\left|\left\langle T\left(u\right)-T\left(\overline{u}\right),v\right\rangle_{V',V}\right|\\&=\sup_{v\in V,\ \|v\|_{\mathbf{W}^{p(x)}(\Omega)}=1}\left|\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)(E_{kh}\left(\nabla u\right)-E_{kh}\left(\nabla\overline{u}\right))\frac{\partial v_i}{\partial x_j}dx\right|\\&\leq\sup_{v\in\mathbf{W}^{p(x)}(\Omega),\ \|v\|_{\mathbf{W}^{p(x)}(\Omega)}=1}\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\\&\left|\int_\Omega a_{ijkh}(x)(E_{kh}\left(\nabla u\right)-E_{kh}\left(\nabla\overline{u}\right))\frac{\partial v_i}{\partial x_j}dx\right|\end{aligned}$$

Thus if $(u_n)_{n\in\mathbb{N}}$ is a sequence of V such that $u_n\to\overline{u}$ in V, we have

$$\left\|T\left(u_n\right)-T\left(\overline{u}\right)\right\|_{V'}\leq 18a\sum_{k,h=1}^{3}\left\|E_{kh}\left(\nabla u_n\right)-E_{kh}\left(\nabla\overline{u}\right)\right\|_{L^{p'(x)}(\Omega)}.$$

For $u\in\mathbf{W}^{p(x)}(\Omega)$, we have $E_{kh},k,h=1$ to 3, are continuous (see theorem 1), consequently $E_{kh}\left(\nabla u_n\right)\to E_{kh}\left(\nabla\overline{u}\right)$ a.e.. We have also, according to the remark 1, $E_{kh}\left(\nabla u\right)$ is bounded in $L^{p(x)}(\Omega)$, so it bounded in $L^{p'(x)}(\Omega)$ because $p(x)>p'(x)$, as soon as $p(x)>3$. So by Lebesgue's dominated convergence theorem $E_{kh}\left(\nabla u_n\right)\to E_{kh}\left(\nabla\overline{u}\right)$ in $L^{p'(x)}\left(\Omega\right),\forall k,h=1$ to 3. We have thus shown that $T\left(u_n\right)\to T\left(\overline{u}\right)$ in V', so T is continuous.
(b) Coercivity of T_n. According to the hypothesis (1) of (4.2), and the coercivity of E_{kh} (see theorem 1), we have:

$$\begin{aligned}\left\langle T\left(u\right).u\right\rangle_{V',V}&=\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial u_i}{\partial x_j}dx\\&\geq C_1\int_\Omega\left|\nabla u\right|^{p(x)}dx.\end{aligned}$$

Now, we use the following inéquality (see section 2)

$$\min\left\{\left\|u\right\|_{L^{p(x)}(\Omega)}^{p_-},\left\|u\right\|_{L^{p(x)}(\Omega)}^{p_+}\right\}\leq\int_\Omega\left|u\left(x\right)\right|^{p(x)}dx\leq\max\left\{\left\|u\right\|_{L^{p(x)}(\Omega)}^{p_-},\left\|u\right\|_{L^{p(x)}(\Omega)}^{p_+}\right\}$$

we obtain

$$\begin{aligned}\left\langle T\left(u\right).u\right\rangle_{V',V}&\geq C_1\min\left\{\left\|\nabla u\right\|_{L^{p(x)}(\Omega)}^{p_-},\left\|\nabla u\right\|_{L^{p(x)}(\Omega)}^{p_+}\right\}\\&\geq C_1\min\left\{\left\|u\right\|_V^{p_-},\left\|u\right\|_V^{p_+}\right\}.\end{aligned}$$

Or, C_1 is a constant depend of α_0 and α.
Consequently, the operator T is coercive. This yields the existence of solution for problem (5.1).

Study of infinite dimension problem

The solution of the problem (5.1) is obtained.
So to show the existence of u a solution of (4.3), we will estimate u_n the solution of (5.1) and then by crossing to the limit when $n \to +\infty$ we will have the solution u of our problem (4.3). Therefore that technique used to show that the limit of the nonlinear term is the desired term.

(a) Estimation on u_n

In view of coercivity, if we substitute v by u_n in (5.1), we obtain:

$$C_1 \int_\Omega |\nabla u_n|^{p(x)}\, dx \le \|f\|_{\left(\mathbf{W}^{p(x)}(\Omega)\right)'} \|u_n\|_{\mathbf{W}^{p(x)}(\Omega)}.$$

On the other hand

$$C_1 \min\left\{\|u_n\|^{p-}_{\mathbf{W}^{p(x)}(\Omega)}, \|u_n\|^{p+}_{\mathbf{W}^{p(x)}(\Omega)}\right\} \le \|f\|_{\left(\mathbf{W}^{p(x)}(\Omega)\right)'} \|u_n\|_{\mathbf{W}^{p(x)}(\Omega)}.$$

(b) Passage to the limit

We deduce from (a) that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{W}^{p(x)}(\Omega)$, so there exists a subsequence denoted again $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ weakly in $\mathbf{W}^{p(x)}(\Omega)$.

The sequence $(E_{kh}(\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $L^{p'(x)}(\Omega)$. Hence there exists $\rho \in L^{p'(x)}(\Omega)$ such that, with a close subsequence,

$$E_{kh}(\nabla u_n) \to \rho \text{ weakly in } L^{p'(x)}(\Omega).$$

Let $v \in \mathbf{W}^{p(x)}(\Omega)$, then there exist $v_n \in V_n$, $n \in \mathbb{N}^*$ such that

$$v_n \to v \text{ in } \mathbf{W}^{p(x)}(\Omega),$$
$$\nabla v_n \to \nabla v \text{ in } \left(L^{p(x)}(\Omega)\right)^9.$$

We substitute v by v_n in (5.1), to obtain:

$$\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_\Omega a_{ijkh}(x) E_{kh}(\nabla u_n(x)) \frac{\partial v_{ni}}{\partial x_j}(x)dx$$
$$= \langle f, v_n\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)', \mathbf{W}^{p(x)}(\Omega)}, \forall v \in V_n.$$

Since $\langle f, v_n\rangle \to \langle f, v\rangle$, $E_{kh}(\nabla u_n) \to \rho$ weakly in $L^{p'(x)}(\Omega)$ and $\frac{\partial v_{ni}}{\partial x_j} \to \frac{\partial v_i}{\partial x_j}$ for $i = 1$ to 3 strongly in $L^{p(x)}(\Omega)$ (because $\nabla v_n \to \nabla v$ in $\left(L^{p(x)}(\Omega)\right)^9$strongly), using the Lemma 3, we obtain

$$\begin{cases} \displaystyle\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_\Omega a_{ijkh}(x) \rho \frac{\partial v_i}{\partial x_j} dx = \\ \langle f, v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)', \mathbf{W}^{p(x)}(\Omega)}, \forall\, v \in \mathbf{W}^{p(x)}(\Omega). \end{cases} \tag{5.2}$$

We tend to conclude that ρ is equal to $E_{kh}(\nabla u)$. Unfortunately, this is not obvious because the E_{kh} are nonlinear.

(c) Limit of nonlinear term

Finally, it remains to prove that

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\rho\frac{\partial v_i}{\partial x_j}dx = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u(x))\frac{\partial v_i}{\partial x_j}dx, \forall v\in \mathbf{W}^{p(x)}(\Omega). \tag{5.3}$$

(I) First, we have

$$\lim_{n\to\infty}\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial u_{ni}}{\partial x_j}dx = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\rho\frac{\partial u_i}{\partial x_j}dx.$$

Indeed

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial u_{ni}}{\partial x_j}dx = \langle f,u_n\rangle \to \langle f,u\rangle .$$

(II) Proof of (5.3)

Let $v\in \mathbf{W}^{p(x)}(\Omega)$, there exist $(v_n)_{n\in\mathbb{N}}$ such that $v_n\in V_n$ for all $n\in\mathbb{N}$ and $v_n\to v$ in $\mathbf{W}^{p(x)}(\Omega)$ when $n\to+\infty$. We will pass to the limit in the term

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n(x))\frac{\partial v_{ni}}{\partial x_j}dx,$$

thanks to the hypothesis (1) of (4.2) and the monotony of E_{kh} (see corrollary 1).
Indeed,

$$\begin{aligned}
0 &\le \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)(E_{kh}(\nabla u_n)-E_{kh}(\nabla v_n))\left(\frac{\partial u_{ni}}{\partial x_j}-\frac{\partial v_{ni}}{\partial x_j}\right)dx = \\
&\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n)\frac{\partial u_{ni}}{\partial x_j}dx - \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u_n)\frac{\partial v_{ni}}{\partial x_j}dx \\
&-\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla v_n)\frac{\partial u_{ni}}{\partial x_j}dx + \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla v_n)\frac{\partial v_{ni}}{\partial x_j}dx \\
&= T_{1,n} - T_{2,n} - T_{3,n} + T_{4,n}.
\end{aligned}$$

It has been seen that in **(I)**:

$$\lim_{n\to+\infty}T_{1,n} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\rho\frac{\partial u_i}{\partial x_j}dx,$$

we have

$$\lim_{n\to+\infty}T_{2,n} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\rho\frac{\partial v_i}{\partial x_j}dx,$$

by a product of a strong convergence in $L^{p(x)}(\Omega)$ and a weak convergence in $L^{p'(x)}(\Omega)$ (Lemma 3).
The same

$$\lim_{n\to+\infty} T_{3,n} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla v)\frac{\partial u_i}{\partial x_j}dx,$$

by a product of a strong convergence of $E_{kh}(\nabla v_n)$ in $L^{p'(x)}(\Omega)$ (because $E_{kh}, k,h=1$ to 3 are continuous and bounded in $L^{p'(x)}(\Omega)$, so by Lebesgue's dominated convergence theorem $E_{kh}(\nabla v_n)\to E_{kh}(\nabla v)$ in $L^{p'(x)}(\Omega)$), and a weak convergence in $L^{p(x)}(\Omega)$.
Finally, we have

$$\lim_{n\to+\infty} T_{4,n} = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla v)\frac{\partial v_i}{\partial x_j}dx,$$

by the product of a strong convergence in $L^{p'(x)}(\Omega)$ and a strong convergence in $L^{p(x)}(\Omega)$.
The passage to the limit in inequality thus gives:

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\left(\rho - E_{kh}(\nabla v)\right)\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial v_i}{\partial x_j}\right)dx \geq 0 \text{ for all } v\in \mathbf{W}^{p(x)}(\Omega).$$

The function test v is now astutely chosen. We take $v = u + \frac{1}{n}w$ with $w\in\mathbf{W}^{p(x)}(\Omega)$ and $n\in\mathbb{N}^*$. We obtain

$$-\frac{1}{n}\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\left(\rho - E_{kh}\left(\nabla u + \frac{1}{n}\nabla w\right)\right)\frac{\partial w_i}{\partial x_j}dx \geq 0$$

so

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\left(\rho - E_{kh}\left(\nabla u + \frac{1}{n}\nabla w\right)\right)\frac{\partial w_i}{\partial x_j}dx \leq 0,$$

but $u+\frac{1}{n}w \to u$ in $\mathbf{W}^{p(x)}(\Omega)$, thus

$$E_{kh}\left(\nabla u + \frac{1}{n}\nabla w\right) \to E_{kh}(\nabla u) \text{ in } L^{p'(x)}(\Omega).$$

By passing to the limit when $n\to+\infty$, we obtains then

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\left(\rho - E_{kh}(\nabla u)\right)\frac{\partial w_i}{\partial x_j}dx \leq 0\ ,\ \forall w\in\mathbf{W}^{p(x)}(\Omega).$$

By the linearity (we can change w in $-w$), we get:

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\left(\rho - E_{kh}(\nabla u)\right)\frac{\partial w_i}{\partial x_j}dx = 0, \forall w\in\mathbf{W}^{p(x)}(\Omega),$$

we deduce that

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)\rho\frac{\partial w_i}{\partial x_j}dx = \sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_\Omega a_{ijkh}(x)E_{kh}(\nabla u)\frac{\partial w_i}{\partial x_j}dx, \forall w\in\mathbf{W}^{p(x)}(\Omega).$$

We have thus proved that u is a solution of (4.3).

Uniqueness

We suppose that $(E_{kh}(\xi) - E_{kh}(\eta))(\xi_{ij} - \eta_{ij}) > 0$, if and only if $\xi_{ij} \neq \eta_{ij}$, and f does not depend to u. Let u_1 and u_2 be two solutions:

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)E_{kh}(\nabla u_l(x))\frac{\partial v_i}{\partial x_j}dx$$
$$= \langle f, v\rangle_{\left(\mathbf{W}^{p(x)}(\Omega)\right)', \mathbf{W}^{p(x)}(\Omega)},\ l = 1, 2;\ \forall v \in \mathbf{W}^{p(x)}(\Omega).$$

Subtracting term to term and substituting v by $u_1 - u_2$, we obtain:

$$\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}\int_{\Omega} a_{ijkh}(x)(E_{kh}(\nabla u_1) - E_{kh}(\nabla u_2))\left(\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}\right)dx = 0.$$

Since

$$M = a_{ijkh}(x)(E_{kh}(\nabla u_1) - E_{kh}(\nabla u_2))\left(\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{2i}}{\partial x_j}\right) \geq 0,\ i, j, k, h = 1 \text{ to } 3.$$

and $M > 0$ if $\frac{\partial u_{1i}}{\partial x_j} \neq \frac{\partial u_{2i}}{\partial x_j}$; we get $\frac{\partial u_{1i}}{\partial x_j} = \frac{\partial u_{2i}}{\partial x_j}$ in $L^{p(x)}(\Omega)$, and by Hölder's inequality we have $u_1 = u_2$ in $\left(W_0^{1,p(x)}(\Omega)\right)^3$.

6 Conclusion

In this work, we consider the nonlinear elasticity system as Leray–Lions's operators with variable exponent, to study the existence and uniqueness of Dirichlet's problem solution by Galerkin techniques and monotone operator theory. It has been found that these techniques adapt well to this type of problems with different boundary conditions.

From a perspective of this work, we will consider the same problem with the boundary conditions Robin, Tresca or Coulomb.

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References

[1] M. B. Benboubker, *Sur certains problèmes elliptiques quasilinéaires non homogènes de type Dirichlet ou Neumann*, Université Sidi Mohamed Ben Abdellah (2013).

[2] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66**, 1383-1406 (2006).

[3] P. G. Ciarlet, *Mathematical elasticity, Vol. I: Three-Dimensional Elasticity*, North-Holland, Amsterdam (1988).

[4] D.V. Cruz-Uribe, A. Fiorenza: *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Birkhäuser, Basel (2013).

[5] R. Dautray, J.L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Vol.1, Masson (1984).

[6] L. Diening, P. Harjulehto, P. Hästo, M. Růžička, *Lebesgue and Sobolev spaces with Variable Exponents*, **2017**, Lecture Notes in Mathematics, Springer, Heidelberg (2017).

[7] A. El Hachimi, S. Maatouk, Existence of periodic solutions for some quasilinear parabolic problems with variable exponents, *Arab. J. Math.*, **6**, 263-280 (2017).

[8] X. Fan, D. Zhao: On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263**, 424-446 (2001).

[9] T. Gallouët, R. Herbin, *Equations aux dérivées partielles*, Université Aix Marseille, (2013).

[10] J. Leray, J. L. Lions, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93**, 97-107 (1965).

[11] J. L. Lions, *Quelques méthode de résolution des problèmes aux limites non linéaires*, Dunod, Paris (1969).

[12] B. Merouani, R. Boufenouche, Trigonometric series adapted for the study of Dirichlet boundary-value problems of Lamé systems, *Electronic Journal of Differential Equations*, **181**, 1-6 (2015).

[13] B. Merouani, Solutions singulières du système de l'élasticité dans un polygone pour différentes conditions aux limites, *Maghreb Math. Rev.*, **5**, 95-112 (1996).

[14] B. Merouani, Quelques problèmes aux limites pour le système de Lamé dans un secteur plan, *C.R.A.S.*, **304**, série I (1987).

[15] J. T. Oden, Existence theorems for a class of problems in nonlinear elasticity, *Journal of Mathematical Analysis and Applications*, **69**, 51-83 (1979).

[16] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math., **1784**, Springer-Verlag, Berlin, (2000).

[17] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Math. USSR*, **29**, 33-36 (1987).

[18] F. Zoubai, B. Merouani, Study of a mixed problem for a nonlinear elasticity system by topological degree, *Stud. Univ. Babeş-Bolyai Math.* (accepted).

[19] F. Zoubai, B. Merouani, On a pure traction problem for the nonlinear elasticity system in Sobolev spaces with variable exponents, *Stud. Univ. Babeş-Bolyai Math.* (accepted).

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